SEQUENTIAL POINT ESTIMATION OF THE POWERS OF AN EXPONENTIAL SCALE PARAMETER

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ABSTRACT. In this paper we consider the bounded risk point estimation problem for the power of scale parameter σ^r of a negative exponential distribution where $r \neq 0$ is any given number and the location parameter μ and scale parameter σ both are unknown. For a preassigned error bound $w > 0$ we want to estimate σ^r by using a random sample of the smallest size such that the risk associated with an estimator is not greater than w. We propose a fully sequential procedure and give the asymptotic expansions of its average sample size and risk. We aso consider a class of sequential estimators based on the idea of bias-correction and make a comparison from the point of view of risk.

1. INTRODUCTION

Let $X_1, X_2, X_3, \ldots$ be independent and identically distributed (i.i.d.) random variables with the probability density function (p.d.f.)

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}\right) I_{(x\geq\mu)}\,, \tag{1.1}$$

where both $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$ are unknown and $I(A)$ denotes the indicator function of the set A. For any given $r \neq 0$ we want to estimate the power of the scale parameter σ^r. Let $\delta_n = \delta_n(X_1, \ldots, X_n)$ be an estimator of σ^r based on a random sample $X_1, \ldots, X_n$ of size n. Then as a loss function we use the squared error loss defined by $L_n = (\delta_n - \sigma^r)^2$. The risk associated with the estimator δ_n is given by $R_n = R_n(\delta_n) = E(L_n)$. Let $w > 0$ be a preassigned error bound for the risk. We want to find the smallest sample size $n = n_0$ which satisfies that $R_n \leq w$. For $n \geq 2$ set

$$T_n = \min\{X_1, \ldots, X_n\} \quad \text{and} \quad \sigma_n = (n-1)^{-1}\sum_{i=1}^{n}(X_i - T_n).$$

We can show that as a function of c the risk of $c\sigma_n^r$ takes the minimal value at $c = c_n \equiv (n-1)^r\Gamma(r+n-1)/\Gamma(2r+n-1)$ provided $n > \max\{1, 1-2r\}$ and that the risk of σ_n^r is equal to that of $c_n\sigma_n^r$ up to the order term $O(n^{-2})$. Thus the estimator σ_n^r is an asymptotically optimal one in this sense. Further, the calculation of σ_n^r is easier than that of $c_n\sigma_n^r$. Therefore we use σ_n^r as an estimator of σ^r in this paper. Our goal is to find an asymptotically smallest sample size n_0 satisfying that $R_n = E(\sigma_n^r - \sigma^r)^2 \leq w$.

Estimation of σ and σ^2 are of great importance. For $r = 1$, namely, for the estimation of the standard deviation by σ_n, Isogai, Saito and Uno (1999a,b) dealt with this bounded risk

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point estimation problem under the squared error and weighted loss functions. Minimum risk point estimation problems for $r = 1$ were considered by Mukhopadhyay and Ekwo (1987), Ghosh and Mukhopadhyay (1989) and Isogai and Uno (1994). Starr and Woodroofe (1972) treated the same problem for $r = 2$. Uno and Isogai (2002) proposed a fully sequential procedure for the estimation of σ^r with normal scale parameter σ. Isogai, Ali and Uno (2003) considered the bounded risk point estimation problem of σ^r with normal scale parameter σ and the minimum risk one with exponential scale parameter. For a review one may refer to Mukhopadhyay (1988), Ghosh and Sen (1991) and Ghosh, Mukhopadhyay and Sen (1997). When we want to estimate the hazard rate σ^{-1} of the negative exponential distribution with p.d.f. in (1.1) we need an estimator of σ^r with $r = -1$. Also, it is of interest to measure mean $\lambda = \mu + \sigma$ in $\sigma-$units and hence to estimate $\mu\sigma^{-1}$. For a normal distribution with mean μ and variance σ^2 both unknown, Sriram (1990) considered the sequential point estimation problem for $\mu\sigma^{-1}$ by using an estimator of σ^{-1}.

We shall now compute the risk $R_n = E(\sigma_n^r - \sigma^r)^2$ to find n_0. We can show that for $n > \max\{1,\, 1-2r\}$

$$R_n < \infty \quad \text{and} \quad R_n = r^2\sigma^{2r}n^{-1} + O(n^{-2}) \quad \text{as} \quad n \to \infty.$$

Ignoring the order term above, we can find the asymptotically smallest sample size n_0 satisfying that $R_n \leq w$. Suppose

$$r^2\sigma^{2r}n^{-1} \leq w, \quad \text{or equivalently,} \quad n \geq \frac{r^2\sigma^{2r}}{w} = n^* \quad \text{(say)}. \tag{1.2}$$

For simplicity n^* is assumed to be an integer. Then $n_0 = n^*$ is the asymptotically best fixed sample size if σ is known. Unfortunately, the asymptotically best fixed sample size procedure n_0 cannot be used since σ is unknown. Further, by Takada (1986,1998) there is no fixed sample size procedure satisfying our condition. Thus we need to find a sequential sampling rule.

In Section 2 we shall propose a fully sequential procedure for this estimation problem and give two theorems concerning the second order approximation to its average sample size and risk associated with our procedure. We shall also consider a class of sequential estimators derived on the basis of the idea of bias-correction and compare them from the point of view of risk. Moreover, we shall provide brief simulation results. Final section will give all proofs of the results in Section 2.

2. RESULTS

In this section we shall propose a fully sequential procedure N motivated by the form of n^* in (1.2) and give two theorems concerning the second order approximation to its average sample size $E(N)$ and risk $R_N = E(\sigma_N^r - \sigma^r)^2$ as $w \to 0$. We shall also consider a class of sequential estimators, including the ordinary estimator σ_N^r, based on the idea of bias-correction. The comparison will be made from the point of view of risk. It will turn out that we can find an appropriate sequential estimator to reduce the risk associated with the ordinary one. Finally, we shall provide brief simulation results.

In this paper we propose the stopping rule defined by

$$N = N_w(r) = \inf\left\{ n \geq m : n \geq \frac{r^2\sigma_n^{2r}}{w} l_n \right\}, \tag{2.1}$$

where m is a starting sample size satisfying that $m > \max\{2,\, 1-2r\}$ and l_x is a given positive function of x on $(0,\infty)$ such that

$$l_x = 1 + \frac{l_0}{x} + o\left(\frac{1}{x}\right) \quad \text{as } x \to \infty \quad \text{with a constant } l_0. \tag{2.2}$$

In Section 3 it will be shown that $P(N_w(r) < \infty) = 1$ for all $w > 0$ and $r \neq 0$. Once the sampling stops at the Nth stage, we estimate σ^r by σ_N^r. Then the risk associated with σ_N^r is given by $R_N = E(\sigma_N^r - \sigma^r)^2$. The following two theorems are concerned with the second order approximation to the average sample size and risk.

