ON NIL RADICALS IN BCI-ALGEBRAS

YOUNG BAE JUN

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Abstract. We verify that the k-nil radical of an obstinate (resp. associative, strong, weakly implicative, implicative, sub-implicative, sub-commutative) ideal is an obstinate (resp. associative, strong, weakly implicative, implicative, sub-implicative, sub-commutative) ideal. We prove that every k-nil radical of a q-ideal and an a-ideal is also a q-ideal and an a-ideal.

1. Introduction

In [4] the notion of nil radical in BCI-algebras was introduced, and various properties were developed in [3, 5, 6, 7, 8, 9]. In this paper, we prove that every k-nil radical of an associative (resp. strong, obstinate) ideal is an associative (resp. strong, obstinate) ideal. Using characterizations of a weakly implicative ideal (resp. an implicative ideal, a q-ideal, an a-ideal, a sub-implicative ideal, a sub-commutative ideal), we show that the k-nil radical of a weakly implicative ideal (resp. an implicative ideal, a q-ideal, an a-ideal, a sub-implicative ideal, a sub-commutative ideal) is a weakly implicative ideal (resp. an implicative ideal, a q-ideal, an a-ideal, a sub-implicative ideal, a sub-commutative ideal).

2. Preliminaries

Recall that a *BCI-algebra* is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms: for every $x, y, z \in X$,

- $((x * y) * (x * z)) * (z * y) = 0$,
- $(x * (x * y)) * y = 0$,
- $x * x = 0$,
- $x * y = 0$ and $y * x = 0$ imply $x = y$.

For any BCI-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X. For any elements x and y of a BCI-algebra X and $k \in \mathbb{N}$, let us write $x * y^k$ instead of $\big(\cdots((x * y) * y)\cdots\big) * y$ in which y occurs k-times. In a BCI-algebra X, the following holds for all $x, y, z \in X$ and $k \in \mathbb{N}$,

(p1) $x * 0 = x$,
(p2) $(x * y) * z = (x * z) * y$,
(p3) $0 * (x * y)^k = (0 * x^k) * (0 * y^k)$,
(p4) $0 * (0 * x)^k = 0 * (0 * x^k)$.

A nonempty subset S of a BCI-algebra X is said to be a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A nonempty subset A of a BCI-algebra X is called an *ideal* of X if it satisfies

- $0 \in A$,

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- $x * y \in A$ and $y \in A$ imply $x \in A$ for all $x, y \in X$.

An ideal A of a BCI-algebra X is said to be *closed* if $0 * x \in A$ for all $x \in A$. Note that an ideal of a BCI-algebra may not be a subalgebra in general, but every closed ideal of a BCI-algebra is a subalgebra. An ideal A of a BCI-algebra X is said to be *strong* if $x * y \in X \setminus A$ for all $x \in A$ and $y \in X \setminus A$. A proper ideal A of a BCI-algebra X is said to be *obstinate* if $x * y \in A$ for all $x, y \in X \setminus A$. A nonempty subset A of a BCI-algebra X is called an *associative ideal* of X if it satisfies

- $0 \in A$,
- $(x * y) * z \in A$ and $y * z \in A$ imply $x \in A$ for all $x, y, z \in X$.

3. Main Results

Throughout this section X is a BCI-algebra and k is a positive integer.

Definition 3.1. ([3, Definition 1]) For any nonempty subset I of X, the set

$$\sqrt[k]{I} := \{x \in X \mid 0 * x^k \in I\}$$

is called the *k-nil radical* of I.

