Scientiae Mathematicae Japonicae Online, **Vol. 8**, (2003), 57–70

COMPLEX MATRIX VARIATE CAUCHY DISTRIBUTION

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Received October 11, 2002; revised November 18, 2002

ABSTRACT. In this article the authors have defined the complex matrix variate Cauchy distribution. The density function has been derived using complex random matrices having dependent normal entries. Some properties of this distribution such as marginal and conditional distributions and distributions of linear and matrix quadratic forms have also been studied.

1 Introduction. In statistical distribution theory, if x and y are independent standard normal variates, then the p.d.f.(probability density function) of the ratio $z = x/y$ is given by

$$\frac{1}{\pi} \cdot \frac{1}{1+z^2},\ -\infty < z < \infty. \tag{1.1}$$

The above density function, that is commonly referred to as the Cauchy density, has been studied in the mathematical world for over three centuries. The curve proportional to $(x^2 + a^2)^{-1}$ has drawn attention of various people including Sir Isaac Newton, Gottfried Leibnitz, Christian Huygens, Guido Grandi and María Agnesi. For the systematic treatment of this distribution the reader is referred to Johnson, Kotz and Balakrishnana [13].

In the matrix variate case, the variables x and y are replaced by the independent random matrices $X\,(p \times r)$ and $Y\,(p \times p)$ with entries having independent standard normal distribution. The matrix variate generalization of the ratio $z = x/y$ is $Z = Y^{-1}X$. The p.d.f. of Z is given by

$$\prod_{i=1}^{p} \frac{\Gamma[(r+p-i+1)/2]}{\pi^{r/2}\Gamma[(p-i+1)/2]} \det(I_p + ZZ')^{-(r+p)/2},\ Z \in \mathbb{R}^{p\times r}. \tag{1.2}$$

Thus (1.2) is the natural matrix variate generalization of (1.1). Recently, Bandekar and Nagar [2] defined and derived this distribution using independent random matrices X and Y having matrix normal distribution with dependent entries. They have also studied several of its properties including marginal and conditional distributions, distributions of linear and matrix quadratic forms.

In this article we propose and study complex matrix variate Cauchy distribution.

The complex matrix variate distributions play an important role in various fields of research. Applications of complex random matrices can be found in multiple time series analysis, nuclear physics and radio communications (Carmeli [3], Krishnaiah [16], Mehta [17] and Smith and Gao [19]). A number of results on the distribution of complex random matrices have also been derived. The complex matrix variate Gaussian distribution was introduced by Wooding [23], Turin [22], and Goodman [5]. The complex Wishart distribution was studied by Goodman [5, 6], Srivastava [20], Hayakawa [10], Chikuse [4] and Gupta and

2000 *Mathematics Subject Classification.* 62H10, 62E15.

Key words and phrases. Cauchy distribution; quadratic form; complex matrix variate; marginal distribution; Moore-Penrose inverse; transformation; unitary group.

Kabe [8]. James [12] and Khatri [14] derived the complex central as well as the noncentral matrix variate beta distributions. Distributional results on quadratic forms involving complex normal variables were given by Khatri [15] and Gupta and Conradie [7]. Systematic treatment of the distributions of complex random matrices was given by Tan [21] which included the Gaussian, Wishart, beta, and Dirichlet distributions.

In Section 2, the complex matrix variate Cauchy distribution is defined in its most general form. This distribution is derived using the representation $Z = Y^{-1}X$ when entries of the independent random matrices X and Y are dependent complex normal variates with zero mean. In Section 3, we study certain properties of the complex matrix variate Cauchy distribution. Section 4 gives distributions of certain matrix quadratic forms involving complex Cauchy matrix. In the last section, we derive the distribution of $Z = Y^{-1}X$ when one of the complex normal random matrices has non zero mean matrix.

2 Complex Matrix Variate Cauchy Distribution. In this section we will define matrix variate Cauchy distribution and derive it using Complex random matrices having complex matrix normal distribution. We first state the following notations and results (Khatri [14], Srivastava [20], Andersen, Højbjerre, Sørensen and Eriksen [1]) that will be utilized in this and subsequent sections. Let $A = (a_{ij})$ be a $p \times p$ matrix of complex numbers. Then, A' denotes the transpose of A; $\bar{A}$ denotes conjugate of A; A^H denotes conjugate transpose of A; $\operatorname{tr}(A) = a_{11} + \cdots + a_{pp}$; $\operatorname{etr}(A) = \exp(\operatorname{tr}(A))$; $\det(A) =$ determinant of A; $\det(A)_+ =$ absolute value of $\det(A)$; $A = A^H > 0$ means that A is Hermitian positive definite and $A^{1/2}$ denotes the unique Hermitian positive definite square root of $A = A^H > 0$. Further, for the partition $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$, $\det(A_{11}) \neq 0$, the Schur compliment is defined as $A_{22\cdot 1} = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Lemma 2.1 *Let $Z\,(p \times n)$ and $W\,(p \times n)$ be complex matrices of functionally independent complex variables and let $G\,(p \times p)$ and $K\,(n \times n)$ be nonsingular matrices. The Jacobian of the transformation $Z = GWK$ is $J(Z \to W) = \det(GG^H)^n \times \det(KK^H)^p$.*

Lemma 2.2 *Let X be a $p \times p$ Hermitian positive definite matrix of functionally independent complex variables. Let $T\,(p \times p)$ be a lower triangular matrix of functionally independent complex variables with real and positive diagonal elements, then the transformation $X = TT^H$ gives $J(X \to T) = 2^p \prod_{j=1}^{p} t_{jj}^{2(p-j)+1}$.*

Lemma 2.3 *Let $T\,(p \times p)$ be a triangular matrix of functionally independent complex variables with real and positive diagonal elements, then $J(T \to T^{-1}) = \prod_{j=1}^{p} t_{jj}^{2p}$.*

Definition 2.1 *The complex multivariate gamma function, denoted by $\Gamma_p(\alpha)$, is defined by*

$$\tilde{\Gamma}_p(\alpha) = \int_{X=X^H>0} \operatorname{etr}(-X) \det(X)^{\alpha-p}\, dX,\ \operatorname{Re}(\alpha) > p-1. \tag{2.1}$$

By evaluating the integral in (2.1), the complex multivariate gamma function can be expressed as product of ordinary gamma functions

$$\tilde{\Gamma}_p(\alpha) = \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma(\alpha - i + 1), \operatorname{Re}(\alpha) > p-1. \tag{2.2}$$

Lemma 2.4 *If X is a matrix of complex elements, of order $p \times n$, $p \leq n$, and of rank p, then there exists a unique triangular matrix $T\,(p \times p)$ with real and positive diagonal elements* *and a semi-unitary matrix $U_1\,(p\times n)$, $U_1U_1^H=I_p$, such that $X=TU_1$. Further, if T is lower triangular, then*

$$J(X\to T,U_1)=\prod_{j=1}^{p}t_{jj}^{2(n-j)+1}\tilde{g}_{n,p}(U_1)$$

and if T is upper triangular, then

$$J(X\to T,U_1)=\prod_{j=1}^{p}t_{jj}^{2(n-p+j-1)+1}\tilde{g}_{n,p}(U_1)$$

where $\tilde{g}_{n,p}(U_1)$ is the function of U_1 only, independent of T.

