UNBOUNDED RADIALLY SYMMETRIC VISCOSITY SOLUTIONS OF SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS

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Abstract. We show the existence of radially symmetric unbounded viscosity solutions of semilinear degenerate elliptic equationsand give the classification of their solutions according to the asymptotic behavior as $|x| \to \infty$.

0 Introduction We consider the following semilinear degenerate elliptic equation :

$$-g(\mid x \mid)\Delta u(x) + u(x) \mid u(x) \mid^{p-1} = f(\mid x \mid) \quad \text{in} \quad \mathbf{R}^N \quad (N \geq 2) \tag{0.1}$$

where $g : [0, \infty) \to [0, \infty)$ is a differentiable and nonnegative function and $p > 1$ is a constant. We are concerned with the existence and structure of continuous viscosity solutions of (0.1) which may not be bounded. In order to study structure of continuous viscosity solutions of (0.1), it is important to investigate whether continuous viscosity solutions of (0.1) are radially symmetry or not.

When f is zero and $g > 0$ there have been lots of works published in this direction, see, for example [1,2,3,4,5,6,7]. For more general equations than our equations (0.1), Li and Ni[5] proved that all bounded positive solutions are radially symmetric and S.D.Taliaferro [6] showed that all solution satisfying $\lim_{|x|\to\infty} u(x) = \infty$ are also radially symmetric. In [5] and [6], combining the asymptotic behavior of the solutions and the moving plane technique, they investigated whether solutions are radially symmetric or not. However, as our equations (0.1) are of degenerate elliptic equations we can not use the moving plane method.

In recent year, we [8] showed the existence and uniqueness of a continuous viscosity solution of Dirichlet problem for the degenerate elliptic equation (0.1) in an open ball with the center at the origin and in [9] studied the existence and uniqueness of solutions of quasilinear degenerate elliptic equations in the open ball. In [8] and [9] we showed that the continuous viscosity solution is unique when a classical radial solution of (0.1) is unique and proved the uniqueness. In this paper we will employ the method used in [8] and [9] to establish the structure of continuous viscosity solutions of (0.1). Therefore, we will introduce an ordinary differential equation associated with the radial solution of (0.1)

$$-g(t)(\frac{d^2y}{dt^2}(t) + \frac{(N-1)}{t}\frac{dy}{dt}(t)) + y(t)|y(t)|^{p-1} = f(t) \quad \text{on} \quad (0, \infty). \tag{0.2}$$

In case f is zero and $g > 0$, the structure of solutions of (0.2) has been studied by many authors; see, for example [10,11,12,13,14]. For a quasilinear equation including our equation (0.2), M.Mizukami, M.naito and H.Usami [9] and T.Tanigawa [12] made a deep investigation the asymptotic behavior of positive solutions as $|x| \to \infty$.

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In this paper, however, we consider the case where the equation is the degenerate type and $f \neq 0$. Moreover, we study the continuous viscosity solution which makes assumptions on neither the state of growth of u as $|x| \to \infty$ nor the positivity of solution.

The main purpose of this paper is to establish the existence and structure of equations (0.2) completely. Moreover, in corollaries of theorems we will state our assertion in respect of continuous viscosity solutions.

Our plan of this paper is as follows. In section 1, we state assumptions and theorems. In section 2, we list some notations and results which will be used throughout this paper. In section 3, as the rate of the polynomial growth of $g(t)$ at $t \to \infty$ is smaller than 2, we will show the uniqueness of the solution of (0.2) and state the uniqueness of the viscosity solution of (0.1). In sections 4 and 5, as $\{$the rate of the growth of $g(t)\} > 2$, we will give the classification of solutions of (0.2) associated with the asymptotic behavior at $|x| \to \infty$ and state the properties of viscosity solutions of (0.1) according to the asymptotic behavior of each viscosity solution of (0.1).

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Professor Tomita passed away soon after we began this joint work. The first author would like to pray for the repose of his soul.

1 Assumptions and Theorems

In this section we state assumptions and theorems.

We now list hypotheses on f and g :

(**H1**) $f(t) \in C^1([0,\infty))$ and there exists the limit such that

$$\lim_{t\to\infty} \frac{f(t)}{t^\gamma} = \infty \text{ or nonnegative constant}$$

for any $\gamma \geq 0$.

(**H'2**) $g \in C^1([0,\infty))$ and $g(t) \geq 0$ for any $t \in [0,\infty)$.

Throughout this paper we assume the conditions (H1) and (H'2).

Remark 1. (1) The limit in (H1) need not be nonnegative. But the proof of our assertion becomes more complicated. Therefore, for simplicity, we assume that is nonnegative.

(2) Since the assumption (H'2) satisfies the assumption (H3) in [8], we can apply the method of the proof of Theorem 2 in [8] to our assertions.

In order to state our theorems we introduce some notations. We denote the set of zero points of $g(t)$ by $\mathcal{Z}$: $\mathcal{Z} = \{t \in [0,\infty) | g(t) = 0\}$ and let $\varphi(t)$ be an implicit function of $y(t)|y(t)|^{p-1} - f(t) = 0$.

In the case of $\sup \mathcal{Z} = \infty$, using the similar method as in the proof of Theorem 2 in [8], we have the following theorem.

Theorem 0 *Let* $\sup \mathcal{Z} = \infty$. *Then there exists a unique continuous viscosity solution* u *of (0.1) . Moreover, it is the radially symmetric solution satisfying* $u(x) = \varphi(|x|)$ *for all* x *such that* $g(|x|) = 0$ *where* $x \neq 0$.

Hence, we have only to study the problem (0.1) under the assumption that $\sup \mathcal{Z} < \infty$. In this way we assume $0 < R = \sup \mathcal{Z} < \infty$.

From Theorem 2 in [8] we obtain that the continuous viscosity solution of the equation (0.1) in $\{x \in R^N; |x| \leq R\}$ satisfies $u(x) = \varphi(R)$ for any $x : |x| = R$. On the other hand we shall consider the following Dirichlet problem:

$$
(1.1) \qquad \begin{cases} -g(|\, x \,|)\Delta u(x) + u(x) \mid u(x) \mid^{p-1} = f(|\, x \,|) & \text{for any} \quad x : |x| > R \\ u(x) = \varphi(R) \quad \text{for any} \quad x : |x| = R. \end{cases}
$$

Connecting the continuous viscosity solutions of (1.1) and the continuous viscosity solutions of the equation (0.1) in $\{x \in R^N; |x| \leq R\}$, we can get all continuous viscosity solutions of (0.1)(See [8]). Then, we have only to investigate the Dirichlet problem of (1.1). Therefore, the equation (0.2) is replaced by the following equation:

$$
(1.2) \qquad \begin{cases} -g(t)(\ddot{y}(t) + \frac{(N-1)}{t}\dot{y}(t)) + y(t)|y(t)|^{p-1} = f(t) \quad \text{on} \quad (R, \infty), \\ y(R) = \varphi(R). \end{cases}
$$

In order to study the structure of solutions of (1.2) more in detail we introduce the following assumption stronger than (H'2):
(**H2**) $g \in C^1([R, \infty))$ and $g(t) > 0$ for any $t \in (R, \infty)$. Moreover,

$$
g(t)^{-1} = d_0 t^{-\ell} + O(t^{-\ell-1}) \quad \text{as} \quad t \to \infty,
$$

where ℓ and d_0 are some positive numbers. For simplicity let $d_0 = 1$.

Theorem 1 *Let $\ell \leq 2$. Under the assumptions (H1) and (H2) there exists a unique classical solution of (1.2).*

Corollary 1 *Let the same conditions as in Theorem 1 be assumed. Then, there exists a unique continuous viscosity solution of (0.1).*

We next state our assertions in the case of $\ell > 2$. We put

$$
\theta = \frac{\ell - 2}{p - 1}.
$$

Theorem 2 *Let $\ell > 2$. Assume $\lim_{t\to\infty} \dfrac{f(t)}{t^{\theta p}} = \infty$. Then our assertions as in Theorem 1 are valid.*

Corollary 2 *Let the same conditions as in Theorem 2 be assumed. Then, our assertions as in Corollary 1 are valid.*

It remains to consider the case $\lim_{t\to\infty} \dfrac{f(t)}{t^{\theta p}} = \kappa^p$ $(0 \leq \kappa < \infty)$.

