FUZZY α-IDEALS OF IS-ALGEBRAS

Zhan Jianming and Tan Zhisong

Received December 6, 2002

Abstract. In this paper, we introduce the concept of fuzzy α-ideals of IS-algebras and investigate some properties.

1 Introduction and Preliminaries In 1966, Iseki introduced the concept of BCK/BCI-algebra. For the general development of BCK/BCI-algebras, the ideal theory plays an important role. In 1993, Jun et al introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup. In 1998, for the convenice of study, Jun et al renamed the BCI-semigroups as the IS-algebra and studied further properties. In this paper, we consider the fuzzification of α-ideals of IS-algebras and study their properties.

By a BCI-algebra we mean an algebra $(X;*,0)$ of type $(2,0)$ sntisfying the following conditions.

(I) $((x*y)*(x*z))*(z*y)=0$

(II) $(x*(x*y))*y=0$

(III) $x*x=0$

(IV) $x*y=0$ and $y*x=0$ imply $x=y$ for all $x,y,z\in X$

In any BCI-algebra X one can define a partial order $\leq$ by putting $x\leq y$ if and only if $x*y=0$.

By an IS-algebra we mean a nonempty set X with two binary operation "$*$" and "$\cdot$" and the constant 0 satisfying the axioms:

(I) $I(X)=(X;*,0)$ is a BCI-algebra

(II) $S(X)=(X;\cdot)$ is a semigroup

(III) The operation "$\cdot$" is distribute over the operation "$*$", that is , $x\cdot(y*z)=(x\cdot y)*(x\cdot z)$ and $(x*y)\cdot z=(x\cdot z)*(y\cdot z)$ for all $x,y,z\in X$.

A nonempty subset A of an IS-algebra X is said to be stable if $xa\in A$ whenever $x\in S(X)$ and $a\in A$.

We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu: X\to[0,1]$. For $t\in[0,1]$, the set $U(\mu;t)=\{x\in X\mid \mu(x)\geq t\}$ is called a level subset of μ. A fuzzy set μ in a BCI-algebra X is called a fuzzy ideal of X if (i) $\mu(0)\geq\mu(x)$, (ii) $\mu(x)\geq\mu(x*y)\wedge\mu(y)$ for all $x,y\in X$. A fuzzy set μ in a semigroup $S(X)=(X,\cdot)$ is said to be fuzzy stable if $\mu(xy)\geq\mu(y)$ for all $x,y\in X$. A fuzzy set μ in an IS-algebra X is called a fuzzy ideal of X if $(F1)$ μ is a fuzzy stable set in $S(X),(F2)$ μ is a fuzzy ideal of a BCI-algebra X.

2 Fuzzy α-ideals

Definition 2.1 A nonempty subset A of an IS-algebra X is called an α-ideal of X if

(I1) A is a stable subset of $S(X)$

(I2) for any $x,y,z\in I(X),(x*z)*(0*y)\in I$ and $z\in I$ imply $y*x\in I$.

2000 *Mathematics Subject Classification.* 06F35, 03G25.

Key words and phrases. α-ideals, Fuzzy α-ideals, IS-algebras.

Definition 2.2 A fuzzy set μ in an IS-algebra X is called a fuzzy α-ideal of X if

(F1) μ is a fuzzy stable set in $S(X)$

(F3) $\mu(y * x) \geq \mu((x * z) * (0 * y)) \wedge \mu(z)$ for all $x, y, z \in X$

Example 2.3 Consider an IS-algebra $X = \{0, a, b, c\}$ with Cayley table as follows:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Let μ be a fuzzy set in X defined by $\mu(0) = f(a) = t_0$ and $\mu(b) = (c) = t_1$, where $t_0, t_1 \in [0, 1]$ and $t_0 > t_1$. By routine calculations give that μ is a fuzzy α-ideal of X.

Theorem 2.4 Any fuzzy α-ideal of X is a fuzzy ideal of X

Proof. Suppose that μ is a fuzzy α-ideal of X. Setting $y = z = 0$ in (F3), it follows that $\mu(0 * x) \geq \mu(x)$ for all $x \in X$. Setting $x = z = 0$ in (F3), it follows that $\mu(y) \geq \mu(0 * (0 * y))$ for all $y \in X$. Hence $\mu(x) \geq \mu(0 * (0 * y)) \geq \mu(0 * x)$ for all $x \in X$.

Thus for any $x, z \in X$, from (F3), we have $\mu(x) \geq \mu(0 * x) \geq \mu((x * z) * (0 * 0)) \wedge \mu(z) = \mu(x * z) \wedge \mu(z)$. Therefore μ satisfies (F2) and combining (F1), μ is a fuzzy ideal of X.

The following example shows that the converse of Theorem 2.4 may not be true.

Example 2.5 Let X be an IS-algebra $X = \{0, 1, 2\}$ with coyley table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

$*$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Define $\mu : X \to [0, 1]$ by $\mu(0) = t_0$ and $\mu(1) = \mu(2) = t_1$, where $t_0, t_1 \in [0, 1]$ and $t_0 > t_1$. It's easy to check that μ is a fuzzy ideal of X, but not a fuzzy α-ideal of X as follows:

$$\mu(2 * 1) = \mu(1) = t_1 \ngeq t_0 = \mu((1 * 0) * (0 * 2)) \wedge \mu(0)$$

Theorem 2.6 Let μ be a fuzzy set in an IS-algebra. Then μ is a fuzzy α-ideal of X if and only if the nonempty level set $U(\mu; t)$ of μ is an α-ideal of X for every $t \in [0, 1]$.