Here $\tilde{g}_{n,p}(U_1)\,dU_1$ defines the invariant measure on the space $U(p,n)=\{U_1(p\times n):U_1U_1^H=I_p\}$. It has been shown by Srivastava [20] that, for $p\le n$,

$$\int_{U(p,n)}\tilde{g}_{n,p}(U_1)\,dU_1=c_1=\frac{2^p\pi^{np}}{\tilde{\Gamma}_p(n)}. \tag{2.3}$$

Thus, $c_1^{-1}\tilde{g}_{n,p}(U_1)\,dU_1$ is the unit invariant measure (denoted by $[dU_1]$) on $U(p,n)$. For $p=n$, the semi-unitary matrix U_1 reduces to a unitary matrix U and the invariant measure defined on the unitary group $U(p)$ is denoted by $[dU]$.

Lemma 2.5 *Let $X\,(p\times n)$ be a complex matrix of rank $p\,(\le n)$ and $f(X)$ be a function of X which depends on X through XX^H only . That is $f(X)=g(XX^H)$ for some function g. Then,*

$$\int_{XX^H=W}f(X)\,dX=\frac{\pi^{np}}{\tilde{\Gamma}_p(n)}\det(W)^{n-p}g(W),\ W=W^H>0. \tag{2.4}$$

Let A be an $m\times n$ matrix of complex entries having full rank. The Moore-Penrose inverse A^+ of A, for $m\ge n$, is defined by $A^+=(A^HA)^{-1}A^H$. If $m\le n$, then A^+ is given by the expression $A^+=A^H(AA^H)^{-1}$. Recently, Olkin [18] has surveyed various methods to derive density of the Moore-Penrose of a real random matrix. In the following theorem the p.d.f. of the Moore-Penrose of a complex random matrix is obtained.

Theorem 2.1 *Let $Z\,(p\times n)$ be a complex random matrix of rank $p\le n$ with p.d.f. $f_Z(Z)$. Then, the p.d.f. of its Moore-Penrose inverse $Y=Z^+=Z^H(ZZ^H)^{-1}$ is given by*

$$\det(Y^HY)^{-2n}f_Z(Y^+),\ Y\in\mathbb{C}^{n\times p} \tag{2.5}$$

where Y^+ is the Moore-Penrose inverse of Y defined by $Y^+=(Y^HY)^{-1}Y^H$.

Proof: We use the transformation $Z=TL$ and $(Z^+)^H=(T^{-1})^HL$ with the Jacobian

$$\begin{aligned} J(Z\to(Z^+)^H) &= J(Z\to T,L)J(T\to T^{-1})J(T^{-1},L\to(Z^+)^H)\\ &= \tilde{g}_{n,p}(L)\prod_{j=1}^{p}t_{jj}^{2(n-j)+1}\prod_{j=1}^{p}t_{jj}^{2p}[\prod_{j=1}^{p}t_{jj}^{-2(n-p+j-1)-1}\tilde{g}_{n,p}(L)]^{-1}\\ &= \det(TT^H)^{2n}=\det((Z^+)^HZ^+)^{-2n}=\det(Y^HY)^{-2n}. \end{aligned} \tag{2.6}$$

Now substituting $Y=Z^+$ with the Jacobian (2.6) in the density of Z, we get the density of Y as $\det(Y^HY)^{-2n}f_Z(Y^+)$, $Y\in\mathbb{C}^{n\times p}$. ∎

Definition 2.2 *The complex random matrix* $X\,(p \times n)$ *is said to have a complex matrix variate normal distribution with mean matrix* $M\,(p\times n)$ *and covariance matrix* $\Sigma \otimes \Psi$ *where* $\Sigma\,(p \times p)$ *and* $\Psi\,(n \times n)$ *are Hermitian positive definite, denoted by* $X \sim \mathbb{C}N_{p,n}(M, \Sigma \otimes \Psi)$, *if its p.d.f is given by*

$$\pi^{-pn} \det(\Sigma)^{-n} \det(\Psi)^{-p} \operatorname{etr}[-\Sigma^{-1}(X-M)\Psi^{-1}(X-M)^H],\ X \in \mathbb{C}^{p\times n}. \tag{2.7}$$

Now, we define complex matrix variate Cauchy distribution.

Definition 2.3 *The complex random matrix* $Z\,(p\times r)$ *is said to have a generalized complex matrix variate Cauchy distribution with parameters* Σ, Ψ *and* M *if its p.d.f. is given by*

$$\begin{aligned}&\frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)} \det(\Sigma)^{-r} \det(\Psi)^{-p}\\ &\qquad \times \det(I_p + \Sigma^{-1}(Z-M)\Psi^{-1}(Z-M)^H)^{-(r+p)},\ Z \in \mathbb{C}^{p\times r},\end{aligned} \tag{2.8}$$

where $\Sigma\,(p \times p)$, $\Psi\,(r\times r)$ *are Hermitian positive definite and* $M \in \mathbb{C}^{p\times r}$.

We will designate this density by $Z \sim \mathbb{C}CAU_{p,r}(M, \Sigma, \Psi)$.

The complex matrix variate Cauchy distribution is a member of the complex matrix variate elliptically contoured family of distributions. When $r = 1$ or $p = 1$ this distribution reduces to a multivariate Cauchy distribution. More specifically when $r = 1$, $Z = \boldsymbol{z}\,(p\times 1)$, $M = \boldsymbol{\mu}\,(p\times 1)$, $\Psi = \psi$ and the density in (2.8) simplifies to

$$\pi^{-p}\,\Gamma(p+1) \det(\psi\Sigma)^{-1} \left(1 + (\boldsymbol{z}-\boldsymbol{\mu})^H (\psi\Sigma)^{-1}(\boldsymbol{z}-\boldsymbol{\mu})\right)^{-(p+1)},\ \boldsymbol{z} \in \mathbb{C}^p$$

which will be denoted by $\boldsymbol{z} \sim \mathbb{C}CAU_p(\boldsymbol{\mu}, \psi\Sigma)$. For $p = 1$ by taking $M^H = \boldsymbol{\nu}\,(r\times 1)$, $\Sigma = \sigma$ it is easily seen that $\boldsymbol{z}^H \sim \mathbb{C}CAU_r(\boldsymbol{\nu}, \sigma\Psi)$.

The complex matrix variate Cauchy density (2.8) can be derived using random matrices having complex matrix variate normal distribution as shown in the following theorems.

Theorem 2.2 *Let* $X \sim \mathbb{C}N_{p,r}(0, I_p \otimes \Psi)$ *independent of* $Y \sim \mathbb{C}N_{p,p}(0, I_p \otimes \Sigma^{-1})$. *Define*

$$Z = Y^{-1}X + M \tag{2.9}$$

where $M\,(p\times r)$ *is a complex matrix of constants. Then,* $Z \sim \mathbb{C}CAU_{p,r}(M, \Sigma, \Psi)$.

