KS-FILTERS IN KS-ALGEBRAS

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Abstract. The notion of a KS-filter of a KS-algebra is introduced, and some related properties are investigated. Conditions for a stable subset to be a KS-filter. KS-filters containing a stable subset are established.

1. Introduction.

In 1993, Jun et al. [1] introduced a new class of algebras related to BCI/BCK-algebras and semigroups, called a *BCI/BCK-semigroup*. In 1998, for the convenience of study, Jun et al. renamed the BCI/BCK-semigroup as the IS/KS-algebra, and studied related properties (see [2]). In this paper, we introduce the notion of KS-filters in KS-algebras, and investigates some of its properties. We give conditions for a stable subset to be a KS-filter. Given a stable subset F of a KS-algebra X, we establish KS-filters containing F.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

- $((x * y) * (x * z)) * (z * y) = 0$,
- $(x * (x * y)) * y = 0$,
- $x * x = 0$,
- $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$. A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$ is called a *BCK-algebra*. In any BCK/BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

Definition 2.1. (Jun et al. [2]) A KS-*algebra* is a non-empty set X with two binary operations "$*$" and "$\cdot$" and constant 0 satisfying the axioms

- $K(X) := (X, *, 0)$ is a BCK-algebra.
- $S(X) := (X, \cdot)$ is a semigroup.
- the operation "$\cdot$" is distributive (on both sides) over the operation "$*$", that is, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$ for all $x, y, z \in X$.

Especially, if $K(X) := (X, *, 0)$ is a BCI-algebra in Definition 2.1, we say that X is an IS-*algebra*. Note that every KS-algebra is an IS-algebra. We shall write the multiplication $x \cdot y$ by xy, for convenience.

Proposition 2.2. (Jun et al. [1]) *Let X be an* IS-*algebra. Then we have*

(i) $0x = x0 = 0$.
(ii) $\forall x, y \in X,\ x \leq y \Rightarrow xz \leq yz,\ zx \leq zy\ \forall z \in X$.

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3. KS-filters

Definition 3.1. A KS-algebra X is said to be *bounded* if there exists a special element $e \in X$ such that $x \leq e$ for all $x \in X$. In this case, we call e the *bound* of X. A KS-algebra X is said to be *star-commutative* (resp. *dot-commutative*) if $x * (x * y) = y * (y * x)$ (resp. $xy = yx$) for all $x, y \in X$.

In what follows let X denote a bounded KS-algebra unless otherwise specified, and we will use the notation $e(x)$ instead of $e * x$ for all $x \in X$ and the bound e of X.

Definition 3.2. A subset F of X is called a *left* (resp. *right*) KS-*filter* of X if it satisfies:
(F1) F is a left (resp. right) stable subset of $S(X)$,
(F2) F contains the bound e of X,
(F3) $e(e(x) * e(y)) \in F$ and $y \in F$ imply $x \in F$.

In the sequel, a KS-filter means a left KS-filter, and a stable subset means a left stable subset.

Proposition 3.3. *Let F be a* KS-*filter of X and let $x \in X$. If there exists $y \in F$ such that $e(x) \leq e(y)$, then $x \in F$.*

Proof. The inequality $e(x) \leq e(y)$ implies that

$$e(e(x) * e(y)) = e(0) = e * 0 = e \in F,$$

and so $x \in F$ by (F3). This completes the proof. □

Corollary 3.4. *Let F be a* KS-*filter of X and let $x, y \in X$ be such that $y \leq x$. If $y \in F$, then $x \in F$.*

Proof. Let $x, y \in X$ be such that $y \leq x$. Then $e(x) \leq e(y)$. It follows from Proposition 3.3 that $x \in F$. □

Proposition 3.5. *Let X be star-commutative and let F be a* KS-*filter of X. Then*

$$\forall x, y \in X,\ x, y \in F \Rightarrow \text{glb}\{x, y\} \in F.$$

Proof. Note that $\text{glb}\{x, y\} = x \wedge y$ for all $x, y \in X$, where $x \wedge y = y * (y * x)$. Let $x, y \in F$. Since

$$\begin{aligned} x &= e(e(x)) \leq e(y * x) = e(y * (y * (y * x))) \\ &= e(y * (x \wedge y)) = e(e(x \wedge y) * e(y)), \end{aligned}$$

it follows from Corollary 3.4 that $e(e(x \wedge y) * e(y)) \in F$ so from (F3) that $x \wedge y \in F$. This completes the proof. □

Theorem 3.6. *Let X be star-commutative and let F be a nonempty subset of X. Then F is a* KS-*filter of X if and only if it satisfies* (F1), (F2) *and*
(F4) $\forall x, y \in X,\ y \in F, e(y * x) \in F \Rightarrow x \in F.$

Proof. The proof is straightarrow because $e(e(x) * e(y)) = e(e(e(y)) * x) = e(y * x)$ for all $x, y \in X$. □

We give conditions for a stable subset of $S(X)$ to be a KS-filter of X.