To obtain the next theorems we assume the following condition of $f(t)$:

$$
(\mathbf{H3}) \qquad f(t) = \kappa^p t^{\theta p} + O(t^{\theta p - 1}) \quad \text{as} \quad t \to \infty.
$$

We now consider solutions of the following equation:

$$
(1.3) \qquad X|X|^{p-1} - \kappa^p - (\theta^2 + (N-2)\theta)X = 0.
$$

The real solutions of the equation (1.3) satisfy one of the following three cases.

(**C1**) It is a positive single solution (ω_+).

(**C2**) These solutions are a positive single solution (ω_+) and nonpositive double solutions (ω_0).

(**C3**) They are a positive single solution (ω_+) and two nonpositive single solutions (ω_0 and ω_-).

Remark 2 The case (C2) implies $\theta^2 + (N-2)\theta - p \mid \omega_0 \mid^{p-1} = 0$ and the case (C3) has $\theta^2 + (N-2)\theta - p \mid \omega_0 \mid^{p-1} > 0$. After this we assume (H2) and (H3).

Theorem 3 *Let the solutions of* (1.3) *satisfy* (C1). *Then there is a unique classical solution of (1.2). Moreover, this solution satisfies* $\lim_{t\to\infty}\frac{y(t)}{t^\theta} = \omega_+$.

Corollary 3 *Under the same assumption as in Theorem 3 there exists a unique continuous viscosity solution of (0.1).*

Theorem 4 *Let the solutions of (1.3) satisfy (C2) or (C3). Then there exist classical solutions of (1.2). Moreover, they have the following asymptotic behavior.*
(1) *Let the solutions of* (1.3) *satisfy* (C2). *Then*

$$\lim_{t\to\infty}\frac{y(t)}{t^\theta} = \omega_+ \text{ or } \omega_0.$$

(2) *Let the solutions of* (1.3) *satisfy* (C3). *Then*

$$\lim_{t\to\infty}\frac{y(t)}{t^\theta} = \omega_+ \text{ or } \omega_0 \text{ or } \omega_-.$$

Moreover, the solution satisfying $\lim_{t\to\infty}\frac{y(t)}{t^\theta} = \omega_+$ *or* ω_- *is only one respectively and a set of solutions satisfying* $\lim_{t\to\infty}\frac{y(t)}{t^\theta} = \omega_0$ *holds the cardinality of continuum.*

Corollary 4 *Under the same assumption as in Theorem 4 there exist continuous viscosity solutions on (0.1) which hold the same asymptotic behavior as in Theorem 4.*

2 Preliminaries At first, we show the properties of the classical solution $y(t)$ of (1.2).

Lemma 2.1 *There exist classical solutions* $y \in C([R,T_y)) \cap C^2((R,T_y))$ *of (1.2) where* T_y *is the life span of a solution* y.

Proof. This lemma is a consequence of Proposition 3.4 in [8].

Lemma 2.2 *For the solution* $y(\cdot)$ *in Lemma 2.1 it holds that* $\lim_{t\to T_y} y(t)$ *exists. (i.e* $\lim_{t\to T_y} y(t) = \pm\infty$ *or* $Const.$)

Proof. Combining the maximum(or minimum) principle and the assumption (H1) we can prove this lemma.

Lemma 2.3 *Let* α *be any real number and any* $P > R$. *Then there exists a unique solution* $y_{\alpha,P}(t)$ *of (1.2) satisfying* $y(P) = \alpha$.

Proof. This lemma is also a consequence of [8].

We denote the life span $T_{y_{\alpha,P}}$ by $T_{\alpha,P}$.

Lemma 2.4 *Let* $\delta_0 > 0$, $\delta_1 > 0$, *and let* $w(t)$ *be a nonnegative continuous function on* $C([T_1,T_w))$ *satisfying*

$$w(t) \geq \delta_0 + \delta_1 \int_{T_1}^{t} s^{-(N-1)} \int_{T_1}^{s} r^{N-3} w(r)\,dr\,ds.$$

Then we have

$$w(t) \geq \frac{\delta_0}{(\lambda_1+\lambda_2)}\left(\lambda_2\left(\frac{t}{T_1}\right)^{\lambda_1} + \lambda_1\left(\frac{t}{T_1}\right)^{-\lambda_2}\right)$$

where $\lambda_1, -\lambda_2 (\lambda_1 > -\lambda_2)$ *are solutions of the following quadratic equation:* $\lambda^2 + (N-2)\lambda - \delta_1 = 0$.

Proof. We consider the following equation:

$$\begin{cases} t^2\ddot{z}(t)+t(N-1)\dot{z}(t)-\delta_1 z(t)=0, \\ z(T_1)=\delta_0,\ \dot{z}(T_1)=0. \end{cases}$$

Then we have

$$z(t)=\frac{\delta_0}{(\lambda_1+\lambda_2)}(\lambda_2(\frac{t}{T_1})^{\lambda_1}+\lambda_1(\frac{t}{T_1})^{-\lambda_2}).$$

Using the comparability theorem we prove this lemma.

Lemma 2.5 *We assume the same conditions as in Lemma 2.4. Moreover,*

$$w(t)\geq \delta_0+\delta_1\int_P^t s^{-(N-1)}\int_P^s r^{N-3}w(r)^p drds.$$

If $T_w=\infty$ we have

$$\lim_{t\to\infty}\frac{w(t)}{t^q}=\infty$$

where q is any positive number.

Proof. From $w(t)\geq\delta_0$ we have $w(t)^p\geq$ (small number)$w(t)$. From lemma 2.4 it follows that $\lim_{t\to\infty}w(t)=\infty$. Replacing P by a sufficiently large number T_1 we see $w(t)^p\geq (large\ Constant)w(t)$ on $[T_1,\infty)$. Using the same method as in the proof of Lemma 2.4 we get $\displaystyle\lim_{t\to\infty}\frac{w(t)}{t^q}=\infty$.

We recall T_w is the life span of w.

Lemma 2.6 *Let $e(t)\in C^1([T,\infty))$ be positive and let $t^{2\beta}e(t)$ be an increasing or decreasing function where $\beta>1$. Suppose that $w(t)\in C^2([T,T_w))$ satisfies the following hypothesis:*

$$\begin{cases} \dot{w}(t)>0, \quad w(t)>0 \quad and \quad \min(e^{1/2}(t),s^{-\beta})w^{(p-1)/2-\epsilon}(t)\notin L_1(T,T_w) \\ \ddot{w}(t)+\frac{\beta}{t}\dot{w}(t)\geq e(t)w(t)^p \end{cases} \tag{2.1}$$

where $0<\epsilon<(p-1)/2$. Then the life span T_w of $w(t)$ is finite.