Proof: The joint density of X and Y is given by

$$\pi^{-p(p+r)} \det(\Psi^{-1}\Sigma)^p \operatorname{etr}\left[-\{Y\Sigma Y^H + X\Psi^{-1}X^H\}\right],\ X \in \mathbb{C}^{p\times r},\ Y \in \mathbb{C}^{p\times p}.$$

Substituting $Z = Y^{-1}X + M$ with Jacobian $J(X \to Z) = \det(YY^H)_+^r$ and integrating out Y we get the density of Z as

$$\begin{aligned}&\pi^{-p(p+r)} \det(\Psi^{-1}\Sigma)^p \int_{Y\in\mathbb{C}^{p\times p}} \det(Y^HY)^r\\ &\qquad\qquad \times \operatorname{etr}\left[-\{\Sigma + (Z-M)\Psi^{-1}(Z-M)^H\}Y^HY\right] dY\\ &= \pi^{-p(p+r)} \det(\Psi^{-1}\Sigma)^p \int_{A=A^H>0}\int_{Y^HY=A} \det(Y^HY)^r\\ &\qquad\qquad \times \operatorname{etr}\left[-\{\Sigma + (Z-M)\Psi^{-1}(Z-M)^H\}Y^HY\right] dY\,dA\\ &= \frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)} \det(\Sigma)^{-r}\det(\Psi)^{-p}\\ &\quad \times \det(I_p + \Sigma^{-1}(Z-M)\Psi^{-1}(Z-M)^H)^{-(r+p)},\ Z\in\mathbb{C}^{p\times r}\end{aligned}$$

where the last step has been obtained by using Lemma 2.5 and complex multivariate Gamma integral. ■

Theorem 2.3 *Let* $X \sim \mathbb{C}N_{p,p+r}(0, I_p \otimes \Psi)$. *Partition* X *and* Ψ *as* $X = (X_{1c}\ X_{2c})$, $X_{1c}\,(p \times p)$ *and* $\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}$, $\Psi_{11}\,(p \times p)$. *Then* $Z = X_{1c}^{-1} X_{2c} \sim \mathbb{C}CAU_{p,r}(\Psi_{11}^{-1}\Psi_{12}, \Psi_{11}^{-1}, \Psi_{22\cdot 1})$.

Proof: The p.d.f. of X is given by

$$\pi^{-p(r+p)} \det(\Psi)^{-p} \operatorname{etr}(-X\Psi^{-1}X^H),\ X \in \mathbb{C}^{p\times(p+r)}.$$

Now, writing the quadratic form in X in its components,

$$X\Psi^{-1}X^H = X_{1c}\Psi_{11}^{-1}X_{1c}^H + (X_{2c} - X_{1c}\Psi_{11}^{-1}\Psi_{12})\Psi_{22\cdot 1}^{-1}(X_{2c} - X_{1c}\Psi_{11}^{-1}\Psi_{12})^H \tag{2.10}$$

and using the result $\det(\Psi) = \det(\Psi_{11})\det(\Psi_{22\cdot 1})$, the density of X is restated as

$$\frac{\operatorname{etr}\left[-X_{1c}\Psi_{11}^{-1}X_{1c}^H - (X_{2c} - X_{1c}\Psi_{11}^{-1}\Psi_{12})\Psi_{22\cdot 1}^{-1}(X_{2c} - X_{1c}\Psi_{11}^{-1}\Psi_{12})^H\right]}{\pi^{p(r+p)}\det(\Psi_{11})^p \det(\Psi_{22\cdot 1})^p},\ X \in \mathbb{C}^{p\times(p+r)}.$$

Substituting $Z = X_{1c}^{-1}X_{2c}$ with the Jacobian $J(X_{2c} \to Z) = \det(X_{1c}X_{1c}^H)_+^r$ above and integrating X_{1c} using Lemma 2.5 we get the desired result. ■

If $X \sim \mathbb{C}N_{p,r}(0, \Sigma\otimes\Psi)$ and $y \sim \mathbb{C}N(0,1)$ are independent, then the p.d.f. of $Z = X/y + M$, where M is a non-random complex matrix, is derived as

$$\begin{aligned} &\frac{\Gamma(rp+1)}{\pi^{rp}} \det(\Sigma)^{-r}\det(\Psi)^{-p} \\ &\qquad \times(1 + \operatorname{tr}\{\Sigma^{-1}(Z-M)\Psi^{-1}(Z-M)^H\})^{-(rp+1)},\ Z \in \mathbb{C}^{p\times r}. \end{aligned} \tag{2.11}$$

By putting the matrix quadratic form in the vec notations it can easily be seen that the density (2.11) is a multivariate Cauchy density.

3 Properties.

In this section we will give certain properties of the matrix variate Cauchy distribution.

It is useful to see that if $Z \sim \mathbb{C}CAU_{p,r}(M, \Sigma, \Psi)$, then $Z^H \sim \mathbb{C}CAU_{r,p}(M^H, \Psi, \Sigma)$.

The marginal distribution of a submatrix of the complex Cauchy matrix is also Cauchy. However, the conditional distribution of a submatrix of a complex Cauchy matrix given the remaining components is complex matrix-t. Thus, before deriving results on marginal and conditional distributions, we define complex matrix variate-t distribution.

Definition 3.1 *The complex random matrix* $T\,(p\times r)$ *is said to have a complex matrix variate* t*-distribution with parameters* $M\,(p\times r)$, $\Sigma\,(p\times p)$, $\Omega\,(r\times r)$, *and* n *if its p.d.f. is given by*

$$\begin{aligned} &\frac{\tilde{\Gamma}_p(n+r+p-1)}{\pi^{rp}\tilde{\Gamma}_p(n+p-1)}\det(\Sigma)^{-r}\det(\Omega)^{-p} \\ &\qquad \times\det(I_p + \Sigma^{-1}(T-M)\Omega^{-1}(T-M)^H)^{-(n+r+p-1)}, T \in \mathbb{C}^{p\times r} \end{aligned} \tag{3.1}$$

where Ω *and* Σ *are Hermitian positive definite matrices and* $n > 0$.

A notation that designates that the complex random matrix T has density function (3.1) is $T \sim \mathbb{C}T_{p,r}(n, M, \Sigma, \Omega)$. When $r = 1$ or $p = 1$, this distribution reduces to a complex multivariate t-distribution. More precisely, when $r = 1$, $T = \boldsymbol{t}\,(p \times 1)$, $M = \boldsymbol{\mu}\,(p \times 1)$, $\Omega = \omega$ and the above density simplifies to

$$\frac{\Gamma(n+p)}{\pi^p \Gamma(n)} \det(\omega\Sigma)^{-1} \Big(1 + (\boldsymbol{t} - \boldsymbol{\mu})^H (\omega\Sigma)^{-1} (\boldsymbol{t} - \boldsymbol{\mu})\Big)^{-(n+p)}, \ \boldsymbol{t} \in \mathbb{C}^p,$$

which is a complex multivariate t-density. For $p = 1$, by taking $M = \boldsymbol{\nu}^H$ and $\Sigma = \sigma$ one can easily see that $r \times 1$ random vector $T^H \equiv \boldsymbol{t}$ has complex multivariate t-density. Now, we turn to our problem of deriving marginal and conditional distributions.

Theorem 3.1 *Let $Z \sim \mathbb{C}CAU_{p,r}(M, \Sigma, \Psi)$ and partition Z, M, Σ and Ψ as*

$$Z = \begin{pmatrix} Z_{1r} \\ Z_{2r} \end{pmatrix} = \begin{pmatrix} Z_{1c} & Z_{2c} \end{pmatrix}, \ M = \begin{pmatrix} M_{1r} \\ M_{2r} \end{pmatrix} = \begin{pmatrix} M_{1c} & M_{2c} \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \ \textit{and} \ \Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix},$$

where for $i, j = 1, 2$, Z_{ir} and M_{ir} are $p_i \times r$, Z_{ic} and M_{ic} are $p \times r_i$, Σ_{ij} is $p_i \times p_j$, Ψ_{ij} is $r_i \times r_j$, $p_1 + p_2 = p$ and $r_1 + r_2 = r$. Then,

