UNIFORM STRUCTURE ON HYPER K-ALGEBRAS

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Abstract. In this note first we define an equivalence relation on a hyper K-algebra H and then we construct a uniform structure on H, when this uniformity gives a topology on H.

1 Introduction

The hyperalgebraic structure theory was introduced by F. Marty [7] in 1934. Imai and Iseki [4] in 1966 introduced the notion of a BCK-algebra. Recently [3] Borzooei, Jun and Zahedi et.al. applied the hyperstructure to BCK-algebras and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. Now, in this note we use this structure and construct a uniform structure on a hyper K-algebra H, which gives a topology on H.

2 Preliminaries

Definition 2.1. [3] Let H be a nonempty set and "$\circ$" be a *hyperoperation* on H, that is "$\circ$" is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$. Then H is called a *hyper K-algebra* if it contains a constant "0" and satisfies the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$
(HK3) $x < x$
(HK4) $x < y,\ y < x \Rightarrow x = y$
(HK5) $0 < x$,
for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.
Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup\limits_{\substack{a\in A\\ b\in B}} a \circ b$ of H.

Theorem 2.2. [3] Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H the following hold:
(i) $(x \circ z) \circ (x \circ y) < y \circ z$, (ii) $x \in x \circ 0$,
(iii) $A \subseteq B$ implies $A < B$, (iv) $(A \circ C) \circ (A \circ B) < B \circ C$,
(v) $(A \circ C) \circ (B \circ C) < A \circ B$,

Definition 2.3. [3] Let I be a nonempty subset of a hyper K-algebra $(H, \circ, 0)$ and $0 \in I$. Then I is called a *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Definition 2.4. [2] We say that the hyper K-algebra H satisfies the *transitive condition* if for all $x, y, z \in H$, $x < y$ and $y < z$ imply that $x < z$.

Definition 2.5. [1] We say that the hyper K-algebra H satisfies the *strong transitive condition* if for all $A, B, C \subseteq H$, $A < B$ and $B < C$ imply that $A < C$.

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Proposition 2.6. [1] Let H satisfies the strong transitive condition. If I is a hyper K-ideal of H and $A, B \subseteq H$, $A \circ B < I$ and $B < I$, then $A < I$.□

3 Uniformity in hyper K-algebras

From now on H is a hyper K-algebra which satisfies the strong transitive condition.

Definition 3.1. Let I be a hyper K-ideal of H. We define the *relation* $\sim_I$ on H as follows:

$$x \sim_I y \text{ if and only if } x \circ y < I \text{ and } y \circ x < I.$$

If A, B are subsets of H then we define $A \sim_I B$ if and only if $\exists a \in A$, $\exists b \in B$ such that $a \sim_I b$.

Proposition 3.2. The relation $\sim_I$ is an equivalence relation on H.

Proof. (i) Since $0 \in x \circ x$, then $x \circ x < I$. Hence $x \sim_I x$.

(ii) Clearly $\sim_I$ is symmetric.

(iii) Let $x \sim_I y$ and $y \sim_I z$. Then $x \circ y < I$, $y \circ x < I$, $y \circ z < I$ and $z \circ y < I$. Since $(x \circ z) \circ (x \circ y) < y \circ z$ and $y \circ z < I$, so by strong transitivity of H we get that $(x \circ z) \circ (x \circ y) < I$. Thus from $(x \circ z) \circ (x \circ y) < I$, $x \circ y < I$ and Proposition 2.6 we conclude that $x \circ z < I$. Similarly $z \circ x < I$, therefore $x \sim_I z$.□

Proposition 3.3. The relation $\sim_I$ is an equivalence relation on $\mathcal{P}^*(H)$.

Proof. (i) Since $x \circ x < I$ for all x in H, so $A \sim_I A$ for any A in $\mathcal{P}^*(H)$.

(ii) Let $A \sim_I B$. Then there exist $a \in A$ and $b \in B$ such that $a \sim_I b$. So $b \sim_I a$ by Proposition 3.2, thus $B \sim_I A$.