Lemma 3.2. ([3, Theorem 2]) *If I is a (closed) ideal of X, then so is $\sqrt[k]{I}$.*

Theorem 3.3. *If I is an associative ideal of X, then so is $\sqrt[k]{I}$.*

Proof. Let $x, y, z \in X$ be such that $(x * y) * z \in \sqrt[k]{I}$ and $y * z \in \sqrt[k]{I}$. Then

$$\big((0 * x^k) * (0 * y^k)\big) * (0 * z^k) = 0 * \big((x * y) * z\big)^k \in I$$

and $(0 * y^k) * (0 * z^k) = 0 * (y * z)^k \in I$. Since I is an associative ideal, it follows that $0 * x^k \in I$, that is, $x \in \sqrt[k]{I}$. Hence $\sqrt[k]{I}$ is an associative ideal of X. $\square$

Theorem 3.4. *If I is a strong ideal of X, then so is $\sqrt[k]{I}$.*

Proof. Let $a \in \sqrt[k]{I}$ and $x \in X \setminus \sqrt[k]{I}$. Then $0 * a^k \in I$ and $0 * x^k \notin I$. Since I is a strong ideal, it follows from (p3) that

$$0 * (a * x)^k = (0 * a^k) * (0 * x^k) \in X \setminus I,$$

that is, $a * x \in X \setminus \sqrt[k]{I}$. Hence $\sqrt[k]{I}$ is a strong ideal of X. $\square$

Theorem 3.5. *Every k-nil radical of an obstinate ideal is also an obstinate ideal.*

Proof. Let I be an obstinate ideal of X. Then I is an ideal of X, and so $\sqrt[k]{I}$ is an ideal of X (see Lemma 3.2). Let $x, y \in X \setminus \sqrt[k]{I}$. Then $0 * x^k \in X \setminus I$ and $0 * y^k \in X \setminus I$. Since I is an obstinate ideal, it follows from (p3) that

$$0 * (x * y)^k = (0 * x^k) * (0 * y^k) \in I$$

so that $x * y \in \sqrt[k]{I}$. This completes the proof. $\square$

By means of [10, Corollary 3], we have the following corollary.

Corollary 3.6. *Every k-nil radical of an obstinate ideal is a closed $(*)$-ideal.*

Definition 3.7. ([14]) A nonempty subset I of X is called a *weakly implicative ideal* of X if it satisfies

- $0 \in I$,
- $z \in I$ and $\big((x * (y * x)) * (0 * (y * x))\big) * z \in I$ imply $x \in I$, for all $x, y, z \in X$.

Note that every weakly implicative ideal is an ideal (see [14, Theorem 1]).

Lemma 3.8. ([14, Theorem 4]) *An ideal I of X is weakly implicative if and only if it satisfies: for all $x, y \in X$,*

- $(x * (y * x)) * (0 * (y * x)) \in I$ *implies* $x \in I$.

Theorem 3.9. *If I is a weakly implicative ideal of X, then so is $\sqrt[k]{I}$.*

Proof. If I is a weakly implicative ideal of X, then I is an ideal of X. Hence $\sqrt[k]{I}$ is an ideal of X (see Lemma 3.2). Let $x, y \in X$ be such that

$$\big(x * (y * x)\big) * \big(0 * (y * x)\big) \in \sqrt[k]{I}.$$

Then

$$\begin{aligned} & \big((0 * x^k) * ((0 * y^k) * (0 * x^k))\big) * \big(0 * ((0 * y^k) * (0 * x^k))\big) \\ = \;& \big(0 * (x * (y * x))^k\big) * \big(0 * (0 * (y * x))^k\big) \\ = \;& 0 * \big((x * (y * x)) * (0 * (y * x))\big)^k \in I. \end{aligned}$$

It follows from Lemma 3.8 that $0 * x^k \in I$ so that $x \in \sqrt[k]{I}$. Hence, by Lemma 3.8, $\sqrt[k]{I}$ is a weakly implicative ideal of X. $\square$

Definition 3.10. ([13, Definition 3.1]) A nonempty subset I of X is called an *implicative ideal* of X if it satisfies

- $0 \in I$,
- $\big(((x * y) * y) * (0 * y)\big) * z \in I$ and $z \in I$ imply
$$x * \big((y * (y * x)) * (0 * (0 * (x * y)))\big) \in I,$$
for all $x, y, z \in X$.

Note that every implicative ideal is an ideal, but not converse (see [13, Theorem 3.7]).

Lemma 3.11. ([13, Theorem 3.4]) *Let I be an ideal of X. Then I is implicative if and only if it satisfies: for all $x, y, z \in X$,*

- $\big((x * y) * y\big) * (0 * y) \in I$ *implies* $x * \big((y * (y * x)) * (0 * (0 * (x * y)))\big) \in I$.