Theorem 1. If $m > m_1(r)$, then as $w \to 0$

$$E(N) = n^* + \rho + l_0 - r(2r+1) + o(1),$$

where

$$m_1(r) = \begin{cases} \max\{2,\, 1+2r\} & \text{if} \quad r > 0 \\ \max\{2,\, 1-2r\} & \text{if} \quad r < 0 \end{cases}$$

and ρ is the constant given in (3.19) with $0 \le \rho \le \frac{1}{2} + 2r^2$.

Theorem 2. If $m > m_2(r)$, then as $w \to 0$

$$n^*\left(\frac{R_N}{w} - 1\right) = 11r^2 + 8r + 1 + \frac{3}{4}(r-1)^2 - \rho - l_0 + o(1),$$

where

$$m_2(r) = \begin{cases} \max\{1+10r, 7+8r\} & \text{if} \quad r > 0 \\ 7 - 14r & \text{if} \quad r < 0. \end{cases}$$

Remark 1. (i) If we take an arbitrary constant l_0 such that

$$l_0 > 11r^2 + 8r + 1 + \frac{3}{4}(r-1)^2 - \rho, \tag{2.3}$$

then from Theorem 2 we have that $R_N < w$ for sufficiently small $w > 0$. Thus our condition on the risk is satisfied. (ii) Theorem 2.1 of Isogai, Saito and Uno (1999a) with $a = 0$ and $b = 1$ is the same as Theorems 1 and 2 with $r = 1$ except for the condition on the starting sample size. The difference of this condition is caused by the fact that this paper deals with all powers. Further, the methods of the proofs are different.

We shall here evaluate the bias of σ_N^r.

Proposition 1. If $m > m_3(r)$, then as $w \to 0$

$$E(\sigma_N^r) - \sigma^r = -\frac{1}{2}\mathrm{sign}(r)(3r+1)(n^*)^{-1/2}w^{1/2} + o(w),$$

where

$$\mathrm{sign}(r) = \begin{cases} 1 & \text{if} \quad r > 0 \\ -1 & \text{if} \quad r < 0 \end{cases} \quad \text{and} \quad m_3(r) = \begin{cases} \max\{1+4r, 3+3r\} & \text{if} \quad r > 0 \\ 3 - 5r & \text{if} \quad r < 0. \end{cases}$$

Taking Proposition 1 into account, we consider a class of sequential estimators $\{\sigma_N^r(k),\ k \in (-\infty, \infty)\}$ for σ^r defined by

$$\sigma_N^r(k) = \sigma_N^r + k\, N^{-1/2} w^{1/2}. \tag{2.4}$$

Then we get the following proposition concerning the bias of $\sigma_N^r(k)$.

Proposition 2. If $m > m_3(r)$, then as $w \to 0$

$$E(\sigma_N^r(k)) = \sigma^r + \{k - \tfrac{1}{2}\mathrm{sign}(r)(3r+1)\}(n^*)^{-1/2} w^{1/2} + o(w).$$

For $k = \frac{1}{2}\mathrm{sign}(r)(3r+1)$, $\sigma_N^r(k)$ is a second-order asymptotically unbiased estimator.

We shall now compare the risk of $\sigma_N^r(k)$ with that of σ_N^r. Let $R_N(k) = E(\sigma_N^r(k) - \sigma^r)^2$.

Theorem 3. If $m > m_2(r)$, then as $w \to 0$

$$\frac{n^*}{w}(R_N(k) - R_N) = k^2 - \mathrm{sign}(r)(5r+1)k + o(1).$$

Remark 2. Let $k = \frac{1}{2}\mathrm{sign}(r)(5r+1)$ for $r \neq 0$. Then we have

$$R_N(k) < R_N \quad \text{for sufficiently small } w > 0 \quad \text{if } r \neq -\tfrac{1}{5}.$$

Thus bias-correction is asymptotically effective in the reduction of the risk for all $r \neq 0$ with using $\sigma_N^r + \frac{1}{2}\mathrm{sign}(r)(5r+1)N^{-1/2}w^{1/2}$ which is not a second-order asymptotically unbiased estimator.

Simulation Results. We shall give brief simulation results which are based on 100,000 repetitions. We choose the constant l_0 satisfying the inequality in (2.3) in Tables 1–3. In Tables 4 and 5 we choose l_0 such that the average sample size $E(N)$ approximately equals the optimal one n^*. Since we do not know any approximate value of ρ between 0 and $\frac{1}{2} + 2r^2$, we use here $\rho = 0$ or $\rho = \frac{1}{2} + 2r^2$ as ρ. Table 5 shows that the smaller $E(N)$ is, the larger R_N is, which justifies Theorems 1 and 2. From these simulation results we might need to improve the stopping rule N in (2.1).

Table 1. $\rho = 0, \quad l_0 > 11r^2 + 8r + 1 + \frac{3}{4}(r-1)^2 - \rho$

$n^* = 100$	$r = -1$	$r = 1$	$r = 2$
$\mu = 0,\ \sigma = 1$	$w = 0.01$	$w = 0.01$	$w = 0.04$
	$m = 22$	$m = 16$	$m = 24$
$l_n = 1 + l_0/n$	$l_0 = 8$	$l_0 = 21$	$l_0 = 62$
	$k = 2$	$k = 3$	$k = 5.5$
$E(N)$	108.630260	116.346200	137.330710
$E(\sigma_N^r)$	0.991965	0.983436	0.951511
$E(\sigma_N^r(k))$	1.011400	1.011567	1.048849
R_N/w	0.963153	0.984520	0.949165
$R_N(k)/w$	0.934459	0.921439	0.800041
$n^*(R_N(k) - R_N)/w$	-2.869400	-6.308015	-14.912360

Table 2. $r = -1, \quad \rho = \frac{1}{2} + 2r^2 = 2.5, \quad l_0 > 11r^2 + 8r + 1 + \frac{3}{4}(r-1)^2 - \rho = 4.5$

$n^* = 40$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$\mu = 0, \quad l_0 = 5$	$w = 0.1$	$w = 0.025$	$w = 0.00625$
$m = 22, \quad k = 2$	$\sigma^r = 2$	$\sigma^r = 1$	$\sigma^r = 0.5$
$E(N)$	45.752540	45.700810	45.738710
$E(\sigma_N^r)$	1.965109	0.982012	0.491141
$E(\sigma_N^r(k))$	2.061243	1.030103	0.515186
R_N/w	0.918271	0.914489	0.925572
$R_N(k)/w$	0.865381	0.859792	0.870948
$n^*(R_N(k) - R_N)/w$	-2.115595	-2.187878	-2.184982