Proof. To prove $T_w<\infty$ by the contradiction we suppose $T_w=\infty$.
We multiply (2.1) by $t^{2\beta}\dot{w}(t)$ and integrate over $[T,t]$ to find

$$\begin{cases} t^{2\beta}(\dot{w}(t))^2-T^{2\beta}(\dot{w}(T))^2 & \geq \frac{2}{p+1}(t^{2\beta}e(t)w^{p+1}(t)-T^{2\beta}e(T)w^{p+1}(T)) \\ & \quad -\frac{2}{p+1}\int_T^t(\frac{d}{ds}s^{2\beta}e(s))w^{p+1}(s)ds=I_1+I_2. \end{cases}$$

If $\dfrac{d}{ds}s^{2\beta}e(s)>0$, from $\dot{w}(t)>0$, we observe that

$$I_2\geq-\frac{2}{p+1}(t^{2\beta}e(t)-T^{2\beta}e(T))w^{p+1}(t).$$

Similarly, if $\dfrac{d}{ds}s^{2\beta}e(s)<0$, we have

$$I_2\geq-\frac{2}{p+1}(t^{2\beta}e(t)-T^{2\beta}e(T))w^{p+1}(T).$$

Then

$$\dot{w}^2(t) > t^{-2\beta}(I_1 + I_2) \geq \frac{2t^{-2\beta}}{(p+1)} \min(t^{2\beta}e(t), T^{2\beta}e(T))w^{p+1}(t)(1 - (\frac{w(T)}{w(t)})^{p+1}).$$

Since $w(t)$ is increasing there exists T_1 such that $T_1 > T$ and $\dfrac{w(T)}{w(T_1)} < 1 - \delta$ where δ is the positive sufficiently small constant.
Thus

$$\frac{\dot{w}(t)}{w^{1+\epsilon}(t)} \geq (\frac{2}{p+1})^{1/2}(1-(1-\delta)^{p+1})t^{-\beta}\min(t^{\beta}e(t)^{1/2}, T^{\beta}e(T)^{1/2})w^{(p-1)/2-\epsilon}(t).$$

Integrating over $[T_1, t]$ we get

$$\frac{1}{\epsilon}\frac{1}{w(T_1)^{\epsilon}} \geq C\int_{T_1}^{t} \min(e(s)^{1/2}, s^{-\beta})w^{(p-1)/2-\epsilon}(s)ds. \tag{2.2}$$

where C is independent of w and t.
Since the right hand side of (2.2) tends to ∞ as $t \to \infty$ we get the contradiction.

Lemma 2.7 *There are positive constants ϵ_0 and K such that*

$$x|x|^{p-1} - y|y|^{p-1} \geq \epsilon_0(x-y)(|x|^{p-1} + |y|^{p-1}) \tag{2.3}$$

and

$$x|x|^{p-1} - y|y|^{p-1} \leq K(x-y)(|x|^{p-1} + |y|^{p-1})$$

for any $x \geq y \in \mathbf{R}$.

Proof. The proof is standard and easy and so we omit it.

3 Proof of Theorem 1 In this section we shall prove Theorem 1. We recall that the function $y_{\alpha,P}$ is the solution of (1.2) satisfying $y(P) = \alpha$ and $T_{\alpha,P}$ is the life span of the solution $y_{\alpha,P}$ of (1.2).

Definition 1 *We define S^{++}, S^{--} and S by*

$$\begin{cases} S^{++} &= \{\alpha : \lim_{t\to T_{\alpha,P}} y_{\alpha,P}(t) = \infty, \quad T_{\alpha,P} < \infty\} \\ S^{--} &= \{\alpha : \lim_{t\to T_{\alpha,P}} y_{\alpha,P}(t) = -\infty, \quad T_{\alpha,P} < \infty\} \\ S &= \{\alpha : T_{\alpha,P} = \infty\} \end{cases}$$

where $P = R + 1$.

Lemma 3.1 *Let $\ell \leq 2$. Then there exist a and b such that $-\infty < b \leq a < \infty$ and*

$$S^{++} = (a, \infty), \quad S = [b, a] \quad \textit{and} \quad S^{--} = (-\infty, b).$$

Proof. From lemma 2.2 we know that $S^{++} \cup S \cup S^{--} = (-\infty, \infty)$.
To prove $S^{++} \neq \phi$ by the contradiction we assume $S^{++} = \phi$. We now choose increasing sequences $\{t_n\}$ and $\{\alpha_n\}$ such that

$$\lim_{n\to\infty} t_n = \infty, \lim_{n\to\infty} \alpha_n = \infty \quad \text{and} \quad \alpha_n > \max_{R+1\leq t\leq t_n} 2|\varphi(t)| + 2.$$

For simplicity we denote by $y_n(t) = y_{\alpha_n,R+1}(t)$ and $f(t,y) = y|y|^{p-1} - f(t)$.
Let n be a sufficiently large number. From the maximum principle it follows that $\dot{y}_n(R+1) \geq 0$. Combining $\alpha_n > \max_{R+1\leq t\leq t_n} 2|\varphi(t)| + 2$ and (2.3) we have $f(t,\alpha_n) > \epsilon_0(\alpha_n - \varphi(t))(|\alpha_n|^{p-1} + |\varphi(t)|^{p-1}) > 0$ for any $t \in [R+1, t_n]$. If there exists $t' \in (R+1, t_n]$ such that $y_n(t') < \alpha_n$, $y_n(t)$ takes its the maximum at a point t" on $[R+1, t']$ and satisfy $y_n(t") \geq \alpha_n$. This is the contradiction from the maximum principle. Then, $y_n(t) \geq \alpha_n$ for any $t : R+1 \leq t \leq t_n$. Thus, it follows that $f(t, y_n(t)) \geq \dfrac{\epsilon_0}{2} y_n^p(t)$ from $y_n(t) - \varphi(t) \geq \dfrac{1}{2}|y_n(t)|$.
Recall that ℓ is smaller than 2. Combining the integral equation associated with (1.2) and the inequality (2.3) we have, for any $t \in [R+1, t_n]$,

$$y_n(t) \geq \alpha_n + \delta_1 \alpha_n^{p-1} \int_{R+1}^{t} s^{-(N-1)} \int_{R+1}^{s} r^{N-3} y_n(r) dr ds \tag{3.1}$$

and

$$t^{N-1} \dot{y}_n(t) \geq \delta_1 \int_{R+1}^{t} s^{N-1-\ell} y_n^p(s) ds, \tag{3.2}$$

where δ_1 is a positive constant independent of n, t. By Lemma 2.4, the inequality (3.1) yields $y_n(t) \geq \dfrac{\alpha_n}{2}(t/(R+1))^{\lambda_1(n)}$ for any $(R+1) \leq t \leq t_n$, where $\lambda_1(n) \to \infty$ as $n \to \infty$. Then, it follows that $y_n(t) > t^{\lambda_1(n)/2}$ for any $t : (R+1)^2 < t < t_n$. Therefore, $t^{-\ell+N-1} y_n^{(p-1)/2-\epsilon}(t) \geq 1$ for sufficiently large n and small $\epsilon > 0$.
From $f(t, y_n(t)) \geq \dfrac{\epsilon_0}{2} y_n^p(t)$, the equation (1.2) implies

$$\ddot{y}_n(t) + \frac{(N-1)}{t} \dot{y}_n(t) \geq \delta t^{-\ell} y_n^p.$$

From (3.2) it follows that $\dot{y}_n(t) > 0$ for large t. Using the similar method to the proof of Lemma 2.6, we have

$$\frac{1}{\epsilon} \frac{1}{y_n((R+1)^2)} \geq Const(t_n - (R+1)^2).$$

For the sufficiently large n the above inequality is a contradiction. Then $S^{++} \neq \phi$.
The similar method used in the proof of $S^{++} \neq \phi$ implies that $S^{--} \neq \phi$.
Assume that $\sup_{\alpha\in S^{++}} T_{\alpha,P} = T$ is finite. Replacing P by $T+1$ and using the similar method to the above mentioned argument, we know that there exists a solution of (1.2) with the finite life span $> T+1$. It is the contradiction. Then $\sup_{\alpha\in S^{++}} T_\alpha = \infty$. Let $\alpha \in S^{++}$ and $\beta > \alpha$. Then we see $\beta \in S^{++}$ for any $\beta \geq \alpha$ from the comparability theorem.
Second we shall prove that S^{++} is an open set by contradiction. Suppose S^{++} is a closed set. From $S^{--} \neq \phi$ there exists $\alpha_0 \ni S^{++}$ such that $\alpha_0 = \min S^{++}$. Then it follows that $T_{\alpha_0,P} \geq \sup_{\alpha\in S^{++}} T_{\alpha,P}$. This contradicts $\sup_{\alpha\in S^{++}} T_{\alpha,P} = \infty$. Thus S^{++} is an open set. S^{--} is also an open set. Therefore S is not empty and a closed set.