(i) $Z_{1c} \sim \mathbb{C}CAU_{p,r_1}(M_{1c}, \Sigma, \Psi_{11})$,

$$\begin{aligned} Z_{2c}|Z_{1c} \sim\ & \mathbb{C}T_{p,r_2}(r_1 + 1, M_{2c} + (Z_{1c} - M_{1c})\Psi_{11}^{-1}\Psi_{12}, \\ & \Sigma(I_p + \Sigma^{-1}(Z_{1c} - M_{1c})\Psi_{11}^{-1}(Z_{1c} - M_{1c})^H), \Psi_{22\cdot 1}), \end{aligned}$$

and (ii) $Z_{1r} \sim \mathbb{C}CAU_{p_1,r}(M_{1r}, \Sigma_{11}, \Psi)$,

$$\begin{aligned} Z_{2r}|Z_{1r} \sim\ & \mathbb{C}T_{p_2,r}(p_1 + 1, M_{2r} + \Sigma_{21}\Sigma_{11}^{-1}(Z_{1r} - M_{1r}), \Sigma_{22\cdot 1}, \\ & (I_r + (Z_{1r} - M_{1r})^H \Sigma_{11}^{-1}(Z_{1r} - M_{1r})\Psi^{-1})\Psi). \end{aligned}$$

Proof: The matrix quadratic form in the density (2.8) can be written as

$$\begin{aligned} (Z - M)\Psi^{-1}(Z - M)^H =\ & (Z_{2c} - M_{2c} - (Z_{1c} - M_{1c})\Psi_{11}^{-1}\Psi_{12})\Psi_{22\cdot 1}^{-1} \\ & (Z_{2c} - M_{2c} - (Z_{1c} - M_{1c})\Psi_{11}^{-1}\Psi_{12})^H \\ & + (Z_{1c} - M_{1c})\Psi_{11}^{-1}(Z_{1c} - M_{1c})^H. \end{aligned} \tag{3.2}$$

Substituting (3.2) in (2.8) and noting that $\det(\Psi) = \det(\Psi_{11})\det(\Psi_{22\cdot 1})$, we can factorize the density of Z as

$$\begin{aligned} & \frac{\tilde{\Gamma}_p(r_1 + p)}{\pi^{pr_1}\tilde{\Gamma}_p(p)} \det(\Sigma)^{-r_1} \det(\Psi_{11})^{-p} \\ & \quad \times \det(I_p + \Sigma^{-1}(Z_{1c} - M_{1c})\Psi_{11}^{-1}(Z_{1c} - M_{1c})^H)^{-(p+r_1)} \\ & \times \frac{\tilde{\Gamma}_p(r_1 + r_2 + p)}{\pi^{pr_2}\tilde{\Gamma}_p(r_1 + p)} \det(\Sigma)^{-r_2} \det(\Psi_{22\cdot 1})^{-p} \\ & \quad \times \det(I_p + \Sigma^{-1}(Z_{1c} - M_{1c})\Psi_{11}^{-1}(Z_{1c} - M_{1c})^H)^{-r_2} \\ & \quad \times \det(I_p + (I_p + \Sigma^{-1}(Z_{1c} - M_{1c})\Psi_{11}^{-1}(Z_{1c} - M_{1c})^H)^{-1}\Sigma^{-1} \\ & \qquad (Z_{2c} - M_{2c} - (Z_{1c} - M_{1c})\Psi_{11}^{-1}\Psi_{12})\Psi_{22\cdot 1}^{-1} \\ & \qquad (Z_{2c} - M_{2c} - (Z_{1c} - M_{1c})\Psi_{11}^{-1}\Psi_{12})^H)^{-(r_1+r_2+p)}, Z_{1c} \in \mathbb{C}^{p \times r_1}, Z_{2c} \in \mathbb{C}^{p \times r_2}. \end{aligned}$$

Thus, from above, it is clear that $Z_{1c} \sim \mathbb{C}CAU_{p,r_1}(M_{1c}, \Sigma, \Psi_{11})$ and the conditional distribution of Z_{2c} given Z_{1c} is complex matrix variate-t, $Z_{2c}|Z_{1c} \sim \mathbb{C}T_{p,r_2}(r_1+1, M_{2c} + (Z_{1c} - M_{1c})\Psi_{11}^{-1}\Psi_{12}, \Sigma(I_p + \Sigma^{-1}(Z_{1c} - M_{1c})\Psi_{11}^{-1}(Z_{1c} - M_{1c})^H), \Psi_{22\cdot 1})$. The proof of (ii) can be obtained by noting that $Z^H = \begin{pmatrix} Z_{1r}^H & Z_{2r}^H \end{pmatrix} \sim \mathbb{C}CAU_{r,p}(M^H, \Psi, \Sigma)$ and using result (i). ■

Using above theorem, the matrix variate Cauchy density can be expressed as the product of complex multivariate-t and complex multivariate Cauchy densities. Setting $r_1 = r - 1$ and $r_2 = 1$ and $Z_{1c} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{r-1})$ and $Z_{2c} \equiv \boldsymbol{z}_r$ in (i), we get

$$\begin{aligned} \boldsymbol{z}_r|Z_{1c} \sim\ & \mathbb{C}T_{p,1}(r, M_{2c} + (Z_{1c} - M_{1c})\Psi_{11}^{-1}\Psi_{12}, \\ & \Sigma(I_p + \Sigma^{-1}(Z_{1c} - M_{1c})\Psi_{11}^{-1}(Z_{1c} - M_{1c})^H), \Psi_{22\cdot 1}), \end{aligned}$$

which is a p-dimensional complex multivariate-t density. Further, from the marginal distribution of Z_{1c}, it can be observed that $\boldsymbol{z}_{r-1}|(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{r-2})$ is also complex multivariate-t. Repeating this procedure $r-1$ times, it is straightforward to show that the density of Z can be written as

$$f(Z) = f_r(\boldsymbol{z}_r|\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{r-1}) f_{r-1}(\boldsymbol{z}_{r-1}|\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{r-2}) \cdots f_2(\boldsymbol{z}_2|\boldsymbol{z}_1) f_1(\boldsymbol{z}_1),$$

where every density on the right hand side is a p-dimensional complex t-density except $f_1(\boldsymbol{z}_1)$ which is a complex multivariate Cauchy density.

Similarly, using Theorem 3.1(ii), it can be established that the density of Z is product of $p-1$ r-dimensional complex multivariate-t densities and a complex multivariate Cauchy density.

In next few theorems we will show that the family of complex matrix variate Cauchy distributions is closed under certain linear transformations.

Theorem 3.2 *Let $Z \sim \mathbb{C}CAU_{p,r}(M, \Sigma, \Psi)$ and $A\,(p \times p)$ and $B\,(r \times r)$ be complex nonsingular matrices, then $AZB \sim \mathbb{C}CAU_{p,r}(AMB, A\Sigma A^H, B^H\Psi B)$*

Proof: Using the transformation $X = AZB$ with the Jacobian $J(Z \to X) = \det(AA^H)^{-r}$ $\det(BB^H)^{-p}$ in the density of Z we get the desired result. ■

Theorem 3.3 *Let $Z \sim \mathbb{C}CAU_{p,r}(M, \Sigma, \Psi)$ and $A\,(t \times p)$ and $B\,(r \times k)$ be a complex matrices of ranks $t(\leq p)$ and $k(\leq r)$ respectively. Then, (i) $AZ \sim \mathbb{C}CAU_{t,r}(AM, A\Sigma A^H, \Psi)$ and (ii) $ZB \sim \mathbb{C}CAU_{p,k}(MB, \Sigma, B^H\Psi B)$.*

Proof: (i) Let A_1 be a complex matrix of order $(t-p) \times p$ such that $A_0 = \begin{pmatrix} A \\ A_1 \end{pmatrix}$ is nonsingular. Now, from Theorem 3.2, we have $A_0 Z = \begin{pmatrix} AZ \\ A_1 Z \end{pmatrix} \sim \mathbb{C}CAU_{p,r}(A_0 M, A_0 \Sigma A_0^H, \Psi)$. Finally, using Theorem 3.1(ii), we get the desired result. To prove (ii), note that $Z^H \sim \mathbb{C}CAU_{r,p}(M^H, \Psi, \Sigma)$. Further, using (i) it is easy to see that $B^H Z^H \sim \mathbb{C}CAU_{k,p}(B^H M^H, B^H\Psi B, \Sigma)$. Finally, $(B^H Z^H)^H = ZB \sim \mathbb{C}CAU_{p,k}(MB, \Sigma, B^H\Psi B)$ ■

Combining results (i) and (ii) given in Theorem 3.3 we state the following result.