Table 3. $r = 1, \quad \rho = \frac{1}{2} + 2r^2 = 2.5, \quad l_0 > 11r^2 + 8r + 1 + \frac{3}{4}(r-1)^2 - \rho = 17.5$

$n^* = 40$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$\mu = 0, \quad l_0 = 18$	$w = 0.00625$	$w = 0.025$	$w = 0.1$
$m = 16, \quad k = 3$	$\sigma^r = 0.5$	$\sigma^r = 1$	$\sigma^r = 2$
$E(N)$	52.049260	52.088870	52.200480
$E(\sigma_N^r)$	0.481075	0.962627	1.928253
$E(\sigma_N^r(k))$	0.514786	1.030025	2.062843
R_N/w	1.004848	1.002715	0.984277
$R_N(k)/w$	0.870552	0.871007	0.862504
$n^*(R_N(k) - R_N)/w$	-5.371849	-5.268332	-4.870901

Table 4. $r = -1, \quad \rho = \frac{1}{2} + 2r^2 = 2.5, \quad l_0 \leq r(2r+1) - \rho = -1.5$

$n^* = 40$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$\mu = 0, \quad l_0 = -2$	$w = 0.1$	$w = 0.025$	$w = 0.00625$
$m = 22, \quad k = 2$	$\sigma^r = 2$	$\sigma^r = 1$	$\sigma^r = 0.5$
$E(N)$	39.236610	39.303210	39.232770
$E(\sigma_N^r)$	1.951794	0.976822	0.487953
$E(\sigma_N^r(k))$	2.056594	1.029155	0.514149
R_N/w	1.034014	1.018080	1.026389
$R_N(k)/w$	0.942045	0.931146	0.935053
$n^*(R_N(k) - R_N)/w$	-3.678757	-3.477360	-3.653437

Table 5. $r = 1, \quad \rho = \frac{1}{2} + 2r^2 = 2.5, \quad l_0 \le r(2r+1) - \rho = 0.5$

$n^* = 40$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$\mu = 0, \quad l_0 = 0$	$w = 0.00625$	$w = 0.025$	$w = 0.1$
$m = 16, \quad k = 3$	$\sigma^r = 0.5$	$\sigma^r = 1$	$\sigma^r = 2$
$E(N)$	37.647620	37.596930	37.616820
$E(\sigma_N^r)$	0.468557	0.936478	1.873344
$E(\sigma_N^r(k))$	0.509244	1.017909	2.036156
R_N/w	1.470935	1.470930	1.473031
$R_N(k)/w$	1.107491	1.103551	1.106660
$n^*(R_N(k) - R_N)/w$	-14.537765	-14.695145	-14.654831

3. PROOFS

In this section we shall give all the proofs of the results in Section 2. First we shall present five lemmas which are useful in proving the results. Throughout this section let M denote a generic positive constant and $W_1, W_2, W_3, \ldots$ be i.i.d. random variables with p.d.f. $f_{0,1}$ in (1.1). Set $\overline{W}_n = n^{-1}\sum_{i=1}^n W_i$ and $Y_{in} = \sigma^{-1}(n-i+1)(X_{n(i)} - X_{n(i-1)})$ for $i = 2, \ldots, n$, where $X_{n(1)} \le X_{n(2)} \le \ldots \le X_{n(n)}$ are the order statistics of $X_1, \ldots, X_n$. Then $Y_{2n}, Y_{3n}, \ldots, Y_{nn}$ are i.i.d. random variables with p.d.f. $f_{0,1}$ and $\sigma_n/\sigma = (n-1)^{-1}\sum_{i=2}^n Y_{in}$. Isogai and Uno (2001) gives the following lemma.

Lemma 1. The following results hold.
(i) $E\left\{\sup_{n\ge 1}(\overline{W}_n)^q\right\} < \infty$ for all $q > 0$.
(ii) $E\left\{\sup_{n\ge m-1}(\overline{W}_n)^{-q}\right\} < \infty$ if $m - 1 > \max\{1, q\}$ for any given $q > 0$.

Let

$$\begin{aligned} t = t_w(r) &= \inf\left\{n \ge m-1 : \frac{n+1}{l_{n+1}}(\overline{W}_n)^{-2r} \ge n^*\right\} \quad \text{for any } r \ne 0 \\ &= \inf\left\{n \ge m-1 : \sum_{i=1}^n W_i \le cn^\alpha L(n)\right\} \quad \text{for any } r > 0, \end{aligned} \tag{3.1}$$

where $m-1 > \max\{1, -2r\}$, $c = (n^*)^{-1/2r}$, $\alpha = 1 + \frac{1}{2r}$, $L(n) = 1 + \frac{L_0}{n} + o(\frac{1}{n})$ and $L_0 = \frac{1-l_0}{2r}$. Let

$$Z_n = \frac{n+1}{l_{n+1}}(\overline{W}_n)^{-2r} \quad \text{for } n \ge m-1 > \max\{1, -2r\}. \tag{3.2}$$

Since $\frac{n+1}{l_{n+1}} = n + (1 - l_0) + o(1)$ as $n \to \infty$, it follows from Taylor's theorem that

$$Z_n = n - 2r\sum_{i=1}^n (W_i - 1) + \xi_n, \tag{3.3}$$

where

$$\begin{aligned} \xi_n &= (1-l_0) - 2r(1-l_0)(\overline{W}_n - 1) + r(2r+1)(n+1-l_0)(\overline{W}_n - 1)^2\eta_n^{-2(r+1)} \\ &\quad + \left\{\frac{n+1}{l_{n+1}} - (n+1-l_0)\right\}(\overline{W}_n)^{-2r}. \end{aligned} \tag{3.4}$$