Proposition 3.2 *S is one point set.*

Proof. Assume α_1, and $\alpha_2 \in S$ such that $\alpha_1 > \alpha_2$. Putting $w(t) = y_{\alpha_1}(t) - y_{\alpha_2}(t)$, we get

$$w(t) > 0, \quad \frac{dw}{dt}(t) > 0, \quad \text{and}$$

$$\ddot{w}(t) + \frac{(N-1)}{t}\dot{w} = g(t)^{-1}(f(t, y_{\alpha_1}(t)) - f(t, y_{\alpha_2}(t))).$$

From (2.3) it follows that

$$\begin{aligned} g(t)^{-1}(f(t,y_{\alpha_1})-f(t,y_{\alpha_2})) &> \delta_1 t^{-\ell}(y_{\alpha_1}-y_{\alpha_2})(|y_{\alpha_1}^{p-1}|+|y_{\alpha_2}|^{p-1}) \\ &> \delta_2 t^{-\ell}|y_{\alpha_1}-y_{\alpha_2}|^p. \end{aligned}$$

Using Lemma 2.4 and Lemma 2.5, we see $t^{-\ell+N-1}w^{(p-1)/2-\epsilon} \geq 1$ for sufficiently large t and small ϵ. Applying Lemma 2.6 to $w(t)$ we obtain that the life span of $w(t)$ is finite. This is a contradiction.

The proofs of Theorem 1 and Corollary 1 are thus accomplished from Proposition 3.2 and Proposition A-3 in Appendix.

4 Proof of Theorem 2

In this section we shall study the case of $\ell > 2$ and $\lim_{t\to\infty}\frac{\varphi(t)}{t^\theta} = \infty$. Let $y(t)$ be the solution of (1.2). We put $y(t) = t^\theta v(t)$, $\varphi(t) = t^\theta \kappa(t)$ and $r_0 = \theta^2 + (N-2)\theta$. From (1.2) we have

$$\begin{cases} \ddot v(t) + \dfrac{2\theta+(N-1)}{t}\dot v(t) = \dfrac{1}{t^2}(g(t)^{-1}t^\ell(v|v|^{p-1} - \kappa^p(t)) - r_0 v(t)), \\ v(R) = \kappa(R). \end{cases} \tag{4.1}$$

We shall show that there exists a unique solution of (4.1) on $[R,\infty)$.

We shall define S_4^{++}, S_4^{--} and S_4 of solutions $\{v_\alpha\}$ of the equation (4.1) by $S^{++} = \{\alpha : \lim_{t\to T_{\alpha,R+1}} v_{\alpha,R+1}(t) = \infty,\quad T_{\alpha,R+1} < \infty\}$, $\quad S^{--} = \{\alpha : \lim_{t\to T_{\alpha,R+1}} v_{\alpha,R+1}(t) = -\infty,\quad T_{\alpha,R+1} < \infty\}$ and $S = \{\alpha : T_{\alpha,R+1} = \infty\}$.

Lemma 4.1 *S_4^{++} and S_4^{--} are not empty.*

Proof. Assume $S_4^{++} = \phi$. From (H2) we have $\inf_{t\geq R} g(t)^{-1}t^\ell \geq \epsilon_2 > 0$. Let Q_0 be a number larger than $\max\{\max_{R\leq t\leq R+1} 2|\kappa(t)|^{p-1}, (\frac{4r_0}{\epsilon_2})^{1-p}\}$. Let $L_0 = R+1$. We choose increasing sequences $\{L_n\}$ and $\{Q_n\}$ such that, for any $n = 1,2,\cdots$,

(1) $L_0 < L_n,\ Q_0 < Q_n$ and $\lim_{n\to\infty} L_n = \infty,\quad \lim_{n\to\infty} Q_n = \infty$.

(2) $Q_n \geq 2\max_{L_0\leq t\leq L_n} |\kappa(t)| + 2$.

Let n be a sufficiently large number and $v_n(t)$ be solutions of (4.1) which are connected $(R,\kappa(R))$ with (L_0, Q_n). The argument used in the proof of Lemma 3.1 imply that $v_n(t) \geq Q_n$. Then, it follows that

$$g(t)^{-1}t^\ell(v_n(t)|v_n(t)|^{p-1} - \kappa^p(t)) - r_0 v_n(t) > \frac{\epsilon_2}{2}(v_n(t)|v_n(t)|^{p-1} - \kappa^p(t)) > \frac{\epsilon_2}{4}Q_n^p$$

for any $t : L_0 \leq t \leq L_n$.

Thus, the equation (4.1) yields

$$\ddot v_n(t) + \frac{2\theta+(N-1)}{t}\dot v_n(t) \geq \frac{\epsilon_2}{4}\frac{v_n(t)^p}{t^2}$$

where any $L_0 \leq t \leq L_n$. The above inequality implies $\dot v_n(t) > 0$ from $\dot v_n(L_0) \geq 0$. By the analogous argument in Lemma 3.1 we have a contradiction. Then $S_4^{++} \neq \phi$. Similarly, we have $S_4^{--} \neq \phi$.

We recall that S^{++}, S^{--} and S are as in Definition 1. On the other hand, it is clear, by the definition of S_4^{++}, S_4^{--} and S_4 , that $S^{++} = (R+1)^\theta S_4^{++}$, $\quad S^{--} = (R+1)^\theta S_4^{--}$ and $S = (R+1)^\theta S_4$.

Lemma 4.2 *S_4 is a nonempty closed set.*

Proof. From Lemma 4.1 it follows that S^{++} and S^{--} are nonempty sets. Using the analogous argument in Lemma 3.1 we know that S^{++} and S^{--} are open sets. Thus it follows that S is the nonempty closed set. Therefore, S_4 is a nonempty closed set.

We denote by $v_0(t)$ the solution of (4.1) satisfying $v(L_0) \in S_4$.

Lemma 4.3 *We have* $\lim_{t\to\infty} v_0(t) = \infty$.

Proof. This lemma is proved by contradiction. Let E be any large number. From $\lim_{t\to\infty} \kappa(t) = \infty$ we can choose a constant L such that $2E+1 < \min_{t>L} \kappa(t)$.
Assume $v_0(L) < E$. If there exists L_1 such that $v_0(L_1) = v_0(L)$ and $v_0(L) > v_0(t)$ for any $L < t < L_1$, $v_0(t)$ holds a locally minimum point. On the other hand, from (2.3), it follows that

$$\text{the right hand side of (4.1)} \leq$$

$$\frac{1}{t^2}\{\epsilon_0\epsilon_2(v_0(t)-\kappa(t))(\mid v_0(t) \mid^{p-1} + |\kappa(t)|^{p-1}) + r_0|v_0(t)|\}.$$

Since $(-\epsilon_0\epsilon_2 E^{p-1} + 2r_0) < 0$ for sufficiently large E, there exists a positive constant δ_0 such that

$$\ddot{v}_0(t) + \frac{2\theta+(N-1)}{t}\dot{v}_0(t) \leq \frac{\delta_0}{4t^2}(v_0(t)-\kappa(t))E^{p-1} \leq \frac{-\delta_0 E^p}{2t^2} \tag{4.2}$$

for any $t : L < t < L_1$ where $\delta_0 = \epsilon_0\epsilon_2$. The existence of the locally minimum point contradicts the inequality of (4.2). Let $v_0(t) < v_0(L) < E$ for any $t > L$. Hence, the inequality (4.2) holds good for any $t \geq L$ and it follows that $\dot{v}_0(L) \leq 0$. Solving the inequality (4.2) we have

$$v_0(t) \leq C_1 - \frac{\delta_0 E^p}{2\theta+N-2}\log\frac{t}{L}$$

where C_1 is a positive constant independent of t. Thus, we know $\lim_{t\to\infty} v_0(t) = -\infty$. Hence, there exists L_1 such that $L_1 > L$ and $v_0(t) < 0$ for any $t \leq L_1$. Moreover,

$$\begin{aligned}\ddot{v}_0(t) + \frac{2\theta+(N-1)}{t}\dot{v}_0(t) &\leq \frac{\delta_0}{2t^2}(v_0(t)-\kappa(t))(\mid v_0(t) \mid^{p-1} + |\kappa(t)|^{p-1}) \\ &\leq \frac{\delta_0}{2t^2}v_0(t) \mid v_0(t) \mid^{p-1}\end{aligned}$$

for any $t > L_1$. If L_1 is sufficiently large, the above inequality yields $\dot{v}_0(t) < 0$ for any $t > L_1$.
From the analogous argument in Lemma 3.1 it follows that the life span of $v_0(t)$ is finite. This is a contradiction. Hence, we have $v_0(L) \geq E$. Assume $v(t') < E$ for some $t' > L$. Repeating the argument of the above, we get the contradiction. Thus the proof of the lemma is completed.