Theorem 3.4 *Let $A(t \times p)$ and $B(r \times k)$ be non-random complex matrices of ranks $t(\leq p)$ and $k\,(\leq r)$ respectively. If $Z \sim \mathbb{C}CAU_{p,r}(M, \Sigma, \Psi)$, then $AZB \sim \mathbb{C}CAU_{t,k}(AMB, A\Sigma A^H, B^H\Psi B)$.*

Corollary 3.4.1 *Let $\boldsymbol{a} \in \mathbb{C}^{p\times 1}$ and $\boldsymbol{b} \in \mathbb{C}^{r\times 1}$ be nonzero vectors. If $Z \sim \mathbb{C}CAU_{p,r}(M, \Sigma, \Psi)$, then $\boldsymbol{a}^H Z \boldsymbol{b} \sim \mathbb{C}CAU_{1,1}(\boldsymbol{a}^H M \boldsymbol{b}, \boldsymbol{a}^H \Sigma \boldsymbol{a}, \boldsymbol{b}^H \Psi \boldsymbol{b})$. Further, if $z = (\boldsymbol{a}^H\Sigma\boldsymbol{a})^{-1/2}(\boldsymbol{b}^H\Psi\boldsymbol{b})^{-1/2}$ $\{\boldsymbol{a}^H(X - M)\boldsymbol{b}\}$, then $z\bar{z} \sim IB(1,1)$.*

The inverted beta distribution designated by $IB(\alpha,\beta)$ used in the above corollary is defined by the p.d.f.

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}},\ \alpha>0,\ \beta>0,\ x>0.$$

Note that the results on linear transformations given above can also be obtained by using the synthetic representation $Z=Y^{-1}X+M$, where the random matrices X and Y are independent, $X\sim N_{p,r}(0,I_p\otimes\Psi)$ and $Y\sim N_{p,p}(0,I_p\otimes\Sigma^{-1})$.

Let $Z=(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_r)$, $M=(\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_r)$, and $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_r)^H$, $\boldsymbol{\alpha}\neq\mathbf{0}$. If $Z\sim\mathbb{C}CAU_{p,r}(M,\Sigma,\Psi)$, then $\alpha_1\boldsymbol{z}_1+\cdots+\alpha_r\boldsymbol{z}_r\sim\mathbb{C}CAU_p(\sum_{i=1}^r\alpha_i\boldsymbol{\mu}_i,(\boldsymbol{\alpha}^H\Psi\boldsymbol{\alpha})\Sigma)$. Furthermore, for $\Psi=I_r$ and $\boldsymbol{\mu}_i=\boldsymbol{\mu}$, $i=1,\ldots,r$, the density of $Z=(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_r)$ simplifies to

$$\frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)}\det(\Sigma)^{-r}\det\left(I_p+\Sigma^{-1}\sum_{i=1}^r(\boldsymbol{z}_i-\boldsymbol{\mu})(\boldsymbol{z}_i-\boldsymbol{\mu})^H\right)^{-(r+p)},\ Z\in\mathbb{C}^{p\times r}.$$

It is easy to see that for the above model the distribution of $\boldsymbol{z}_0=r^{-1}\sum_{i=1}^r\boldsymbol{z}_i$ is complex multivariate Cauchy, more specifically, $\sqrt{r}\,\boldsymbol{z}_0\sim CAU_p(\sqrt{r}\,\boldsymbol{\mu},\Sigma)$. The distribution of $A=\sum_{i=1}^r(\boldsymbol{z}_i-\boldsymbol{z}_0)(\boldsymbol{z}_i-\boldsymbol{z}_0)^H$ can be derived using orthogonal transformation (Gupta and Nagar [9, p. 141]). Let $Y=ZU=\begin{pmatrix}\boldsymbol{y}_1 & Y_2\end{pmatrix}$ where U $(r\times r)$ is an orthogonal matrix with elements in the first column all equal to $r^{-1/2}$, $\boldsymbol{y}_1$ and Y_2 are of order $p\times 1$ and $(p\times(r-1))$ respectively. Then $\boldsymbol{y}_1=\sqrt{r}\,\boldsymbol{z}_0$, $\sum_{i=1}^r(\boldsymbol{z}_i-\boldsymbol{\mu})(\boldsymbol{z}_i-\boldsymbol{\mu})^H=r(\boldsymbol{z}_0-\boldsymbol{\mu})(\boldsymbol{z}_0-\boldsymbol{\mu})^H+Y_2Y_2^H$, $A=Y_2Y_2^H$ and the Jacobian $J(Z\to\boldsymbol{y}_1,Y_2)=1$. Making these substitutions in the above density, the joint density of $\sqrt{r}\,\boldsymbol{z}_0$ and Y_2 is obtained as

$$\frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)}\det(\Sigma)^{-r}\det(I_p+r\Sigma^{-1}(\boldsymbol{z}_0-\boldsymbol{\mu})(\boldsymbol{z}_0-\boldsymbol{\mu})^H+\Sigma^{-1}Y_2Y_2^H)^{-(r+p)},$$

where $\boldsymbol{z}_0\in\mathbb{C}^{p\times 1}$ and $Y_2\in\mathbb{C}^{p\times(r-1)}$. Now, using Lemma 2.5, the joint density of $\sqrt{r}\,\boldsymbol{z}_0$ and A is derived as

$$\frac{\tilde{\Gamma}_p(r+p)}{\pi^{p}\tilde{\Gamma}_p(p)\tilde{\Gamma}_p(r-1)}\det(\Sigma)^{-r}\det(A)^{r-p-2}$$
$$\times\det(I_p+r\Sigma^{-1}(\boldsymbol{z}_0-\boldsymbol{\mu})(\boldsymbol{z}_0-\boldsymbol{\mu})^H+\Sigma^{-1}A)^{-(r+p)},\ \boldsymbol{z}_0\in\mathbb{C}^p,\ A=A^H>0.$$

Finally, integration of $\boldsymbol{z}_0$ above yields the marginal density of A as

$$\frac{\tilde{\Gamma}_p(r+p-1)}{\tilde{\Gamma}_p(p)\tilde{\Gamma}_p(r-1)}\det(\Sigma)^{-(r-1)}\det(A)^{r-p-2}\det(I_p+\Sigma^{-1}A)^{-(r+p-1)},\ A=A^H>0$$

which is a generalized inverted complex matrix variate beta (complex matrix variate beta type II) distribution.

4 Matrix Quadratic Forms.

In this section we study the distributions of quadratic forms of the type ZAZ^H, where $A=A^H>0$ is a non-random complex matrix and the complex random matrix Z has a complex matrix variate Cauchy distribution.