Here η_n is a random variable such that $|\eta_n - 1| < \left|\overline{W}_n - 1\right|$. From Lemma 4.1 and Corollary 1.4 of Woodroofe (1982) and the strong law of large numbers (SLLN) we obtain

$$P\left(t_w(r) < \infty\right) = 1 \quad \text{for all } w > 0 \text{ and } r \neq 0, \quad t/n^* \xrightarrow{a.s.} 1 \quad \text{as } w \to 0,$$

$$\overline{W}_t \xrightarrow{a.s.} 1 \quad \text{and} \quad (n^*)^{1/2}(\overline{W}_t - 1) \xrightarrow{d} Z \quad \text{as } w \to 0, \tag{3.5}$$

where $\xrightarrow{a.s.}$ and $\xrightarrow{d}$ denote almost sure convergence and convergence in distribution, respectively, and Z is a random variable according to the standard normal distribution $N(0,1)$ throughout Section 3. It is known that $N \stackrel{d}{=} t+1$ for all $w > 0$ where $X \stackrel{d}{=} Y$ means that X and Y have the same distribution, which, together with the above results, yields that $P\left(N_w(r) < \infty\right) = 1$ for all $w > 0$ and $r \neq 0$. From the definition of t in (3.1) we get

$$(t/n^*)^{-p} \leq \left(\frac{t+1}{tl_{t+1}}\right)^p (\overline{W}_t)^{-2rp} \leq M(\overline{W}_t)^{-2rp} \tag{3.6}$$

if $m > \max\{2, 1-2r\}$ and that $t/n^* \leq l_t(\overline{W}_{t-1})^{2r} I_{\{t \geq m\}} + ((m-1)/n^*) I_{\{t=m-1\}} \leq M(\overline{W}_{t-1})^{2r} I_{\{t \geq m\}} + m$ for $0 < w \leq w_0$, where w_0 satisfies that $n^* \geq 1$ for all $0 < w \leq w_0$. Thus by using c_r-inequality we obtain

$$(t/n^*)^p \leq M(\overline{W}_{t-1})^{2rp} I_{\{t \geq m\}} + M. \tag{3.7}$$

From (3.6), (3.7) and Lemma 1 we obtain

Lemma 2. Let $p > 0$. Then
(i) $\{(t/n^*)^{-p},\ w > 0\}$ is uniformly integrable if $m > \max\{2,\ 1+2rp,\ 1-2r\}$.
(ii) For some $w_0 > 0$, $\{(t/n^*)^p,\ 0 < w \leq w_0\}$ is uniformly integrable if $m > \max\{2,\ 1-2rp,\ 1-2r\}$.

From Theorem 2 of Chow, Hsiung and Lai (1979) we have

Lemma 3. For $p > 1$, if $m > \max\{2,\ 1-2rp,\ 1-2r\}$ then
$\left\{\left((n^*)^{-\frac{1}{2}}\left|\sum_{i=1}^t (W_i - 1)\right|\right)^p,\ 0 < w \leq w_0\right\}$ is uniformly integrable for some $w_0 > 0$.

Lemma 4.5 of Isogai and Uno (2001) gives

Lemma 4. Let η_t be any random variable lying between 1 and $\overline{W}_t$. Then for any s

$$\sup_{w>0} E(\eta_t^s) \leq M \quad \text{if } m > \max\{2, 1-2r,\ 1-s\}.$$

We shall now give the lemma concerning all the conditions (C1)-(C6) of Aras and Woodroofe (1993).

Remark 3. In the notation of Aras and Woodroofe (1993) set
$\mathbf{b} = 1, \quad \mathbf{c} = -2r, \quad \mathbf{X}_i = W_i - 1, \quad a = n^*, \quad \Sigma = 1, \quad \xi = 1 - l_0 + r(2r+1)Z^2$
and $\mathbf{S_n} = \sum_{i=1}^n (W_i - 1)$.

Lemma 5. If $m > m_0(r;p)$, then all the conditions (C1)–(C6) of Aras and Woodroofe (1993) with any $p \geq 3$, $\alpha \geq 3/2$ and $0 < \varepsilon_0, \varepsilon_1 < 1$ are satisfied, where $m_0(r;p) = \max\{2,\ 1+2pr,\ 1-2r\}$.

Proof. It is obvious that (C1) is satisfied for all $p \geq 3$. By making use of Example 1.8, 4.1(i) and Lemmas 1.4 and 4.1 of Woodroofe (1982) and the central limit theorem (CLT) we can show that $\xi_n,\ n \geq 1$, are slowly changing where ξ_n is defined by (3.4), from which (C4) holds. We shall show (C5). Let any $\alpha \in [\frac{3}{2}, \infty)$ and any $\varepsilon \in (0,1)$ be fixed. Let $A_n{}'$ denote the complement of $A_n \equiv \{|\overline{W}_n - 1| \leq \varepsilon\}$. By the Marcinkiewicz-Zygmund inequality we get

$$E\left|\overline{W}_n - 1\right|^q \leq M\, n^{-q/2} \quad \text{for any } q \geq 2. \tag{3.8}$$

Since $\{\overline{W}_n - 1, n \geq 1\}$ is a reverse martingale, it follows from Doob's maximal inequality, Markov's inequality and (3.8) that $P\left(\bigcup_{k=n}^{\infty} A_k^{'}\right) \leq \varepsilon^{-5} E\left\{\sup_{k \geq n} |\overline{W}_k - 1|^5\right\} \leq M\, n^{-5/2}$, which yields the first assertion of (C5). Next we shall show the second assertion of (C5). From (3.4) we get

$$\begin{aligned}
&\max_{k \leq n} |\xi_{n+k} I_{A_{n+k}}|^\alpha \\
\leq\ & M\Bigg[|1 - l_0|^\alpha \max_{k \leq n} I_{A_{n+k}} + 2^\alpha |r(1 - l_0)|^\alpha \max_{k \leq n} \left(|\overline{W}_{n+k} - 1| I_{A_{n+k}}\right)^\alpha \\
& + |r(2r+1)|^\alpha \max_{k \leq n} \left|(n + k + 1 - l_0)(\overline{W}_{n+k} - 1)^2 \eta_{n+k}^{-2(r+1)} I_{A_{n+k}}\right|^\alpha \\
& + \max_{k \leq n} \left|\left(\frac{n+k+1}{l_{n+k+1}} - (n + k + 1 - l_0)\right)(\overline{W}_{n+k})^{-2r} I_{A_{n+k}}\right|^\alpha \Bigg] \\
\equiv\ & M\left[J_{1n} + 2^\alpha \left|r(1 - l_0)\right|^\alpha J_{2n} + \left|r(2r+1)\right|^\alpha J_{3n} + J_{4n}\right], \quad \text{say.}
\end{aligned} \tag{3.9}$$