Proposition 4.4 *We assume that* $\ell > 2$ *and* $\lim_{t\to\infty}\frac{\varphi(t)}{t^\theta} = \infty$. *Then there exists a unique solution of* (1.2).

Proof. From Lemma 4.2 we know the existence of the solution of (1.2). For the proof of uniqueness of the solution by contradiction we put $v(t) = \dfrac{y_1(t)-y_2(t)}{t^\theta}$ where y_1 and y_2 are solutions of (1.2). From the comparison theorem we can assume $y_1(t) > y_2(t)$ for any $t > R$. Then, it follows that $v(t) > 0$ for any $t > R$. Noting (2.3) we have the following inequality:

$$\ddot{v}(t) + \frac{2\theta+(N-1)}{t}\dot{v}(t) + \frac{\theta^2+(N-2)\theta}{t^2}v(t) \geq \frac{\delta_0}{t^2}v(t)(|v_1(t)|^{p-1} + |v_2(t)|^{p-1})$$

where $v_i(t) = \dfrac{y_i(t)}{t^\theta}$, $\quad i = 1,2$ and $\delta_0 = \epsilon_0\epsilon_2$.
Let L be a sufficiently large number. Then, Lemma 4.3 and the above inequality implies

$$\ddot{v}(t) + \frac{2\theta + (N-1)}{t}\dot{v}(t) \geq \frac{\delta_0}{4t^2}v(t)(|v_1(t)|^{p-1} + |v_2(t)|^{p-1})$$

for any $t : t \geq L$. This inequality implies $\dot{v}(t) > 0$ for any $t : t \geq L$. Noting $|v_1(t)|^{p-1} + |v_2(t)|^{p-1} > Const\ v(t)^{p-1}$ and using the analogous argument to the proof of Lemma 3.1 we know that the life span of $v(t)$ is finite. This is a contradiction. The proofs of Theorem

2 and Corollary 2 are now accomplished by Proposition 4.4 and Proposition A-3.

5 Proofs of Theorems 3,4 and Corollaries 3,4 Throughout this section we assume that $\ell > 2$, $\quad \lim_{t\to\infty} \dfrac{\varphi(t)}{t^\theta} = \kappa$. We shall use the analogous argument to the proof of Theorem 2.

We remark S^{++}, S^{--} and S are as in Definition 1.

Lemma 5.1 *S^{++}, S^{--} are nonempty open sets and S is a nonempty closed set.*

Proof. By the same reason as in Lemmas 4.1 and 4.2 we can prove this lemma.

Let $y(t)$ be the solution of (1.2). We put $y(t) = v(t)t^\theta$ and $\varphi(t) = \kappa(t)t^\theta$. Then our assumptions (H2) and (H3) imply $g(t)^{-1}t^\ell = 1 + O(t^{-1})$ and $\kappa(t) = \kappa + O(t^{-1})$ as $t \to \infty$. Thus the equation (4.1) is replaced with the following equation:

$$\ddot{v}(t) + \frac{2\theta + (N-1)}{t}\dot{v}(t) = \frac{1}{t^2}\{v|v|^{p-1} - \kappa^p - (\theta^2 + (N-2)\theta)v(t) + \frac{f_2(t,v)}{t}\} \tag{5.1}$$

for any $t : t \geq R+1$ where $|f_2(t,v)| \leq \mathrm{Const}(|v(t)|^p + 1)$.

Lemma 5.2 *Let $y(R+1) \in S$, then we see $v(t) \in L^\infty(R+1,\infty)$.*

Proof. To prove this, we shall use contradiction. Let M be a sufficiently large number. We assume there exists a large number t_0 such that $v(t_0) \geq M$. Then we may assume $\dot{v}(t_0) \geq 0$. If there exists $t_1 > t_0$ such that $v(t_1) = v(t_0)$ and $v(t) > v(t_0)$ for any $t : t_0 \leq t \leq t_1$, we have a locally maximum point t_2 in (t_0, t_1). As the right hand term of (5.1) is positive for $t = t_2$, the above result contradicts the maximum principle. Then, $v(t) \geq M$ for any $t \geq t_0$. Thus, we see

$$\ddot{v}(t) + \frac{2\theta + (N-1)}{t}\dot{v}(t) \geq \frac{\delta_0}{t^2}|v(t)|^p$$

for any $t \geq t_0$. Using the analogous argument as in lemma 2.6 we obtain that the life span of $v(t)$ is finite. This is a contradiction. Then, $v(t)$ is upper bounded. Likewise, we obtain that $v(t)$ is lower bounded. The proof is complete.

For simplicity we denote $F(z) = \{z|z|^{p-1} - \kappa^p - (\theta^2 + (N-2)\theta)z\}$.

Lemma 5.3 *Assume that the life span of the solution y is infinite. Then we have*

$$\overline{\lim}_{t\to\infty} v(t) \leq \omega_+.$$

If ω_- exists we have

$$\underline{\lim}_{t\to\infty} v(t) \geq \omega_-.$$

Proof. We will show $\overline{\lim}_{t\to\infty} v(t) \le \omega_+$ by contradiction. We assume that $\overline{\lim}_{t\to\infty} v(t) \ge \omega_+ + \epsilon_0$. Then, the solution $v(t)$ is larger than $\omega_+ + \epsilon_0/2$ for large t, or there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$, $v(t_n) \ge \omega_+ + \epsilon/2$ and t_n are locally maximum points of $v(t)$. We first consider the former case. Since $v(t) > \omega_+ + \epsilon_0/2$ for large t, it follows that $F(v(t)) > \delta_0 > 0$. On the other hand, from Lemma 5.2, there exists a constant M such that $|f_2(t, v(t))| < M$ for large t. Then, the right hand side of (5.1) is larger than $(\delta_0 - M/t)/t^2 > \delta_0/(2t^2)$. Then, it follows that

$$\ddot{v}(t) + \frac{2\theta + (N-1)}{t}\dot{v}(t) \ge \delta_0/(2t^2).$$

Solving the inequality we get $v(t) > C_1 + C_2 \log t$ for large t. This contradicts Lemma 5.2. Second, let the sequence $\{t_n\}$ exist. If n is sufficiently large, it follows that $F(v(t_n)) - M/t_n > 0$. By the maximum principle we see that the existence of locally maximum points t_n contradicts that the right hand part of (5.1) is positive. Thus we have $\overline{\lim}_{t\to\infty} v(t) \le \omega_+$. Similarly, we also get $\underline{\lim}_{t\to\infty} v(t) \ge \omega_-$.