Theorem 4.1 *Let $Z\sim\mathbb{C}CAU_{p,r}(M,\Sigma,\Psi)$. Then*
(i) if $r\geq p$, the p.d.f. of $W_1=(Z-M)\Psi^{-1}(Z-M)^H$ is given by

$$\frac{\tilde{\Gamma}_p(r+p)}{\tilde{\Gamma}_p(p)\tilde{\Gamma}_p(r)}\det(\Sigma)^{-r}\det(W_1)^{r-p}\det(I_p+\Sigma^{-1}W_1)^{-(p+r)},\ W_1=W_1^H>0,$$

(ii) if $r \leq p$, the p.d.f. of $W_2 = (Z-M)^H \Sigma^{-1}(Z-M)$ is given by

$$\frac{\tilde{\Gamma}_r(r+p)}{\tilde{\Gamma}_r(p)\tilde{\Gamma}_r(r)} \det(\Psi)^{-p} \det(W_2)^{p-r} \det(I_r + \Psi^{-1}W_2)^{-(r+p)},\ W_2 = W_2^H > 0.$$

Proof: Let $W_1 = (Z-M)\Psi^{-1}(Z-M)^H$ and $\Lambda = (Z-M)\Psi^{-1/2}$. Then $W_1 = \Lambda\Lambda^H$ and $\Lambda \sim \mathbb{C}CAU_{p,r}(0, \Sigma, I_r)$. Now, using (2.8), the p.d.f. of W_1 is obtained as

$$\begin{aligned}
&\frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)} \det(\Sigma)^{-r} \int_{\Lambda\Lambda^H = W_1} \det(I_p + \Sigma^{-1}\Lambda\Lambda^H)^{-(r+p)}\, d\Lambda \\
&= \frac{\tilde{\Gamma}_p(r+p)}{\tilde{\Gamma}_p(p)\tilde{\Gamma}_p(r)} \det(\Sigma)^{-r} \det(W_1)^{r-p} \det(I_p + \Sigma^{-1}W_1)^{-(r+p)},\ W_1 = W_1^H > 0,
\end{aligned}$$

where the last step has been obtained by using Lemma 2.5.

(ii) Similar to the proof of (i). ■

Thus, the quadratic forms in complex Cauchy matrix have generalized inverted complex matrix variate beta distribution. Using Bartlett's decomposition of an inverted beta matrix (Tan [21], Gupta and Nagar [9, p. 195]) and above result, we can easily infer the following result.

Theorem 4.2 *Let $Z \sim \mathbb{C}CAU_{p,r}(M, \Sigma, \Psi)$.*

(i) If $r \geq p$, then $\det(\Sigma^{-1}(Z-M)\Psi^{-1}(Z-M)^H) \sim \prod_{i=1}^p x_i$, where $x_1, \dots, x_p$ are independently distributed, $x_i \sim IB(r-i+1, i)$, $i = 1, \dots, p$.

(ii) If $r \leq p$, then $\det(\Psi^{-1}(Z-M)^H\Sigma^{-1}(Z-M)) \sim \prod_{i=1}^r y_i$, where $y_1, \dots, y_r$ are independently distributed, $y_i \sim IB(p-i+1, i)$, $i = 1, \dots, r$.

Our next two theorems give expressions in terms of complex matrix variate Dirichlet distributions generalizing Theorem 4.1.

Theorem 4.3 *Let the complex random matrix $Z\,(p \times r)$ be partitioned as $Z = (Z_1, \dots, Z_k)$, $Z_i(p \times r_i)$, $r_i \geq p$, $i = 1, \dots, k$ and $r_1 + \cdots + r_k = r$. Define $W_{1i} = Z_i Z_i^H$, $i = 1, \dots, k$. If $Z \sim \mathbb{C}CAU_{p,r}(0, \Sigma, I_r)$, then the p.d.f. of $(W_{11}, \dots, W_{1k})$ is given by*

$$\frac{\tilde{\Gamma}_p(r+p)}{\prod_{i=1}^k \tilde{\Gamma}_p(r_i)\tilde{\Gamma}_p(p)} \det(\Sigma)^{-r} \prod_{i=1}^k \det(W_{1i})^{r_i - p} \det\left(I_p + \Sigma^{-1}\sum_{i=1}^k W_{1i}\right)^{-(r+p)},$$

where $W_{1i} = W_{1i}^H > 0$, $i = 1, \dots, k$.

Theorem 4.4 *Let the $p \times r$ complex random matrix Z be partitioned as $Z = (Z_1^H, \dots, Z_\ell^H)^H$, $Z_i(p_i \times r)$, $p_i \geq r$, $i = 1, \dots, \ell$ and $p_1 + \cdots + p_\ell = p$. Define $W_{2i} = Z_i^H Z_i$, $i = 1, \dots, \ell$. If $Z \sim \mathbb{C}CAU_{p,r}(0, I_p, \Psi)$, then the p.d.f. of $(W_{21}, \dots, W_{2\ell})$ is given by*

$$\frac{\tilde{\Gamma}_r(p+r)}{\prod_{i=1}^\ell \tilde{\Gamma}_r(p_i)\tilde{\Gamma}_r(r)} \det(\Psi)^{-p} \prod_{i=1}^\ell \det(W_{2i})^{p_i - r} \det\left(I_r + \Psi^{-1}\sum_{i=1}^\ell W_{2i}\right)^{-(p+r)},$$

where $W_{2i} = W_{2i}^H > 0$, $i = 1, \dots, \ell$.

The distributions of quadratic forms of the type $(Z-M)A(Z-M)^H$, where $Z \sim \mathbb{C}CAU_{p,r}(M, \Sigma, \Psi)$ and $A = A^H > 0$ is a non-random matrix, can also be derived. Since the derivation of the density function involves integration of hypergeometric function of

Hermitian matrix argument over unitary group, we here define hypergeometric function of Hermitian matrix argument and give results on integration over unitary group. For further details the reader is referred to James [12], Khatri [15], Hayakawa [11], Chikuse [4] and Gupta and Nagar [9].

The generalized hypergeometric functions of one Hermitian matrix and two Hermitian matrices are defined by

$$ {}_r\tilde{F}_s(a_1,\dots,a_r;b_1,\dots,b_s;X)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_r]_\kappa}{[b_1]_\kappa\cdots[b_s]_\kappa}\frac{\tilde{C}_\kappa(X)}{k!} \tag{4.1}$$

and

$$ {}_r\tilde{F}_s^{(p)}(a_1,\dots,a_r;b_1,\dots,b_s;X,Y)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_r]_\kappa}{[b_1]_\kappa\cdots[b_s]_\kappa}\frac{\tilde{C}_\kappa(X)\tilde{C}_\kappa(Y)}{\tilde{C}_\kappa(I_p)\,k!} \tag{4.2}$$

respectively, where a_i, $i=1,\dots,r$; b_j, $j=1,\dots,s$ are arbitrary complex numbers, X $(p\times p)$ and Y $(p\times p)$ are Hermitian matrices, $\tilde{C}_\kappa(X)$ is the zonal polynomial of $p\times p$ Hermitian matrix X corresponding to the partition κ and $\sum_\kappa$ denotes summation over all partitions $\kappa=(k_1,\dots,k_p), k_1\geq\cdots\geq k_p\geq 0$ and $[a]_\kappa=\prod_{j=1}^{p}(a-j+1)_{k_j}$. Conditions for convergence of these series are available in the literature. It may be noted here that exponential function ${}_0F_0$, Bessel function ${}_0F_1$, Confluent hypergeometric function ${}_1F_1$ and Gauss hypergeometric function ${}_2F_1$ are particlular cases of the generalized hypergeometric functions defined above. Further

$$\begin{aligned}\int_{U(p)} {}_r\tilde{F}_s(a_1,\dots,a_r;b_1,\dots,b_s;UXU^HY)\,[dU]\\ = {}_r\tilde{F}_s^{(p)}(a_1,\dots,a_r;b_1,\dots,b_s;X,Y)\end{aligned} \tag{4.3}$$

where $[dU]$ is the unit invariant measure over the unitary group $U(p)$.