Clearly $\{J_{1n}, n \geq 1\}$ is uniformly integrable. We can easily show that $J_{2n} \leq n^{-\alpha} \max_{k \leq 2n} \left|\sum_{i=1}^k (W_i - 1)\right|^\alpha$, which implies $E\{(J_{2n})^2\} \leq n^{-2\alpha} E\left\{\max_{k \leq 2n} |\sum_{i=1}^k (W_i - 1)|^{2\alpha}\right\}$. Since $\{\sum_{i=1}^k (W_i - 1),\ 1 \leq k \leq 2n\}$ is a martingale, by using Doob's maximal inequality and (3.8) we have that $\sup_{n \geq 1} E\{(J_{2n})^2\} \leq M\ \sup_{n \geq 1} n^{-2\alpha} E\{|\sum_{i=1}^{2n} (W_i - 1)|^{2\alpha}\} \leq M$. Thus $\{J_{2n}, n \geq 1\}$ is uniformly integrable. Since $|\eta_n - 1| < |\overline{W}_n - 1|$, we get that on the set A_{n+k} $\eta_{n+k}^{-2(r+1)} < (1 - \epsilon)^{-2(r+1)}$ for $r \geq -1$ and that $\eta_{n+k}^{-2(r+1)} < (1 + \epsilon)^{-2(r+1)}$ for $r < -1$. Set $C_1 = \max\{(1 - \varepsilon)^{-2(r+1)}, (1 + \varepsilon)^{-2(r+1)}\} (> 0)$. Then we have that $0 < \eta_{n+k}^{-2(r+1)} < C_1$ on A_{n+k}, which gives

$$\begin{aligned}
J_{3n} &\leq C_1^\alpha \max_{k \leq n} |n + k + 1 - l_0|^\alpha |\overline{W}_{n+k} - 1|^{2\alpha} \\
&\leq M \max_{k \leq n} \left|(n + k)(\overline{W}_{n+k} - 1)^2\right|^\alpha \leq M \max_{k \leq 2n} \left|\frac{1}{\sqrt{2n}} \sum_{i=1}^k (W_i - 1)\right|^{2\alpha}.
\end{aligned}$$

Thus from Proposition 1 of Aras and Woodroofe (1993) $\{J_{3n}, n \geq 1\}$ is uniformly integrable. Set $C_2 = \max\{(1-\varepsilon)^{-2r}, (1+\varepsilon)^{-2r}\} (> 0)$. Then since $0 < (\overline{W}_{n+k})^{-2r} \leq C_2$ on A_{n+k}, we get that $J_{4n} \leq M$ for all $n \geq 1$. Hence, $\{J_{4n}, n \geq 1\}$ is uniformly integrable. Therefore, from the uniform integrabilities of $\{J_{in}, n \geq 1\}$ $(i = 1, 2, 3, 4)$ and (3.9), $\{\max_{k \leq n} |\xi_{n+k} I_{A_{n+k}}|^\alpha, n \geq 1\}$ is uniformly integrable which concludes the second assertion of (C5). We shall show (C3). Let any $\varepsilon_1 \equiv \varepsilon \in (0, 1)$ be fixed and let

$$J_{1n} = -2r(1 - l_0)(\overline{W}_n - 1), \quad J_{2n} = r(2r+1)(n + 1 - l_0)(\overline{W}_n - 1)^2 \eta_n^{-2(r+1)}$$

$$\text{and} \quad J_{3n} = \left\{\frac{n+1}{l_{n+1}} - (n + 1 - l_0)\right\} (\overline{W}_n)^{-2r}.$$

Then we get that $\xi_n = (1 - l_0) + J_{1n} + J_{2n} + J_{3n}$ in (3.4) and

$$\begin{aligned} P\{\xi_n < -(1-\epsilon)n\} &\leq P\{1 - l_0 < -\tfrac{1}{4}(1-\varepsilon)n\} + \textstyle\sum_{k=1}^{3} P\{J_{kn} < -\tfrac{1}{4}(1-\varepsilon)n\} \\ &\equiv K_{0n} + \sum_{i=1}^{3} K_{in}, \quad \text{say.} \end{aligned} \tag{3.10}$$

It is obvious that

$$\sum_{n=1}^{\infty} nK_{0n} < \infty. \tag{3.11}$$

From (3.8) we get that $K_{1n} \leq Mn^{-3}$, which gives

$$\sum_{n=1}^{\infty} nK_{1n} < \infty. \tag{3.12}$$

For $r > 0$ or $r \leq -\frac{1}{2}$ it is clear that $K_{2n} = 0$ for large n, which yields

$$\sum_{n=1}^{\infty} nK_{2n} < \infty. \tag{3.13}$$

Let $-\frac{1}{2} < r < 0$. It is easy to see that $K_{2n} \leq ME\left\{(\overline{W}_n - 1)^6 \eta_n^{-6(r+1)}\right\}$. Cauchy-Schwarz's inequality, (3.8), convexity and Lemma 1 give that $E\left\{(\overline{W}_n - 1)^6 \eta_n^{-6(r+1)}\right\} \leq Mn^{-3}\left\{E\left(\eta_n^{-12(r+1)}\right)\right\}^{1/2} \leq M\,n^{-3}$ if $n > 12(r+1)$. Hence we get (3.13). From Lemma 1 we have that $K_{3n} \leq P\left\{M(\overline{W}_n)^{-2r} \geq \frac{1}{4}(1-\varepsilon)n\right\} \leq Mn^{-3}E\left\{(\overline{W}_n)^{-6r}\right\} \leq Mn^{-3}$, which gives

$$\sum_{n=1}^{\infty} nK_{3n} < \infty. \tag{3.14}$$

Combining (3.10)–(3.14) we obtain (C3). We shall show (C2). Let any $p \geq 3$ and $\varepsilon_0 \equiv \varepsilon \in (0,1)$ be fixed. From (3.2) and c_r-inequality we have

$$\begin{aligned} \left\{\left(Z_n - \frac{n}{\varepsilon}\right)^+\right\}^p &\leq M\left\{n\left((\overline{W}_n)^{-2r} - \varepsilon^{-1}\right)^+\right\}^p + M\left\{\left(\frac{n+1}{l_{n+1}} - n\right)^+ (\overline{W}_n)^{-2r}\right\}^p \\ &\equiv M\,J_{1n} + M\,J_{2n}, \quad \text{say} \quad \text{for all } n \geq m-1, \end{aligned} \tag{3.15}$$

where $x^+ = \max(0,x)$. We shall prove the uniform integrability of $\{J_{1n}, n \geq m-1\}$. Let $s > p$ and $u > 1$ with $u^{-1} + v^{-1} = 1$. Then it follows from Hölder's inequality that

$$\begin{aligned} \sup_{n \geq m-1} E\{(J_{1n})^{s/p}\} &\leq \sup_{n \geq m-1} n^s E\left[\{(\overline{W}_n)^{-2r} I((\overline{W}_n)^{-2r} \geq \varepsilon^{-1})\}^s\right] \\ &\leq \left[\sup_{n \geq m-1} E\left\{(\overline{W}_n)^{-2sur}\right\}\right]^{1/u} \times \sup_{n \geq m-1}\left\{n^s\left(P\{(\overline{W}_n)^{-2r} \geq \varepsilon^{-1}\}\right)^{1/v}\right\}. \end{aligned} \tag{3.16}$$