(iii) Let $A \sim_I B$ and $B \sim_I C$. Then there exist $a \in A$, $b, b' \in B$, and $c \in C$ such that $a \sim_I b$ and $b' \sim_I c$. So we have $a \circ b < I$, $b \circ a < I$, $b' \circ c < I$ and $c \circ b' < I$. Now by Theorem 2.2 (i) we have $(A \circ C) \circ (A \circ B) < B \circ C$. Since $b' \circ c \subseteq B \circ C$ and $b' \circ c < I$, we get that $B \circ C < I$. Therefore the strong transitivity of H implies that $(A \circ C) \circ (A \circ B) < I$. Also $a \circ b < I$, gives that $A \circ B < I$. Thus by Proposition 2.6 we conclude that $A \circ C < I$. Similarly $C \circ A < I$. So there are $a' \in A$ and $c' \in C$ such that $a' \circ c' < I$. Since $c' \circ a' \subseteq C \circ A$ and $C \circ A < I$, then $c' \circ a' < I$ by Theorem 2.2 (iii) and strong transitivity of H. That is $a' \sim_I c'$, therefore $A \sim_I C$.□

Lemma 3.4. Let $A, B \in \mathcal{P}^*(H)$, and I be a hyper K-ideal of H. Then $A \circ B < I$ and $B \circ A < I$ imply that $A \sim_I B$.

Proof. Since $A \circ B < I$, there are $a \in A$ and $b \in B$ such that $a \circ b < I$. We have $b \circ a \subseteq B \circ A$ so $b \circ a < B \circ A$. Now $b \circ a < B \circ A$ and $B \circ A < I$ imply that $b \circ a < I$, by strong transitivity of H. Thus $a \sim_I b$, which implies that $A \sim_I B$.□

Theorem 3.5. The relation $\sim_I$ is a congruence relation on H.

Proof. By considering Proposition 3.2, it is enough to show that If $x \sim_I y$ and $u \sim_I v$, then $x \circ u \sim_I y \circ v$. Since $u \sim_I v$, we have $v \circ u < I$ and $u \circ v < I$. So $(x \circ u) \circ (x \circ v) < v \circ u$ and $v \circ u < I$ imply that $(x \circ u) \circ (x \circ v) < I$. Similarly $(x \circ v) \circ (x \circ u) < I$. Therefore by Lemma 3.4

$$x \circ u \sim_I x \circ v. \tag{1}$$

Similarly from $(x \circ v) \circ (y \circ v) < x \circ y$, $(y \circ v) \circ (x \circ v) < y \circ x$, $x \circ y < I$ and $y \circ x < I$ we can see that

$$x \circ v \sim_I y \circ v. \tag{2}$$

Since $\sim_I$ is an equivalence relation on $\mathcal{P}^*(H)$, then (1) and (2) imply that $x \circ u \sim_I y \circ v$.□

Let X be a non empty set and U and V be any subsets of $X \times X$. We let
$U \diamond V = \{(x, y) \in X \times X \mid \text{for some } z \in X, (x, z) \in U \text{ and } (z, y) \in V\}$,
$U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\}$,
$\Delta = \{(x, x) \in X \times X \mid x \in X\}$.

Definition 3.6. [5] By a *uniformity* on X we shall mean a non empty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:
(U_1) $\Delta \subseteq U$ for any $U \in \mathcal{K}$,
(U_2) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$,
(U_3) if $U \in \mathcal{K}$, then there exist a $V \in \mathcal{K}$, such that $V \diamond V \subseteq U$,
(U_4) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$,
(U_5) if $U \in \mathcal{K}$, and $U \subseteq V \subseteq X \times X$ then $V \in \mathcal{K}$.

The pair (X,$\mathcal{K}$) is called a *uniform structure*.

Theorem 3.7. Let I be a hyper K-ideal of H and $U_I = \{(x, y) \in X \times X \mid x \sim_I y\}$. If

$$\mathcal{K}^* = \{U_I \mid I \text{ is a hyper } K\text{-ideal of } H\}$$

then $\mathcal{K}^*$ satisfies the conditions $(U_1) - (U_4)$.