Theorem 4.5 *Let $Z\sim\mathbb{C}CAU_{p,r}(M,\Sigma,\Psi)$ and A be a Hermitian positive definite constant matrix of suitable order. Then*
(i) if $p\leq r$, the p.d.f. of $W=(Z-M)A(Z-M)^H$ is given by

$$\begin{aligned}&\frac{\tilde{\Gamma}_p(r+p)}{\tilde{\Gamma}_p(r)\tilde{\Gamma}_p(p)}\det(\Sigma)^{-r}\det(A\Psi)^{-p}\det(W)^{r-p}\det(I_p+\Sigma^{-1}W)^{-(r+p)}\\&\times {}_1\tilde{F}_0^{(r)}\left(r+p;(I_p+\Sigma^{-1}W)^{-1}\Sigma^{-1}W,\Lambda_1\right),\ W=W^H>0,\end{aligned}$$

(ii) if $p\geq r$, the p.d.f. of $W=(Z-M)^HA(Z-M)$ is given by

$$\begin{aligned}&\frac{\tilde{\Gamma}_r(r+p)}{\tilde{\Gamma}_r(r)\tilde{\Gamma}_r(p)}\det(A\Sigma)^{-r}\det(\Psi)^{-p}\det(W)^{p-r}\det(I_r+\Sigma^{-1}W)^{-(r+p)}\\&\times {}_1\tilde{F}_0^{(p)}\left(r+p;(I_r+\Psi^{-1}W)\Psi^{-1}W,\Lambda_2\right),\ W=W^H>0,\end{aligned}$$

where $\Lambda_1=I_r-A^{-1/2}\Psi^{-1}A^{-1/2}$, $\Lambda_2=I_p-A^{-1/2}\Sigma^{-1}A^{-1/2}$, and ${}_1\tilde{F}_0^{(\alpha)}$ is the hypergeometric function of two Hermitian matrix arguments.

Proof: Note that $W=(Z-M)A(Z-M)^H=YY^H$ where $Y=(Z-M)A^{1/2}\sim\mathbb{C}CAU_{p,r}(0,\Sigma,A^{1/2}\Psi A^{1/2})$. The p.d.f. of W is therefore derived as

$$\begin{aligned}&\frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)}\det(\Sigma)^{-r}\det(A\Psi)^{-p}\\&\times\int_{YY^H=W}\det(I_p+\Sigma^{-1}YA^{-1/2}\Psi^{-1}A^{-1/2}Y^H)^{-(r+p)}dY.\end{aligned} \tag{4.4}$$

Writing $\det(I_p+\Sigma^{-1}YA^{-1/2}\Psi^{-1}A^{-1/2}Y^H)^{-(r+p)} = \det(I_p+\Sigma^{-1}YY^H)^{-(r+p)}\,{}_1\tilde{F}_0(r+p; Y^H(I_p+\Sigma^{-1}YY^H)^{-1}\Sigma^{-1}Y\Lambda_1)$ in (4.4) we have

$$\begin{aligned}(4.5)\qquad &\frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)}\det(\Sigma)^{-r}\det(A\Psi)^{-p}\int_{YY^H=W}\det(I_p+\Sigma^{-1}YY^H)^{-(r+p)}\\ &\times {}_1\tilde{F}_0(r+p;Y^H(I_p+\Sigma^{-1}YY^H)^{-1}\Sigma^{-1}Y\Lambda_1)\,dY.\end{aligned}$$

The integral in (4.5) is invariant under the transformation $\Lambda_1 \to U\Lambda_1U^H$, $U \in U(r)$. Hence, replacing Λ_1 by $U\Lambda_1U^H$, $U \in U(r)$ in (4.5) and integrating over the unitary group $U(r)$ using (4.3), we obtain the density of W as

$$\begin{aligned}&\frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)}\det(\Sigma)^{-r}\det(A\Psi)^{-p}\int_{YY^H=W}\det(I_p+\Sigma^{-1}YY^H)^{-(r+p)}\\ &\times {}_1\tilde{F}_0^{(r)}(r+p;Y^H(I_p+\Sigma^{-1}YY^H)^{-1}\Sigma^{-1}Y,\Lambda_1)\,dY.\end{aligned}$$

Finally, using Lemma 2.5 we obtain the desired result. The proof of (ii) follows similar steps. ■

It may be remarked here that by substituting $A=\Psi$ in Theorem 4.5(i) and $A=\Sigma$ in Theorem 4.5(ii) one can obtain results (i) and (ii) respectively of Theorem 4.1.

Using Theorem 2.1, the density of the Moore-Penrose inverse of a complex Cauchy matrix is derived in the following theorem.

Theorem 4.6 *If $Z \sim \mathbb{C}CAU_{p,r}(0,\Sigma,\Psi), p \le r$, then the p.d.f. of $Y = Z^+ = Z^H(ZZ^H)^{-1}$ is derived as*

$$\begin{aligned}&\frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}(p)}\det(\Sigma)^{-r}\det(\Psi)^{-p}\det(Y^HY)^{-2r}\\ &\qquad\times\det(I_p+\Sigma^{-1}(Y^HY)^{-1}Y^H\Psi^{-1}Y(Y^HY)^{-1})^{-(r+p)},\ Y\in\mathbb{C}^{r\times p}.\end{aligned}$$

5 Non-Central Complex Matrix Variate Cauchy Distribution. If the complex random matrices X and Y are independent, $X \sim \mathbb{C}N_{p,r}(0, I_p\otimes\Psi)$ and $Y \sim \mathbb{C}N_{p,p}(0, I_p\otimes\Sigma^{-1})$, then $Z = Y^{-1}X \sim \mathbb{C}CAU_{p,r}(0,\Sigma,\Psi)$. In this section we will obtain distribution of the ratio $Z=Y^{-1}X$ where now one of the random matrices has non zero mean matrix. The distribution so obtained is called the non-central matrix variate Cauchy distribution. We will necessarily need the following results.

Lemma 5.1 *For $X\,(p\times n)$ of rank $p \le n$ and $L\,(p\times n)$,*

$$\int_{XX^H=A}\operatorname{etr}[2\operatorname{Re}(LX^H)]\,dX = \frac{\pi^{np}}{\tilde{\Gamma}_p(n)}\det(A)^{n-p}\,{}_0\tilde{F}_1(n;LL^HA)$$

where ${}_0\tilde{F}_1$ is the Bessel function of Hermitian matrix argument.