Let $r > 0$. Since $m - 1 > \max\{1, 2pr\}$, we can choose $s > p$ and $u > 1$ such that $m-1 > \max\{1, 2sur\}$. Letting $\delta = 1 - \varepsilon^{1/2r} (> 0)$, it follows from (3.8) that $P\{(\overline{W}_n)^{-2r} \geq$

$\varepsilon^{-1}\} \le P\{|\overline{W}_n - 1| \ge \delta\} \le M n^{-sv}$ for all $n \ge 1$. Hence from Lemma 1 and (3.16) we get the uniform integrability of $\{J_{1n}, n \ge m-1\}$. Similarly, we can show the uniform integrability of $\{J_{1n}, n \ge m-1\}$ for $r < 0$. In the same way we can show the uniform integrability of $\{J_{2n}, n \ge m-1\}$. Thus, from (3.15) (C2) is satisfied. Finally we shall show (C6). Let any $(u,v) \ne (0,0)$ be fixed. Set $S_n = \sum_{i=1}^n (W_i - 1)$ and $S_n^* = \frac{S_n}{\sqrt{n}}$. From (2.2) and (3.4) we have

$$\begin{aligned} uS_n^* + v\xi_n &= uS_n^* + v\{1 - l_0 + r(2r+1)(S_n^*)^2\} \\ &\quad + vr(2r+1)(S_n^*)^2 \left\{ \left(1 + \frac{1-l_0}{n}\right) \eta_n^{-2(r+1)} - 1 \right\} + o_p(1) \,. \end{aligned} \tag{3.17}$$

Since $S_n^* \xrightarrow{d} Z$ and $\eta_n \xrightarrow{a.s.} 1$ as $n \to \infty$, from (3.17) we have that $uS_n^* + v\xi_n \xrightarrow{d} uZ + v\{1 - l_0 + r(2r+1)Z^2\}$ as $n \to \infty$. Thus by the Cramér-Wold device we obtain that $(S_n^*, \xi_n) \xrightarrow{d} (Z, \xi)$ as $n \to \infty$, which shows (C6). Therefore the proof of Lemma 5 is complete. $\square$

We are now in a position to prove all the results of Section 2. Throughout the proof below set $p = 3$ and $\alpha = 3/2$ in Lemma 5.

Proof of Theorem 1. Let $r > 0$. From the results of Woodroofe (1977) we get

$$E(N) = E(t) + 1 = n^* + \rho - \nu + 1 + o(1) \quad \text{as } w \to 0, \tag{3.18}$$

where

$$\rho = \frac{1}{2} + 2r^2 - \sum_{k=1}^{\infty} \frac{1}{k} E\left[\{k - 2rk(\overline{W}_k - 1)\}^-\right] \quad \text{with } x^- \equiv \max(0, -x) \tag{3.19}$$

and hence $0 \le \rho \le \frac{1}{2} + 2r^2$. Let $r < 0$. From Theorem 1 of Aras and Woodroofe (1993) with $m_0(r;3)$ in Lemma 5 and Remark 3 we have (3.18). Therefore the proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. It follows from Lemma 3.4 of Isogai, Saito and Uno (1999a) that $P\left\{\frac{\sigma_k}{\sigma} \le x | N = k\right\} = P\left\{\overline{W}_{k-1} \le x | t = k-1\right\}$ for all $x > 0$. Hence by (1.2) we have

$$\frac{R_N}{w} = \frac{n^*}{r^2} E\{(\overline{W}_t)^r - 1\}^2 \,. \tag{3.20}$$

By using Taylor's theorem we get

$$\begin{aligned} E\{(\overline{W}_t)^r - 1\}^2 &= r^2 E(\overline{W}_t - 1)^2 + 2\beta_r r E(\overline{W}_t - 1)^3 + \beta_r^2 E(\overline{W}_t - 1)^4 \\ &\quad + 2r\gamma_r E\{(\overline{W}_t - 1)^4 \eta_t^{r-3}\} + 2\beta_r\gamma_r E\{(\overline{W}_t - 1)^5 \eta_t^{r-3}\} \\ &\quad + \gamma_r^2 E\{(\overline{W}_t - 1)^6 \eta_t^{2(r-3)}\}, \end{aligned} \tag{3.21}$$

where $\beta_r = \frac{1}{2} r(r-1), \gamma_r = \frac{1}{6} r(r-1)(r-2)$ and η_t is a random variable such that $|\eta_t - 1| <$

$\left|\overline{W}_t - 1\right|$. From (3.20) and (3.21) we have

$$
\begin{aligned}
n^*\left(\frac{R_N}{w}-1\right) &= \{(n^*)^2E(\overline{W}_t-1)^2-n^*\}+\frac{2}{r}\beta_r(n^*)^2E(\overline{W}_t-1)^3\\
&\quad+\frac{1}{r^2}\beta_r^2(n^*)^2E(\overline{W}_t-1)^4+\frac{2}{r}\gamma_r(n^*)^2E\{(\overline{W}_t-1)^4\eta_t^{r-3}\}\\
&\quad+\frac{2}{r^2}\beta_r\gamma_r(n^*)^2E\{(\overline{W}_t-1)^5\eta_t^{r-3}\}\\
&\quad+\frac{1}{r^2}\gamma_r^2(n^*)^2E\{(\overline{W}_t-1)^6\eta_t^{2(r-3)}\}\,. \qquad (3.22)
\end{aligned}
$$

It follows from Corollary 1 of Aras and Woodroofe (1993), Remark 3 and (3.18) that

$$(n^*)^2E(\overline{W}_t-1)^2-n^* = 22r^2-3r+1-\rho-l_0+o(1) \quad \text{as } w\to 0 \qquad (3.23)$$

if $m > m_0(r;\,3)$. By Theorem 3 of Aras and Woodroofe (1993) we have that as $w \to 0$

$$(n^*)^2E(\overline{W}_t-1)^3 = -12r+2+o(1) \quad \text{and} \quad (n^*)^2E(\overline{W}_t-1)^4 = 3+o(1)\,. \qquad (3.24)$$