Proposition 5.4 *Let the solutions of the equation (1.3) satisfy with (C1). Then there exists a unique solution $y(t)$ of (1.2). Moreover, the solution satisfies $\lim\limits_{t\to\infty} \dfrac{y(t)}{t^\theta} = \omega_+$.*

Proof. From Lemma 5.1 it is trivial to prove the existence of the solution. Then, we will show $\lim\limits_{t\to\infty} \dfrac{y(t)}{t^\theta} = \omega_+$. Since Lemma 5.3 implies $\overline{\lim}_{t\to\infty} \dfrac{y(t)}{t^\theta} \le \omega_+$ it suffices to verify $\underline{\lim}_{t\to\infty} \dfrac{y(t)}{t^\theta} \ge \omega_+$. Replacing ω_- by ω_+ and using the similar argument as the proof of $\underline{\lim}_{t\to\infty} v(t) \ge \omega_-$ in Lemma 5.3, we have our assertion. We will next show the uniqueness. Assume there exist different two solutions $y_i(t)$ of (1.2) satisfying $\lim\limits_{t\to\infty} \dfrac{y_i(t)}{t^\theta} = \omega_+ (i = 1, 2)$. Let $v(t) = \dfrac{y_1(t) - y_2(t)}{t^\theta}$ and $v_i(t) = \dfrac{y_i(t)}{t^\theta}$. Then we have

$$\begin{aligned} \ddot{v}(t) \quad &+ \frac{2\theta + (N-1)}{t}\dot{v}(t) + \frac{\theta^2 + (N-2)\theta}{t^2} v(t) \\ &= g(t)^{-1} t^{\theta p - \theta} (v_1(t)(|v_1(t)|^{p-1} - v_2(t)|v_2(t)|^{p-1}) = I. \end{aligned}$$

On the other hand we see $v(t) > 0$ and $\dot{v}(t) > 0$ for any $t > R$ from the maximum principle. Moreover, since a sole value of tangent linear of (1.3) at $X = \omega_+$ is positive we see $p|\omega_+|^{p-1} - (\theta^2 + (N-2)^\theta) > 0$. Thus, using $g(t)^{-1} t^\ell = 1 + O(t^{-1})$ and $\lim_{t\to\infty} v_i(t) = \omega_+$ we see that there are positive small constant δ and large number M such that

$$I \ge v(t)(p|\omega_+|^{p-1} - (\theta^2 + (N-2)\theta) - \delta - \frac{M}{t}).$$

Therefore, $p|\omega_+|^{p-1} - (\theta^2 + (N-2)\theta) > 0$ implies

$$\ddot{v}(t) + \frac{2\theta + (N-1)}{t}\dot{v}(t) \ge (\frac{\delta_0}{t^2} - \frac{M}{t^3})v(t) \ge \frac{\delta_0}{2t^2} v(t)$$

for the sufficiently large number t where δ_0 is a small positive number. From Lemma 2.4 we have $\lim_{t\to\infty} v(t) = \infty$. This is a contradiction.

The proofs of Theorem 3 and Corollary 3 are now accomplished by Proposition 5.4 and Proposition A-3 in Appendix.

Lemma 5.5 *Let the solution of the equation (1.3) satisfy with (C2) or (C3). If $y(R+1) \in S$ there exist solutions of (1.2) respectively. Moreover, these solutions satisfy:*

$$\lim_{t\to\infty} \frac{y(t)}{t^\theta} = \omega_+ \quad or \quad \omega_0 = \omega_- \quad in\ (C2)\ case$$

and

$$\lim_{t\to\infty} \frac{y(t)}{t^\theta} = \omega_+ \quad or \quad \omega_0 \quad or \quad \omega_- \quad in\ (C3)\ case.$$

Proof. From Lemma 5.1 it suffices to show the behavior of the solutions as $t \to \infty$. Recall that $v(t) = \dfrac{y(t)}{t^\theta}$ where $y(t)$ and $v(t)$ are solutions of the equations (1.2) and (5.1) respectively. Using the integral representation of $\dot{v}(t)$ and noting Lemma 5.3 we have

$$| t^\beta \dot{v}(t)(t) - t_0^\beta \dot{v}(t_0) | \leq Const \int_{t_0}^{t} s^{\beta-2} ds$$

where $\beta = (2\theta + (N-1))$. Then from $\beta > 1$ we know $|t\dot{v}(t)| <$ Const for sufficiently large number t.
Putting $t = e^s$ and $v^*(s) = v(t)$, the equation (5.1) is rewritten in the following equation:

$$\begin{aligned} &\ddot{v}^*(s) + (\beta - 1)\dot{v}^*(s) \\ &= \{v^*|v^*|^{p-1}(s) - \kappa^p - (\theta^2 + (N-2)\theta)v^*(s)\} + \frac{f_2(s, v^*)}{e^s}. \end{aligned} \tag{5.2}$$

We next show $\lim\limits_{s\to\infty} \dot{v}^*(s) = 0$. Let $s_1 < s_2$ be large numbers. We first remark that

$$\int_{s_1}^{s_2} F(v^*(s))\dot{v}^*(s) ds = \int_{v^*(s_1)}^{v^*(s_2)} F(w) dw.$$

Moreover, $|f_2(t, v^*(s))\dot{v}^*(s)| < M$ from Lemma 5.3 and $|\dot{v}^*(s)| = |t\dot{v}(t)| <$ Const.
Multiplying (5.2) by $\dot{v}^*(s)$ and integrating this equation over (s_1, s_2) we have

$$\begin{aligned} | (\dot{v}^*(s_2)^2 \quad & -\dot{v}^*(s_1)^2)/2 + (\beta-1)\textstyle\int_{s_1}^{s_2} \dot{v}^*(\xi)^2 d\xi \\ & - \int_{v^*(s_1)}^{v^*(s_2)} (w|w|^{p-1} - \kappa^p - (\theta^2 + (N-2)\theta) w dw \ | \\ & \leq M \int_{s_1}^{s_2} e^{-s} ds. \end{aligned} \tag{5.3}$$

Since $\displaystyle\int_{v^*(s_1)}^{v^*(s_2)} F(w) dw$ is bounded for any s_2, s_1, from Lemma 5.3, we get

$$| \int_{s_1}^{s} \dot{v}^*(\xi)^2 d\xi \ | \leq \text{Const.}$$

for any $s_2 > s_1$. Thus it follows that

$$\lim_{s_2, s_1 \to \infty} \int_{s_1}^{s_2} \dot{v}^*(\xi)^2 d\xi = 0. \tag{5.4}$$

By contradiction we will show $\lim\limits_{s\to\infty} \dot{v}^*(s) = 0$.
We assume that $\underline{\lim}_{s\to\infty} \dot{v}^*(s) < \overline{\lim}_{s\to\infty} \dot{v}^*(s)$. Then, there exist sequences $\{s_{1,n}\}$ and $\{s_{2,n}\}$ such that $s_0 < s_{1,n} < s_{2,n} < s_{1,n+1}$ and

$$\delta_0 \leq \dot{v}^*(s_{2,n}) - \dot{v}^*(s_{1,n}) \quad \text{and} \quad |\dot{v}^*(s)| \geq \delta_1 \quad \text{for any} \quad s \in (s_{1,n}, s_{2,n})$$

where s_0 is the sufficiently large number and δ_0, δ_1 are sufficiently small positive numbers independent of n. Since $|f_2(s, v^*)| \leq$ Const, the equation (5.2) implies $|\ddot{v}^*(s)| \leq$ Const for any $s > s_0$. Then, it follows that

$$\delta_0 < |\dot{v}^*(s_{2,n}) - \dot{v}^*(s_{1,n})| = |\ddot{v}^*(\xi)(s_{2,n} - s_{1,n})| \leq C_0(s_{2,n} - s_{1,n}).$$

Then, since

$$\text{Const} \geq \int_{s_1}^{\infty} \dot{v}^*(s)^2 ds > \sum_{n=1}^{\infty} \int_{s_{1,n}}^{s_{2,n}} \dot{v}^*(s)^2 ds \geq \sum_{n=1}^{\infty} \delta_1^2 \frac{\delta_0}{C_0} = \infty,$$

we have a contradiction. Thus, we have $\lim_{s\to\infty} \dot{v}^*(s) = C$. If $C \neq 0$ we have a contradiction form (5.4). Therefore, we obtain

$$\lim_{s\to\infty} \dot{v}^*(s) = 0. \tag{5.5}$$

We next show the claim in this lemma. Let ϵ be any positive number. From (5.3),(5.4) and (5.5) there exists a large number s_0 such that

$$\left| \int_{v^*(s_1)}^{v^*(s_2)} F(w)dw \right| \leq \epsilon \quad \textit{for any } s_2, s_1 > s_0. \tag{5.6}$$