Theorem 3.12. *Every k-nil radical of an implicative ideal is also an implicative ideal.*

Proof. Let I be an implicative ideal of X. Then I is an ideal, and so $\sqrt[k]{I}$ is an ideal of X. For the convenience of notations, we use 0_x instead of $0 * x^k$. Then $0_x * 0_y = 0_{x*y}$ by (p3). Let $x, y \in X$ be such that $\big((x * y) * y\big) * (0 * y) \in \sqrt[k]{I}$. Then

$$\big((0_x * 0_y) * 0_y\big) * (0 * 0_y) = 0_{((x*y)*y)*(0*y)} = 0 * \big(((x * y) * y) * (0 * y)\big)^k \in I,$$

which implies

$$\begin{aligned} & 0 * \big(x * ((y * (y * x)) * (0 * (0 * (x * y))))\big)^k \\ = \;& 0_{x*((y*(y*x))*(0*(0*(x*y))))} \\ = \;& 0_x * \big(0_{y*(y*x)} * (0 * (0 * 0_{x*y}))\big) \\ = \;& 0_x * \big((0_y * (0_y * 0_x)) * (0 * (0 * (0_x * 0_y)))\big) \in I \end{aligned}$$

by Lemma 3.11. Hence

$$x * \big((y * (y * x)) * (0 * (0 * (x * y)))\big) \in \sqrt[k]{I},$$

and thus $\sqrt[k]{I}$ is an implicative ideal of X by Lemma 3.11. $\square$

Definition 3.13. ([11, Definition 3.1]) A nonempty subset I of X is called a *q-ideal* of X if it satisfies:

- $0 \in I$,
- $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$, for all $x, y, z \in X$.

Lemma 3.14. ([11, Theorem 3.3]) *Every q-ideal is an ideal.*

Lemma 3.15. ([11, Theorem 3.5]) *Let I be an ideal of X. Then the following are equivalent.*

(i) *I is a q-ideal of X.*
(ii) *$x * (0 * y) \in I$ implies $x * y \in I$, for all $x, y \in X$.*
(iii) *$x * (y * z) \in I$ implies $(x * y) * z \in I$, for all $x, y, z \in X$.*

Theorem 3.16. *Every k-nil radical of a q-ideal is also a q-ideal.*

Proof. Let I be a q-ideal of X. Then I is an ideal of X, and so $\sqrt[k]{I}$ is an ideal of X. Let $x, y \in X$ be such that $x * (0 * y) \in \sqrt[k]{I}$. Using (p3) and (p4), we have

$$(0 * x^k) * \big(0 * (0 * y^k)\big) = (0 * x^k) * \big(0 * (0 * y)^k\big) = 0 * \big(x * (0 * y)\big)^k \in I.$$

Since I is a q-ideal, it follows from (p3) and Lemma 3.15 that

$$0 * (x * y)^k = (0 * x^k) * (0 * y^k) \in I$$

so that $x * y \in \sqrt[k]{I}$. Therefore $\sqrt[k]{I}$ is a q-ideal of X by Lemma 3.15. □

Definition 3.17. ([11, Definition 4.1]) A nonempty subset I of X is called an *a-ideal* of X if it satisfies:

- $0 \in I$,
- $(x * z) * (0 * y) \in I$ and $z \in I$ imply $y * x \in I$, for all $x, y, z \in X$.

Lemma 3.18. ([11, Theorem 4.3]) *Any a-ideal is an ideal.*

Lemma 3.19. ([11, Theorem 4.4]) *Let I be an ideal of X. Then the following are equivalent.*

(i) *I is an a-ideal of X.*
(ii) *$(x * z) * (0 * y) \in I \Rightarrow y * (x * z) \in I$.*
(iii) *$x * (0 * y) \in I \Rightarrow y * x \in I$.*