It follows from (3.5) that $(n^*)^2(\overline{W}_t-1)^4\eta_t^{r-3} \xrightarrow{d} Z^4$ as $w \to 0$. If $\{(n^*)^2(\overline{W}_t-1)^4\eta_t^{r-3}, 0 < w \le w_0\}$ is uniformly integrable for some $w_0 > 0$ when $m > m_4(r)$, then we have

$$(n^*)^2E\{(\overline{W}_t-1)^4\eta_t^{r-3}\} = 3+o(1) \quad \text{as } w\to 0 \quad \text{if } m > m_4(r), \qquad (3.25)$$

where

$$m_4(r) = \begin{cases} \max\{1+8r, 4+7r\} & \text{if} \quad r > 0 \\ 4-9r & \text{if} \quad r < 0\,. \end{cases}$$

To prove the uniform integrability it is sufficient to show that for all $r \neq 0$

$$\sup_{0<w\le w_0} E\left|(n^*)^2(\overline{W}_t-1)^4\eta_t^{r-3}\right|^{\alpha} < \infty \quad \text{for some } \alpha > 1\,. \qquad (3.26)$$

Let $\alpha > 1$, $s > 1$ with $s^{-1}+u^{-1} = 1$ and $v > 1$ with $v^{-1}+q^{-1} = 1$. By Hölder's inequality we get

$$
\begin{aligned}
&\sup_{0<w\le w_0} E\left|(n^*)^2(\overline{W}_t-1)^4\eta_t^{r-3}\right|^{\alpha}\\
&\le \sup_{0<w\le w_0}\left\{E(t/n^*)^{-4\alpha s v}\right\}^{1/sv} \times \sup_{0<w\le w_0}\left\{E\left((n^*)^{-1/2}\left|\sum_{i=1}^{t}(W_i-1)\right|\right)^{4\alpha s q}\right\}^{1/sq}\\
&\quad \times \sup_{0<w\le w_0}\left\{E\left(\eta_t^{(r-3)\alpha u}\right)\right\}^{1/u}\,. \qquad (3.27)
\end{aligned}
$$

Let $r \ge 3$. Since $m > 1+8r$, we can choose $\alpha > 1, (s,u)$ and (v,q) such that $m > 1+8r\alpha s v$. Hence by Lemmas 2–4 and (3.27) we have (3.26) for some $w_0 > 0$. Setting $s = 1+(3-r)/(8r)$ and $u = 1+8r/(3-r)$ for $0 < r < 3$ and $s = 1-(3-r)/(8r)$ and $u = 1-8r/(3-r)$ for $r < 0$, we can show (3.26) for some $w_0 > 0$ by the same way as above. Thus (3.26) holds for all $r \neq 0$. Similarly, we can show

$$(n^*)^2E\left\{(\overline{W}_t-1)^6\eta_t^{2(r-3)}\right\} = o(1) \quad \text{as } w\to 0 \quad \text{if } m > m_2(r) \qquad (3.28)$$

and

$$(n^*)^2 E\left|(\overline{W}-1)^5 \eta_t^{r-3}\right| = o(1) \quad \text{as } w \to 0 \quad \text{if } m > m_4(r). \tag{3.29}$$

Combining (3.22)–(3.25), (3.28), (3.29) and the fact that $m_2(r) > m_4(r) > m_0(r;3)$ we obtain Theorem 2. Therefore the proof is complete. $\square$

Proof of Proposition 1. By Taylor's theorem and (1.2) we have

$$\begin{aligned} E(\sigma_N^r) - \sigma^r &= \frac{1}{|r|}(n^*)^{-1/2} w^{1/2} \left[r n^* E(\overline{W}_t - 1) \right. \\ &\quad \left. + \tfrac{1}{2} r(r-1) n^* E\{(\overline{W}_t - 1)^2 \eta_t^{r-2}\} \right], \end{aligned} \tag{3.30}$$

where η_t is a random variable such that $|\eta_t - 1| < |\overline{W}_t - 1|$. Let $r > 0$. By Wald's equation we get that $n^* E(\overline{W}_t - 1) = -E\{t^{-1}(t-n^*)(S_t - t)\}$. It follows from the results of Woodroofe (1977) and Lemmas 2 and 3 that $t^{-1}(t-n^*)(S_t-t) \xrightarrow{d} 2rZ^2$ as $w \to 0$ and that $\{t^{-1}(t-n^*)(S_t-t),\ 0 < w \le w_0\}$ is uniformly integrable if $m > \max\{2,\ 1+3r\}$. Thus we have

$$n^* E(\overline{W}_t - 1) = -2r + o(1) \quad \text{as } w \to 0 \tag{3.31}$$

if $m > \max\{2,\ 1+3r\}$ for $r > 0$. For $r < 0$ Theorem 2 of Aras and Woodroofe (1993) gives (3.31) if $m > \max\{2,\ 1-2r\}$. Since from (3.5) $n^*(\overline{W}_t - 1)^2 \eta_t^{r-2} \xrightarrow{d} Z^2$ as $w \to 0$, we can show

$$n^* E\{(\overline{W}_t - 1)^2 \eta_t^{r-2}\} = 1 + o(1) \quad \text{as } w \to 0 \quad \text{if } m > m_3(r). \tag{3.32}$$

Thus, combining (3.30)–(3.32) we obtain Proposition 1. $\square$

Proof of Proposition 2. Let $m > m_3(r)$. From Proposition 1 we have

$$\begin{aligned} E(\sigma_N^r(k)) = \sigma^r &\quad + \quad \left[kE\left\{((t+1)/n^*)^{-1/2}\right\} - style\frac{1}{2}\operatorname{sign}(r)(3r+1) \right] \\ &\quad \times \quad (n^*)^{-1/2} w^{1/2} + o(w) \quad \text{as } w \to 0. \end{aligned} \tag{3.33}$$

By using Lemma 2, the inequality that $((t+1)/n^*)^{-1/2} \le (t/n^*)^{-1/2}$ and (3.5) we have that $E\left\{((t+1)/n^*)^{-1/2}\right\} = 1 + o(1)$ as $w \to 0$. Thus, from (3.33) we obtain Proposition 2. $\square$

Proof of Theorem 3. Let $m > m_2(r)\,(> m_3(r))$. By using (2.4) we can easily see

$$\begin{aligned} \frac{n^*}{w}(R_N(k) - R_N) &= 2k\, n^* w^{-1/2} E\{(\sigma_N^r - \sigma^r) N^{-1/2}\} + k^2 n^* E(N^{-1}) \\ &\equiv I_1 + I_2, \quad \text{say.} \end{aligned} \tag{3.34}$$