Proof. (U_1): Since $0 \in x \circ x$, hence $x \circ x < I$ for any hyper K-ideal I of H. Thus $x \sim_I x$ for any x in H, hence $\Delta \subseteq U_I$ for all $U_I \in \mathcal{K}^*$.

(U_2): For any $U_I \in \mathcal{K}^*$, we have

$$(x, y) \in (U_I)^{-1} \Longleftrightarrow (y, x) \in U_I \Longleftrightarrow y \sim_I x \Longleftrightarrow x \sim_I y \Longleftrightarrow (x, y) \in U_I.$$

Hence $(U_I)^{-1} = U_I \in \mathcal{K}^*$.

(U_3): For any $U_I \in \mathcal{K}^*$, the transitivity of $\sim_I$ implies that $U_I \diamond U_I \subseteq U_I$.

(U_4): For any $U_I, U_J \in \mathcal{K}^*$, we claim that $U_I \cap U_J = U_{I \cap J}$. Let $(x, y) \in U_I \cap U_J$. Then $x \sim_I y$ and $x \sim_J y$. So we have $x \circ y < I$, $y \circ x < I$, $x \circ y < J$ and $y \circ x < J$. Thus there exist $t \in x \circ y$ and $i \in I$ such that $t < i$, that is $0 \in t \circ i$. So $t \circ i < I$. Therefore $i \in I$ implies that $t \in I$. Hence $t \in (x \circ y) \cap I$. Similarly we can see that there is $r \in H$ such that $r \in (x \circ y) \cap J$. Since $0 \in (x \circ y) \circ (x \circ y)$, so

$$(x \circ y) \circ (x \circ y) < J \tag{3}$$

we have $t \circ r \subseteq (x \circ y) \circ (x \circ y)$, thus (3) and the strong transitivity of H imply that $t \circ r < J$. Since $r \in J$ we get that $t \in J$, hence $t \in I \cap J \cap (x \circ y)$. So $x \circ y < I \cap J$. Similarly $y \circ x < I \cap J$, therefore $x \sim_{I \cap J} y$. Thus $(x, y) \in U_{I \cap J}$. Conversely, let $(x, y) \in U_{I \cap J}$. Then $x \sim_{I \cap J} y$, hence $x \circ y < I \cap J$ and $y \circ x < I \cap J$. We have $I \cap J < I, J$. So the strong transitivity of H implies that $x \circ y < I$, $y \circ x < I$, $x \circ y < J$ and $y \circ x < J$. Thus $x \sim_I y$ and

$x \sim_J y$, therefore $(x, y) \in (U_I \cap U_J)$. So $U_I \cap U_J = U_{I\cap J}$, Since $I \cap J$ is a hyper K-ideal of H, thus $U_I \cap U_J \in \mathcal{K}^*$.□

Theorem 3.8. Let $\mathcal{K} = \{U \subseteq X \times X \mid U_I \subseteq U$ for some $U_I \in \mathcal{K}^*\}$. Then $\mathcal{K}$ satisfies a uniformity on H and the pair $(H,\mathcal{K})$ is a uniform structure.

Proof. By applying Theorem 3.7 we can show that $\mathcal{K}$ satisfies the conditions $(U_1) - (U_4)$. Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$. Then there exists a $U_I \subseteq U \subseteq V$, which means that $V \in \mathcal{K}$. This proves the theorem.□

Given a $x \in H$ and $U \in \mathcal{K}$, we define

$$U[x] := \{y \in H \mid (x, y) \in U\}.$$

Theorem 3.9. For any x in H, The collection $\mathcal{U}_x := \{U[x] \mid U \in \mathcal{K}\}$, forms a neighborhood base at x , making H a topological space.

Proof. Note that $x \in U[x]$ for each $x \in H$. Since $U_1[x] \cap U_2[x] = (U_1 \cap U_2)[x]$, the intersection of neighborhoods is also a neighborhood. Finally, if $U[x] \in \mathcal{U}_x$ then by (U_3) there exists a $V \in \mathcal{K}$ such that $V \diamond V \subseteq U$. Hence for any $y \in V[x]$, we can check that $V[y] \subseteq U[x]$, this completes the prove of theorem.□

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