We assume $a = \overline{\lim}_{s\to\infty} v^*(s) > \underline{\lim}_{s\to\infty} v^*(s) = b$.
From (5.6) we have $\left| \int_b^a F(w)dw \right| < 2\epsilon$. Then it follows that $\left| \int_b^a F(w)dw \right| = 0$. On the other hand Lemma 5.3 yields $\omega_- \leq b < a \leq \omega_+$.
We first consider the case (C2). Since $F(w) < 0$ for any $w : b < w < a$ it follows that $\int_b^a F(w)dw < 0$. Then it is a contradiction. Thus, we have $a = b$.
We second study the case (C3). Since $\left| \int_b^a F(w)dw \right| = 0$ we find that $\omega_- \leq b < \omega_0 < a \leq \omega_+$. Let δ be a positive sufficiently small number. Then, there exist sufficiently large numbers s_1, s_2, s_3 such that $s_0 < s_1 < s_3 < s_2$ and $v^*(s_1) = b + \delta < v^*(s_3) = \omega_0 < a - \delta = v^*(s_2)$. Then, we get $\left| \int_{v^*(s_3)}^{v^*(s_2)} F(w)dw \right| \geq Const$ or $\left| \int_{v^*(s_1)}^{v^*(s_3)} F(w)dw \right| \geq Const$. This contradicts the inequality (5.6). Then it follows that $a = b$.
Thus there exists $\lim_{s\to\infty} v^*(s) = a$. If $F(a) \neq 0$ we know that $\lim_{s\to\infty} \ddot{v}^*(s) = F(a) \neq 0$ from (5.2) and (5.5). Then, $\lim_{s\to\infty} v^*(s) = \infty$ or $-\infty$. This is a contradiction. Therefore, $F(a) = 0$.

Definition 2 *We define S^+, S^- and S^0 by*

$$\begin{cases} S^+ = \{\alpha \in S : \lim_{t\to\infty} \dfrac{y_\alpha(t)}{t^\theta} = \omega_+\} \\ S^0 = \{\alpha \in S : \lim_{t\to\infty} \dfrac{y_\alpha(t)}{t^\theta} = \omega_0\} \\ S^- = \{\alpha \in S : \lim_{t\to\infty} \dfrac{y_\alpha(t)}{t^\theta} = \omega_-\}. \end{cases}$$

Proposition 5.6 *Assume $\omega_- < \omega_0$. Then we have*
(1) $S^+(resp.)S^-)$ is a nonempty one point set $= \{\alpha_+\}(resp.\{\alpha_-\}\ )$
(2) S^0 is a nonempty open set $= (\alpha_-, \alpha_+)$.

Proof. We first show that S^+ is nonempty. Let $\alpha_1 > \alpha_2 \in S^{++}$. By the maximum principle it follows that $y_{\alpha_1}(t) > y_{\alpha_2}(t)$ for any $t \in (R, \min\{T_{\alpha_1}, T_{\alpha_2}\})$. In the proof of Lemma 3.1 $\sup\limits_{\alpha \in S^{++}} T_\alpha = \infty$ was shown. On the other hand, noting $\underline{\lim}_{t\to\infty}\varphi(t) \geq 0$ and using the minimal principle we get that there exists a constant L independent of $\alpha \in S^{++}$ and $t > R$ such that $y_\alpha(t) > -L$. Hence, it follows that

$$\underline{\lim}_{\alpha\to\alpha_0} y_\alpha(t) = y_{\alpha_0}(t) \quad \text{locally uniformly on } [R,\infty)$$

where $\alpha_0 = \inf_{\alpha\in S^{++}} \alpha$. Moreover, $y_{\alpha_0}(t)$ is a solution of (1.2) and $y_{\alpha_0}(t) \geq -L$. Thus it follows that $\alpha_0 \in S^+$ form Lemma 5.5.

Second we will show that S^+ is one point set by contradiction. Let be $\alpha_1 > \alpha_2 \in S^+$. For the simplicity we denote $y_{\alpha_i} = y_i (i = 1, 2)$. From the comparison theorem we have $y_1(t) - y_2(t) = \delta(t) > 0$, $\dot{y}_1(t) - \dot{y}_2(t) > 0$ for any $t \in (R,\infty)$ where $\delta(t)$ is the increasing continuous function on (R,∞). The mean value theorem implies

$$y_1(t)|y_1(t)|^{p-1} - y_2(t)|y_2(t)|^{p-1} = p|\zeta(t)|^{p-1}t^\theta(y_1(t) - y_2(t))$$

where $y_1(t) < \zeta(t)t^\theta < y_2(t)$. Since $|\zeta(t)| > \omega_+ - \epsilon_0$ for sufficiently small ϵ_0 and large t, it follows that there exists a small positive constant ϵ such that

$$g(t)^{-1}(y_1(t)|y_1(t)|^{p-1} - y_2(t)|y_2(t)|^{p-1}) > \frac{1}{t^2}(1-\epsilon)p\omega_+^{p-1}(y_1(t) - y_2(t)).$$

Hence, from the integral equation associated with (1.2), we get

$$y_1(t) - y_2(t) \geq \delta(t_1) + (1-\epsilon)p\omega_+^{p-1}\int_{t_1}^t s^{-(N-1)}\int_{t_1}^s r^{N-3}(y_1 - y_2)(r)drds$$

where t_1 is sufficiently large number. Then, Lemma 2.4 yields

$$y_1(t) - y_2(t) \geq c_0(\frac{t}{t_1})^{\lambda_1} \tag{5.7}$$

where λ_1 is a positive solution of the equation $\lambda^2 + (N-2)\lambda - p\omega_+^{p-1}(1-\epsilon) = 0$. Since a slope value of a tangent line of (1.3) at $X = \omega_+$ is positive, we get $p\omega_+^{p-1}(1-\epsilon) - (\theta^2 + (N-2)\theta) > 0$. Then, $\lambda_1 > \theta$. On the other hand, since $y_1(R+1)$ and $y_2(R+1)$ belong to S^+, it follows that $y_1(t) - y_2(t) = O(t^\theta)$. This contradicts the inequality (5.7). Thus S^+ is one point set.

In a similar way, we also obtain that S^- is at most one point.

Next, we will show that S^- is not empty by contradiction.

We assume S^- is empty. If S^0 is also empty, there exist $\alpha_+ \in S^+$ and a sequence $\{\alpha_n\} \subset S^{--}$ such that

$$\lim_{n\to\infty} y_{\alpha_n}(t) = y_{\alpha_+}(t)$$
$$\lim_{n\to\infty} \dot{y}_{\alpha_n}(t) = \dot{y}_{\alpha_+}(t) \quad \text{locally uniformly on } [R,\infty).$$

Then, for a sufficiently large n, we find that $y_{\alpha_n}(t_0) > 0$ and $\dot{y}_{\alpha_n}(t_0) > 0$ where t_0 is a sufficiently large number. On the other hand we may assume that $y^*(t) = \frac{1}{2}(w_0 + w_{-1})t^\theta$ is an super solution of (1.2) on $[t_0,\infty)$. Moreover, we see that $y^*(t_0) < 0$ and $\dot{y}^*(t_0) < 0$. Then, $y_{\alpha_n}(t) - y^*(t)$ holds the maximum point in (t_0, T_{α_n}) where T_{α_n} is the life span of y_{α_n} This contradicts the maximum principle. Hence, S^0 is not empty.

We next assume that S^0 is closed. Then, $\alpha_0 = \inf S^0$ belongs to S^0.

Thus, $\lim_{t\to\infty}\dfrac{y_{\alpha_0}(t)}{t^\theta} = \omega_0$. From Cauchy's mean value theorem there exists a sequence

$\{t_m\}$ such that $\lim_{m\to\infty} \dfrac{\dot{y}_{\alpha_0}(t_m)}{t_m^{\theta-1}} = \theta\omega_0$. On the other hand, we have there exists a sequence $\{\alpha_n\} \subset S^{--}$ such that $\{y_{\alpha_n}\}$ and $\{\dot{y}_{\alpha_n}\}$ are locally uniformly convergent to y_{α_0} and $\dot{y}_{\alpha_0}$ on $[R,\infty)$ respectively. Then, we obtain that there exist sufficiently large numbers n and m such that $y_{\alpha_n}(t_m) > \dfrac{1}{2}(\omega_- + \omega_0)t_m^\theta$ and $\dot{y}_{\alpha_n}(t_m) > \dfrac{\theta}{2}(\omega_- + \omega_0)t_m^{\theta-1}$. Thus, it follows that $y_{\alpha_n}(t) - y_0^*(t)$ holds the maximum point in (t_0, T_{α_n}). We remark that $y_0^*(t)$ is the super solution of (1.2) on $[t_0,\infty)$. This contradicts the maximum principle theorem. Thus, S^0 is open. Therefore, S is not the closed set. This results contradicts Lemma 5.1 Thus S^- is not empty.