Theorem 3.7. *Let X be star-commutative that satisfies the equality $x * y = (x * y) * y$ for all $x, y \in X$. Let F be a stable subset of $S(X)$ such that*
(i) $\forall x, y \in X,\ x \in F,\ x \leq y \Rightarrow y \in F.$
(ii) $\forall x, y \in X,\ x, y \in F \Rightarrow \text{glb}\{x, y\} \in F.$
Then F is a KS-*filter of X.*

Proof. Since $x \leq e$ for all $x \in X$ and hence $x \in F$, it follows from (i) that F contains the bound e of X. Let $x, y \in X$ be such that $e(e(x) * e(y)) \in F$ and $y \in F$. Note that

$$x * y = (x * y) * y \Leftrightarrow y \wedge x = x * e(y)$$

for all $x, y \in X$. Thus

$$\begin{aligned} x \wedge y &= y * (y * x) = e(y * x) * e(y) \\ &= e(y * x) \wedge y = e(e(x) * e(y)) \wedge y \in F \end{aligned}$$

by (ii). Since $x \wedge y \leq x$, it follows from (i) that $x \in F$. Hence F is a KS-filter of X. □

Lemma 3.8. *For any $r_1, \cdots, r_n, x, y, z \in X$, we have*

$$\begin{aligned} & r_n(r_{n-1}(\cdots(r_1 x)\cdots)) \leq r_n(r_{n-1}(\cdots(r_1 y)\cdots)) \\ & \quad \Rightarrow r_n(r_{n-1}(\cdots(r_1(x * z))\cdots)) \leq r_n(r_{n-1}(\cdots(r_1(y * z))\cdots)). \end{aligned}$$

Proof. It is straightforward by the mathematical induction. □

Theorem 3.9. *Let X be dot-commutative such that $e(kx) = ke(x)$ for all $k, x \in X$ and let F be a stable subset of X. Then the set*

$$\begin{aligned} \Omega_1 := \{x \in X \mid b_n(b_{n-1}(\cdots(b_1((\cdots((e(x) * e(a_1)) * e(a_2)) * \cdots) * e(a_n)))\cdots)) = 0 \\ \text{for some } a_1, a_2, \cdots, a_n \in F \text{ and } b_1, b_2, \cdots, b_n \in X \setminus \{0\}\} \end{aligned}$$

is a KS-*filter of X containing F.*

Proof. Obviously, Ω_1 contains the bound e of X. Let $k \in X$ and $x \in \Omega_1$. Then there exist $a_1, a_2, \cdots, a_n \in F$ and $r_1, r_2, \cdots, r_n \in X \setminus \{0\}$ such that

$$r_n(r_{n-1}(\cdots(r_1((\cdots((e(x) * e(a_1)) * e(a_2)) * \cdots) * e(a_n)))\cdots)) = 0.$$

Since F is stable, we have $ka_1, ka_2, \cdots, ka_n \in F$. It follows that

$$\begin{aligned} & r_n(r_{n-1}(\cdots(r_1((\cdots((e(kx) * e(ka_1)) * e(ka_2)) * \cdots) * e(ka_n)))\cdots)) \\ & \quad = r_n(r_{n-1}(\cdots(r_1(k((\cdots((e(x) * e(a_1)) * e(a_2)) * \cdots) * e(a_n))))\cdots)) \\ & \quad = k(r_n(\cdots(r_1((\cdots((e(x) * e(a_1)) * e(a_2)) * \cdots) * e(a_n)))\cdots)) \\ & \quad = k0 = 0. \end{aligned}$$

Hence $kx \in \Omega_1$, and so Ω_1 is stable. Assume that $e(e(x) * e(y)) \in \Omega_1$ and $y \in \Omega_1$. Then there exist $a_1, \cdots, a_n, b_1, \cdots, b_m \in F$ and $r_1, \cdots, r_n, s_1, \cdots, s_m \in X \setminus \{0\}$, where $n \geq m$, such that