Proof: Transform $X = TU_1$ where U_1 is $p\times n$, $U_1U_1^H = I_p$, and $T\,(p\times p)$ is a lower triangular complex matrix with positive diagonal elements, with Jacobian, from Lemma 2.4,

$J(X \to T, U_1) = \prod_{i=1}^p t_{ii}^{2(n-i)+1} \tilde{g}_{n,p}(U_1)$. Then

$$\begin{aligned}
&\int_{XX^H=A} \operatorname{etr}[2\operatorname{Re}(LX^H)]\,dX \\
&= \int_{TT^H=A} \prod_{i=1}^p t_{ii}^{2(n-i)+1} \int_{U(p,n)} \operatorname{etr}[2\operatorname{Re}(T^H L U_1^H)] \tilde{g}_{n,p}(U_1)\,dU_1\,dT \\
&= \frac{2^p \pi^{np}}{\tilde{\Gamma}_p(n)} \int_{TT^H=A} \prod_{i=1}^p t_{ii}^{2(n-i)+1} \int_{U(p,n)} \operatorname{etr}[2\operatorname{Re}(T^H L U_1^H)]\,[dU_1]\,dT \\
&= \frac{2^p \pi^{np}}{\tilde{\Gamma}_p(n)} \int_{TT^H=A} \prod_{i=1}^p t_{ii}^{2(n-i)+1}\, {}_0\tilde{F}_1(n, LL^H TT^H)\,dT
\end{aligned}$$

where the last two lines have been obtained by using (2.3) and the result (James [12], Hayakawa [11]),

$$\int_{U(p,n)} \operatorname{etr}[2\operatorname{Re}(ZU_1^H)]\,[dU_1] = {}_0\tilde{F}_1(n, ZZ^H).$$

Now, transforming $TT^H = A$ with Jacobian $J(T \to A) = 2^{-p} \prod_{i=1}^p t_{ii}^{-2(p-i)-1}$ we get the final result. ■

Lemma 5.2 *Let X and Y be $p \times p$ Hermitian positive definite and Hermitian matrices respectively. Then*

$$\begin{aligned}
&\int_{S=S^H>0} \operatorname{etr}(-XS) \det(S)^{a-p}\, {}_0\tilde{F}_1(b; SY)\,dS \\
&= \tilde{\Gamma}_p(a) \det(X)^{-a}\, {}_1\tilde{F}_1(a; b; X^{-1}Y),\ \operatorname{Re}(a) > p-1
\end{aligned}$$

where ${}_1\tilde{F}_1$ is the confluent hypergeometric function of Hermitian matrix argument.

We now derive desired distributional results.

Theorem 5.1 *Let $X \sim \mathbb{C}N_{p,r}(M, I_p \otimes \Psi)$ independent of $Y \sim \mathbb{C}N_{p,p}(0, I_p \otimes \Sigma^{-1})$. Define $Z = Y^{-1}X$. Then, the p.d.f. of Z is given by*

$$\begin{aligned}
&\frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)} \det(\Psi^{-1}\Sigma)^p \operatorname{etr}\left(-M\Psi^{-1}M^H\right) \det(\Sigma + Z\Psi^{-1}Z^H)^{-(p+r)} \\
&\times {}_1\tilde{F}_1\left(p+r; p; Z\Psi^{-1}M^H M \Psi^{-1} Z^H (\Sigma + Z\Psi^{-1}Z^H)^{-1}\right),\ Z \in \mathbb{C}^{p\times r}
\end{aligned}$$

where ${}_1\tilde{F}_1$ is the confluent hypergeometric function of Hermitian matrix argument.

Proof: The joint density of X and Y is given by

$$\pi^{-p(p+r)} \det(\Psi^{-1}\Sigma)^p \operatorname{etr}\left[-\{Y\Sigma Y^H + (X-M)\Psi^{-1}(X-M)^H\}\right],\quad X \in \mathbb{C}^{p\times r},\ Y \in \mathbb{C}^{p\times p}.$$

Substituting $Z = Y^{-1}X$ with the Jacobian $J(X \to Z) = \det(YY^H)_+^r$ in the joint density of X and Y, we get the joint density of Z and Y as

$$\begin{aligned}
&\pi^{-p(p+r)} \det(\Psi^{-1}\Sigma)^p \operatorname{etr}\left(-M\Psi^{-1}M^H\right) \det(YY^H)_+^r \\
&\times \operatorname{etr}\left[-\{\Sigma + Z\Psi^{-1}Z^H\}Y^H Y + 2\operatorname{Re}(Z\Psi^{-1}M^H Y)\right],\ Z \in \mathbb{C}^{p\times r},\ Y \in \mathbb{C}^{p\times p}.
\end{aligned}$$

Now, integrating out Y we get the density of Z as

$$\begin{aligned}
&\pi^{-p(p+r)} \det(\Psi^{-1}\Sigma)^p \operatorname{etr}\left(-M\Psi^{-1}M^H\right) \\
&\times \int_{Y\in\mathbb{C}^{p\times p}} \operatorname{etr}\left[-(\Sigma + Z\Psi^{-1}Z^H)Y^HY + 2\operatorname{Re}(Z\Psi^{-1}M^HY)\right] \det(Y^HY)^r \, dY \\
&= \pi^{-p(p+r)} \det(\Psi^{-1}\Sigma)^p \operatorname{etr}\left(-M\Psi^{-1}M^H\right) \int_{A=A^H>0} \operatorname{etr}\left[-(\Sigma + Z\Psi^{-1}Z^H)A\right] \\
&\quad \times \det(A)^r \int_{Y^HY=A} \operatorname{etr}[2\operatorname{Re}(Z\Psi^{-1}M^HY)] \, dY \, dA \\
&= \frac{\det(\Psi^{-1}\Sigma)^p}{\pi^{rp}\tilde{\Gamma}_p(p)} \operatorname{etr}\left(-M\Psi^{-1}M^H\right) \int_{A=A^H>0} \operatorname{etr}\left[-(\Sigma + Z\Psi^{-1}Z^H)A\right] \\
&\quad \times \det(A)^{r+p-p} {}_0\tilde{F}_1\left(p; Z\Psi^{-1}M^HM\Psi^{-1}Z^HA\right) \, dA \\
&= \frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)} \det(\Psi^{-1}\Sigma)^p \operatorname{etr}\left(-M\Psi^{-1}M^H\right) \det(\Sigma + Z\Psi^{-1}Z^H)^{-(p+r)} \\
&\quad \times {}_1\tilde{F}_1\left(p+r; p; Z\Psi^{-1}M^HM\Psi^{-1}Z^H(\Sigma + Z\Psi^{-1}Z^H)^{-1}\right), \; Z \in \mathbb{C}^{p\times r}
\end{aligned}$$

where last two steps have been obtained by using Lemma 5.1 and Lemma 5.2. ■

Theorem 5.2 *Let* $X \sim \mathbb{C}N_{p,r}(0, I_p \otimes \Psi)$ *independent of* $Y \sim \mathbb{C}N_{p,p}(M, I_p \otimes \Sigma^{-1})$. *Define* $Z = Y^{-1}X$. *Then, the p.d.f. of* Z *is given by*

$$\begin{aligned}
&\frac{\tilde{\Gamma}_p(r+p)}{\pi^{rp}\tilde{\Gamma}_p(p)} \det(\Psi^{-1}\Sigma)^p \operatorname{etr}\left(-M\Sigma M^H\right) \det(\Sigma + Z\Psi^{-1}Z^H)^{-(p+r)} \\
&\quad \times {}_1\tilde{F}_1\left(p+r; p; \Sigma M^HM\Sigma(\Sigma + Z\Psi^{-1}Z^H)^{-1}\right), \; Z \in \mathbb{C}^{p\times r}
\end{aligned}$$

where ${}_1\tilde{F}_1$ *is the confluent hypergeometric function of Hermitian matrix argument.*

Proof: Similar to the proof of Theorem 5.1. ■

Acknowledgments: The research work of DKN was supported by the Comité para el Desarrollo de la Investigación, Universidad de Antioquia research grant no. IN387CE.

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