Proof. Suppose that μ is a fuzzy α-ideal of X. Let $x \in S(X)$ and $y \in U(\mu; t)$. Then $\mu(y) \geq t$ and so $\mu(xy) \geq \mu(y) \geq t$, which implies that $xy \in U(\mu; t)$, thence $U(\mu; t)$ is a stable subset of $S(X)$. Let $x, y, z \in X$ be such that $(x * z) * (0 * y) \in U(\mu; t)$. Then $\mu((x*z)*(0*y)) \geq t$ and $\mu(z) \geq t$. It follows that $\mu(y*x) \geq \mu((x*z)*(0*y)) \wedge \mu(z) \geq t$, so that $y*x \in U(\mu; t)$. Hence $U(\mu; t)$ is an α-ideal of X. Conversely, assume that the nonempty level set $U(\mu; t)$ of μ is an α-ideal of X for every $t \in [0, 1]$. If there are $x_0, y_0 \in S(X)$ such that $\mu(x_0 y_0) < \mu(y_0)$, then by taking $t_0 = (\mu(x_0 y_0) + \mu(y_0))/2$, we have $\mu(x_0 y_0) < t_0 < \mu(y_0)$. It follows that $y_0 \in U(\mu; t_0)$ and $x_0 y_0 \notin U(\mu; t_0)$. This is a contradiction. Therefore μ is a fuzzy stable set in $S(X)$. Suppose that $\mu(y_0 * x_0) < \mu((x_0 * z_0) * (0 * y_0)) \wedge \mu(z_0)$ for some $x_0, y_0, z_0 \in X$. Putting $s_0 = (\mu(y_0 * x_0) + \mu((x_0 * z_0) * (0 * y_0)) \wedge \mu(z_0))/2$, then $\mu(y_0 * x_0) < s_0 < \mu((x_0 * z_0) * (0 * y_0)) \wedge \mu(z_0)$, which shows that $(x_0 * z_0) * (0 * y_0) \in U(\mu; s_0), z_0 \in U(\mu; s_0)$ but $y_0 * x_0 \notin U(\mu; s_0)$. This is impossible. Hence μ is a fuzzy α-ideal of X.

Theorem 2.7 Let A be an α-ideal of an IS-algebra X and let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t_0 & \text{if} \quad x \in A \\ t_1 & \text{otherwise} \end{cases}$$

where $t_0 > t_1$ in $[0, 1]$. Then μ is a fuzzy α-ideal of X, and $U(\mu; t_0) = A$.

Proof. Notice that

$$U(\mu;t_0)=\begin{cases}\emptyset & \text{if} \quad t_0<t\\ A & \text{if} \quad t_1<t\leq t_0\\ X & \text{if} \quad t\leq t_1\end{cases}$$

It follows from Theorem 2.6 that μ is a fuzzy α-ideal of X. Clearly, we have $U(\mu;t_0)=A$.

Theorem 2.7 suggests that any α-ideal of an IS-algebra X can be realized as a level α-ideal of some fuzzy α-ideal of X, we now consider the converse of Theorem 2.7.

Theorem 2.8 For a nonempty subset A of an IS-algebra X, let μ be a fuzzy set in X which is given in Theorem 2.7. If μ is a fuzzy α-ideal of X. then A is an α-ideal of X.
Proof. Assume that μ ia a fuzzy α-ideal of X and let $x\in S(X)$ and $y\in A$. Then $\mu(xy)\geq \mu(y)=t_0$ and so $xy\in U(x_0;t_0)=A$. Hence A is a stable subset of $S(X)$. Let $x,y,z\in I(X)$ be such that $(x*z)*(0*y)\in A$ and $z\in A$. It follows that $\mu(y*x)\geq\mu((x*z)*0*y))\wedge\mu(z)=t_0$. So that $x\in U(\mu;t_0)=A$. This competes the proof.

For a subset A of X, we call

$$\chi(x)=\begin{cases}1 & \text{if} \quad x\in A\\ 0 & \text{otherwise}\end{cases}$$

the characteristic function of A. Clearly χ_I is a fuzzy set of X.

Theorem 2.9 Let A be a subset of X. Then χ_I is a fuzzy α-ideal of X if and only if A is an α-ideal of X.
Proof. It's straightforward by using Theorem 2.7 and Theorem 2.8.

References

[1] Y. B. Jun, S. M. Hong & E. H. Roh,*BCI-semigroups*, Honam Math. J. 15(1993), 59-64

[2] Y. B. Jun & E. H. Roh, *Fuzzy p&φ-ideals in IS-algebras*, J.Fuzzy Math , 7(1999), 473-480.

[3] E. H. Roh, Y. B. Jun & W. H. Shim, *Fuzzy associative φ-ideals of IS-algerbas*, Int. J. Math & Math Sci. 24 (2000), 729-735.

[4] Y. L. Liu & X. H. Zhang, *Fuzzy α-ideals of BCI-algebras*, Adv in Math. (in Chinese), 31(2002), 65-73.

[5] Y. L. Liu & J. Meng, *q-ideals and α-ideals in BCI-algebras*, SEA Bull Math. 24(2000), 243-253.

[6] L. A. Zadeh, *Fuzzy sets*, Inform & control 8(1965), 338-353

Department of Mathematics, Hubei Institute for Nationalities,Enshi, Hubei Province, 445000.P.R.China
E-mail:zhanjianming@hotmail.com