Proposition 5.7 *Let* $\omega_- = \omega_0$.
(1) $S^+ = \{\alpha_+\}$ *is a nonempty one point set,*
(2) $S = [\alpha_-.\alpha_+)$ *is a nonempty set .*

Proof. From the similar argument in Proposition 5.6, (1) is trivial.
We first consider in the case of $\dfrac{1}{2} < \theta$ or $N > 2$. Let K and t_1 be sufficiently large number. We denote by $y_+(t) = \omega_0 t^\theta + Kt^{\theta-1}$. We will show that $y_+(t)$ is an super solution of (1.2). Since a slope value of a tangent line of (1.3) at $X = \omega_0$ is zero, we find

$$\omega_0|\omega_0|^{p-1} - \kappa^p - (\theta^2 + (N-2)\theta)\omega_0) = 0 \quad \text{and} \quad p|\omega_0|^{p-1} - (\theta^2 + (N-2)\theta) = 0.$$

Then, it follows that $p|\omega_0|^{p-1} - ((\theta-1)^2 + (N-2)(\theta-1)) > 0$. Hence, noting $(1 - t^{-\ell}g(t)) = O(1/t)$ and $((f(t)/t^{\theta p}) - \kappa^p) = O(1/t)$ and using the Taylor expansion theorem we see

$$\begin{aligned}
&-g(t)(\ddot{y}_+(t) + \tfrac{(N-1)}{t}\dot{y}_+(t)) + y_+(t)|y_+(t)|^{p-1} - f(t)\\
&> t^{\theta-3+\ell}\{K(p|\omega_0|^{p-1} - ((\theta-1)^2 + (N-2)(\theta-1)))\\
&\qquad -M_1 - M_2\frac{K}{t} - M_3\frac{K^2}{t}\}
\end{aligned}$$

where $M_i (i = 1,2,3)$ are constants independent of K and t. If K is sufficiently large and $\dfrac{K^2}{t}$ is sufficiently small, the right hand side of the above inequality is positive. Then $y_+(t)$ is the super solution of (1.2).
We second consider in the case of $0 < \theta \le \dfrac{1}{2}$ and $N = 2$. We denote by $y_+(t) = \omega_0 t^\theta + \log t$. We use the similar method as the case of the above. Then, we have

$$\begin{aligned}
&-g(t)(\ddot{y}_+(t) + \tfrac{1}{t}\dot{y}_+(t)) + y_+(t)|y_+(t)|^{p-1} - f(t)\\
&> p|\omega_0|^{p-1}t^{\theta p-\theta}\log t - M_1 t^{\theta-3+\ell} - M_2 t^{\theta p-1} - M_3 t^{\theta p-2\theta}(\log t)^2\\
&> \frac{1}{2}p|\omega_0|^{p-1}t^{\theta p-\theta}\log t
\end{aligned}$$

for the large t where $M_i (i = 1,2,3)$ are positive constants independent of t. Then $y_+(t)$ is the super solution of (1.2).
Therefore, in both cases there exists the super solution of (1.2) such that $\lim_{t\to\infty} \dfrac{y_+(t)}{t^\theta} = \omega_+$. Using the similar argument of the proof of Proposition 5.6, we complete the proof.

The proofs of Theorem 4 and Corollary 4 are now accomplished by Proposition 5.6 and Proposition 5.7.

Appendix. We shall prove that the uniqueness of a classical solution implies the uniqueness of a continuous viscosity solution. As this argument is in our paper [8] we will give an outline of the proof in this paper.

Lemma A-1 *There exists a solution y of (1.2) satisfying $y(T_1) = \alpha$ and $y(T_2) = \beta$ where any $T_1, T_2 : R < T_1 < T_2$ and any $\alpha, \beta \in R^1$. If $T_1 = R$, we have the solution of (1.2) such that $y(T_1) = \varphi(R)$ and $y(T_2) = \beta$.*
Proof. See Proposition 3.1 and Lemma 3.2 in [8].

Let $u(x) \in C(R^N - B_R)$ be an arbitrary viscosity solution of (0.1). Define for $x \in R^N - B_R$

$$\overline{U}(x) = \sup\{u(Qx); Q \in O(N)\} \quad \text{and} \quad \underline{U}(x) = \inf\{u(Qx); Q \in O(N)\},$$

where $O(N)$ denotes the set of orthogonal $N \times N$ matrices. Since $O(N)$ is compact and closed (in the matrix norm), we see $\overline{U}(x) = \max\{u(Qx); Q \in O(N)\}$ and $\underline{U}(x) = \min\{u(Qx); Q \in O(N)\}$.

Lemma A-2 *$\overline{U}(x)$ and $\underline{U}(x)$ are continuous on $R^N - \overline{B_R}$. Moreover,*
(i) *$\overline{U}$ is a radial viscosity subsolution of (0.1).*
(ii) *$\underline{U}$ is a radial viscosity supersolution of (0.1).*
Proof. See Section 4 in [8].

As $\overline{U}(x)$ and $\underline{U}(x)$ are radial functions we denote by $\overline{U}(|x|) = \overline{U}(x)$ and $\underline{U}(|x|) = \underline{U}(x)$.

Proposition A-3 *Suppose that the solution of (1.2) is unique. Then, the continuous viscosity solution (0.1) is also unique.*
Proof. Connecting the continuous viscosity solution of (0.1) and thecontinuous viscosity solution in [8] we can get all continuous viscosity solutions of (0.1). (See [8]) Then, we have only to study in case of the domain $D = \{x \in R^N; |x| \geq R\}$ and the boundary condition $u(x) = \varphi(R)$ for any $|x| = R$. Then we shall consider the following equation:

$$(\mathrm{A}-1) \qquad \begin{cases} -g(|\, x\, |)\Delta u(x) \quad +u(x) \mid u(x) \mid^{p-1} = f(|x|) & \text{on any} \quad |x| > R \\ \qquad u(x) \qquad\qquad\qquad = \varphi(R) & \text{on any} \quad |x| = R. \end{cases}$$

Let $\overline{U}(x)$ and $\underline{U}(x)$ be functions associated with the solution of (A-1). We shall prove $\overline{U}(x) = \underline{U}(x)$ by contradiction. Assume that $\overline{U}(t_1) > \underline{U}(t_1)$ where $t_1 > R$. From Lemma A-1 there exist solutions $y_1(t)$ and y_2 of (1.2) such that $y_1(R) = \varphi(R), y_1(t_1) = \dfrac{1}{3}(\overline{U}(t_1) + 2\underline{U}(t_1))$ and $y_2(R) = \varphi(R), y_2(t_1) = \dfrac{1}{3}(2\overline{U}(t_1) + \underline{U}(t_1))$ respectively. From the maximum principle and the minimum principle it follows that $\overline{U}(t) > y_2(t) > y_1(t) > \underline{U}(t)$ for any $t : t > t_1$. Then the life spans of $y_1(t)$ and $y_2(t)$ are infinity. The existence of solutions $y_2(t) > y_1(t)$ on $[R, \infty)$ contradicts the hypothesis in this proposition.

Proposition A-4 *A continuous radial viscosity solution $u(t)$ of (0.1) is a classical solution of (0.2).*
Proof. Let $R < T_1 < T_2$. We denote the radial solution of (1.2) connected $(T_1, u(T_1))$ and $(T_2, u(T_2))$ by $y_{1,2}$. Since $y_{1,2}$ is a viscosity solution $\in C^2$ of (0.1) on $x : T_1 < |x| < T_2$, by the maximum principle, it follows that $u(x) = y_{1,2}(|x|)$. Then, the proof is complete.

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