Since $E\left\{(N/n^*)^{-1}\right\} = E\left\{((t+1)/n^*)^{-1}\right\} = 1 + o(1)$, we get

$$I_2 = k^2 + o(1) \quad \text{as } w \to 0. \tag{3.35}$$

It is easy to see

$$\begin{aligned} I_1 &= 2k(n^*)^{1/2} w^{-1/2} \left[E\left\{ \left((N/n^*)^{-1/2} - 1\right)(\sigma_N^r - \sigma^r) \right\} + E(\sigma_N^r - \sigma^r) \right] \\ &\equiv 2k(n^*)^{1/2} w^{-1/2} (I_{11} + I_{12}), \quad \text{say.} \end{aligned} \tag{3.36}$$

From Proposition 1

$$2k(n^*)^{1/2}w^{-1/2}I_{12} = -k\,\mathrm{sign}(r)(3r+1)+o(1) \quad \text{as } w\to 0. \tag{3.37}$$

We shall evaluate I_{11}. Taking the conditional expectation into consideration, we obtain

$$I_{11} = \frac{1}{|r|}(n^*)^{1/2}w^{1/2}E\left[\left\{((t+1)/n^*)^{-1/2}-1\right\}\left\{(\overline{W_t})^r-1\right\}\right].$$

Hence

$$\begin{aligned} 2k(n^*)^{1/2}w^{-1/2}I_{11} &= -k\,\mathrm{sign}(r)E\left[-\frac{2n^*}{r}\left\{((t+1)/n^*)^{-1/2}-1\right\}\left\{(\overline{W_t})^r-1\right\}\right] \\ &\equiv -k\,\mathrm{sign}(r)E(J), \quad \text{say.} \end{aligned} \tag{3.38}$$

By Taylor's theorem we have

$$\left\{((t+1)/n^*)^{-1/2}-1\right\}\left\{(\overline{W_t})^r-1)\right\} = -\frac{r}{2}(n^*)^{-1}(t+1-n^*)(\overline{W_t}-1)\varphi_t^{-3/2}\eta_t^{r-1},$$

where φ_t and η_t are random variables such that

$$|\varphi_t-1|<|(t+1)/n^*-1| \quad \text{and} \quad |\eta_t-1|<|\overline{W_t}-1|. \tag{3.39}$$

Thus we get that $J=(t+1-n^*)(\overline{W_t}-1)\varphi_t^{-3/2}\eta_t^{r-1}$. Let $Q_w = Z_t - n^*$. Then (3.3) gives that $t+1-n^* = 2rt(\overline{W_t}-1)+(Q_w-\xi_t+1)$, which yields

$$J = 2rt(\overline{W_t}-1)^2\varphi_t^{-3/2}\eta_t^{r-1} + (Q_w-\xi_t+1)(\overline{W_t}-1)\varphi_t^{-3/2}\eta_t^{r-1}. \tag{3.40}$$

From Proposition 3 of Aras and Woodroofe (1993) with $\rho=E(R)$ we have that $Q_w-\xi_t+1 \xrightarrow{d} R-\xi+1$ as $w\to 0$. Hence, by (3.5), (3.39) and (3.40) we obtain that $J\xrightarrow{d} 2rZ^2$ as $w\to 0$. Suppose that $\{J,\, 0<w\le w_0\}$ is uniformly integrable for some $w_0>0$. Then we get that $E(J)=2r+o(1)$ as $w\to 0$, which, together with (3.38), yields that $2k(n^*)^{1/2}w^{-1/2}I_{11} = -2kr\,\mathrm{sign}(r)+o(1)$ as $w\to 0$. Thus, combining this result and (3.34)–(3.37) we obtain Theorem 3. In the remainder of this proof we shall show the uniform integrability of $\{J,\, 0<w\le w_0\}$. Let $t^*\equiv (n^*)^{-1/2}(t-n^*+1)$. From Proposition 8 of Aras and Woodroofe (1993) $\{|t^*|^\gamma,\, 0<w\le w_0\}$ for any $\gamma\in(0,2]$ is uniformly integrable for some $w_0\in(0,\infty)$. We can show that $\varphi_t^{-3/2}\le (t/n^*)^{-3/2}$ on $\{(t+1)/n^*\le 1/2\}$ and that $(t/n^*)^{-1}\varphi_t^{-3/2}<8$ on $\{(t+1)/n^*>1/2\}$. Thus

$$\begin{aligned} |J| &= |J|I((t+1)/n^*\le 1/2)+|J|I((t+1)/n^*>1/2) \\ &\le (t/n^*)^{-5/2}|t^*|\left|(n^*)^{-1/2}\sum_{i=1}^t(W_i-1)\right|\eta_t^{r-1} + M|t^*|\left|(n^*)^{-1/2}\sum_{i=1}^t(W_i-1)\right|\eta_t^{r-1} \\ &\equiv J_1+J_2, \quad \text{say.} \end{aligned}$$

It is sufficient to show that $\{J_i,\, 0<w\le w_0\}$ for $i=1,2$ are uniformly integrable. Let $\alpha>1$, $s>1$ with $s^{-1}+u^{-1}=1$ and $v>1$ with $v^{-1}+q^{-1}=1$. By Hölder's inequality we get

$$\begin{aligned} &\sup_{0<w\le w_0} E|J_2|^\alpha \\ &\le \sup_{0<w\le w_0}\{E|t^*|^{\alpha sv}\}^{1/sv} \times \sup_{0<w\le w_0}\left\{E\left((n^*)^{-1/2}\left|\sum_{i=1}^t(W_i-1)\right|\right)^{\alpha sq}\right\}^{1/sq} \\ &\quad \times \sup_{0<w\le w_0}\left\{E\left(\eta_t^{(r-1)\alpha u}\right)\right\}^{1/u}. \end{aligned}$$

Let

$$m_5(r) = \begin{cases} 2 & \text{if} \quad r > 0 \\ 3 - 6r & \text{if} \quad r < 0. \end{cases}$$

By Lemmas 3 and 4 and an argument similar to that in (3.25) we can show the uniform integrability of $\{J_2, 0 < w \leq w_0\}$ if $m > m_5(r)$. Let

$$m_6(r) = \begin{cases} \max\{1 + 10r, 3 + 8r\} & \text{if} \quad r > 0 \\ 3 - 6r & \text{if} \quad r < 0. \end{cases}$$

Then, similarly we can show the uniform integrability of $\{J_1, 0 < w \leq w_0\}$ if $m > m_6(r)$. It is clear that $m_2(r) \geq m_6(r) \geq m_5(r)$. Therefore the proof of Theorem 3 is complete. $\square$

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