Theorem 3.20. *Every k-nil radical of an a-ideal is also an a-ideal.*

Proof. Let I be an a-ideal of X. Then I is an ideal of X, and thus $\sqrt[k]{I}$ is an ideal of X. Let $x, y \in X$ be such that $x * (0 * y) \in \sqrt[k]{I}$. Then

$$(0 * x^k) * \big(0 * (0 * y^k)\big) = (0 * x^k) * \big(0 * (0 * y)^k\big) = 0 * \big(x * (0 * y)\big)^k \in I,$$

which implies from (p3) and Lemma 3.19 that $0 * (y * x)^k = (0 * y^k) * (0 * x^k) \in I$. Hence $y * x \in \sqrt[k]{I}$, and so $\sqrt[k]{I}$ is an a-ideal of X by Lemma 3.19. □

Definition 3.21. ([12, Definition 3.1]) A nonempty subset I of X is called a *sub-implicative ideal* of X if it satisfies:

- $0 \in I$,
- $\big((x * (x * y)) * (y * x)\big) * z \in I$ and $z \in I$ imply $y * (y * x) \in I$, for all $x, y, z \in X$.

Lemma 3.22. ([12, Theorem 3.5]) *Every sub-implicative ideal is an ideal.*

Lemma 3.23. ([12, Theorem 3.4]) *Let I be an ideal of X. Then I is a sub-implicative ideal of X if and only if*

$$\big(x * (x * y)\big) * (y * x) \in I \Rightarrow y * (y * x) \in I$$

for all $x, y \in X$.

Theorem 3.24. *If I is a sub-implicative ideal of X, then so is $\sqrt[k]{I}$.*

Proof. Let I be a sub-implicative ideal of X. Then I is an ideal of X, and so $\sqrt[k]{I}$ is an ideal of X. Let $x, y \in X$ be such that $\big(x*(x*y)\big)*(y*x) \in \sqrt[k]{I}$. Then

$$\big((0*x^k)*((0*x^k)*(0*y^k))\big)*\big((0*y^k)*(0*x^k)\big) = 0*\big((x*(x*y))*(y*x)\big)^k \in I.$$

Since I is sub-implicative, it follows from (p3) and Lemma 3.23 that

$$0*\big(y*(y*x)\big)^k = (0*y^k)*\big((0*y^k)*(0*x^k)\big) \in I$$

so that $y*(y*x) \in \sqrt[k]{I}$. Hence, by Lemma 3.23, $\sqrt[k]{I}$ is a sub-implicative ideal of X. □

Definition 3.25. ([12, Definition 3.9]) A nonempty subset I of X is called a *sub-commutative ideal* of X if it satisfies:

- $0 \in I$,
- $\big(y*(y*(x*(x*y)))\big)*z \in I$ and $z \in I$ imply $x*(x*y) \in I$, for all $x, y, z \in X$.

Lemma 3.26. ([12, Theorem 3.13]) *Every sub-commutative ideal is an ideal.*

Lemma 3.27. ([12, Theorem 3.12]) *Let I be an ideal of X. Then I is a sub-commutative ideal of X if and only if*

$$y*\big(y*(x*(x*y))\big) \in I \;\Rightarrow\; x*(x*y) \in I$$

for all $x, y \in X$.

Theorem 3.28. *If I is a sub-commutative ideal of X, then so is $\sqrt[k]{I}$.*

Proof. Let I be a sub-commutative ideal of X. Then I is an ideal of X, and so $\sqrt[k]{I}$ is an ideal of X. Let $x, y \in X$ be such that $y*\big(y*(x*(x*y))\big) \in \sqrt[k]{I}$. Then

$$0_y*\big(0_y*(0_x*(0_x*0_y))\big) = 0_{y*(y*(x*(x*y)))} = 0*\big(y*(y*(x*(x*y)))\big)^k \in I.$$

Since I is sub-commutative, it follows from Lemma 3.27 that

$$0*\big(x*(x*y)\big)^k = 0_{x*(x*y)} = 0_x*0_{x*y} = 0_x*(0_x*0_y) \in I$$

so that $x*(x*y) \in \sqrt[k]{I}$. Therefore $\sqrt[k]{I}$ is a sub-commutative ideal of X. □

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Department of Mathematics Education, Gyeongsang National University, Chinju (Jinju) 660-701, Korea

E-mail address: `ybjun@nongae.gsnu.ac.kr`