(1) $$r_n(r_{n-1}(\cdots(r_1((\cdots((e(e(e(x) * e(y))) * e(a_1)) * e(a_2)) * \cdots) * e(a_n)))\cdots)) = 0,$$

(2) $$s_m(s_{m-1}(\cdots(s_1((\cdots((e(y) * e(b_1)) * e(b_2)) * \cdots) * e(b_m)))\cdots)) = 0.$$

Note that (1) is equivalent to the following:

(3) $$r_n(r_{n-1}(\cdots(r_1((\cdots((e(x) * e(y)) * e(a_1)) * e(a_2)) * \cdots) * e(a_n)))\cdots)) = 0,$$

which implies that

$$\begin{aligned} 0 &= r_n(r_{n-1}(\cdots(r_1(((\cdots((e(x) * e(a_1)) * e(a_2)) * \cdots) * e(a_n)) * e(y)))\cdots)) \\ &= r_n(r_{n-1}(\cdots(r_1((\cdots((e(x) * e(a_1)) * e(a_2)) * \cdots) * e(a_n)))\cdots)) * \\ & \qquad r_n(r_{n-1}(\cdots(r_1 e(y))\cdots)), \end{aligned}$$

that is,

$$\begin{aligned} & r_n(r_{n-1}(\cdots(r_1((\cdots((e(x) * e(a_1)) * e(a_2)) * \cdots) * e(a_n)))\cdots)) \\ & \quad \leq r_n(r_{n-1}(\cdots(r_1 e(y))\cdots)). \end{aligned}$$

It follows from Lemma 3.8 that

$$\begin{aligned} & r_n(r_{n-1}(\cdots(r_1((\cdots(((\cdots((e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))* \\ & \qquad e(b_1))*\cdots)*e(b_m)))\cdots)) \\ & \leq \quad r_n(r_{n-1}(\cdots(r_1((\cdots(e(y)*e(b_1))*\cdots)*e(b_m)))*\cdots)) \end{aligned}$$

so from Proposition 2.2(ii) that

$$\begin{aligned} & s_m(s_{m-1}(\cdots(s_1(r_n(r_{n-1}(\cdots(r_1((\cdots(((\cdots((e(x)*e(a_1))*e(a_2))*\cdots)* \\ & \qquad e(a_n))*e(b_1))*\cdots)*e(b_m)))\cdots))))\cdots)) \\ \leq \quad & s_m(s_{m-1}(\cdots(s_1(r_n(r_{n-1}(\cdots(r_1((\cdots(e(y)*e(b_1))*\cdots)*e(b_m)))\cdots))))\cdots)) \\ = \quad & r_n(r_{n-1}(\cdots(r_1(s_m(s_{m-1}(\cdots(s_1((\cdots(e(y)*e(b_1))*\cdots)*e(b_m)))\cdots))))\cdots)) \\ = \quad & 0. \end{aligned}$$

Hence

$$\begin{aligned} & s_m(s_{m-1}(\cdots(s_1(r_n(r_{n-1}(\cdots(r_1((\cdots(((\cdots((e(x)*e(a_1))*e(a_2))*\cdots)* \\ & \quad e(a_n))*e(b_1))*\cdots)*e(b_m)))\cdots))))\cdots))=0, \end{aligned}$$

which shows that $x \in \Omega_1$. Therefore Ω_1 is a KS-filter of X. It is clear that $F \subseteq \Omega_1$. This completes the proof. □

Theorem 3.10. *Let X be dot-commutative such that $e(kx) = ke(x)$ for all $k, x \in X$ and let F be a stable subset of X. Then the set*

$$\begin{aligned} \Omega_2 := \{x \in X \mid & r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n)) = 0 \\ & \text{for some } r_1, r_2, \cdots, r_n \in X \setminus \{0\} \text{ and } a_1, a_2, \cdots, a_n \in F\} \end{aligned}$$

is a KS*-filter of X containing F.*

Proof. Clearly, Ω_2 contains the bound e of X. Let $x, y \in X$ be such that $e(e(x)*e(y)) \in \Omega_2$ and $y \in \Omega_2$. Then there exist $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m \in F$ and $r_1, r_2, \cdots, r_n, s_1, s_2, \cdots, s_m \in X \setminus \{0\}$, where $n \geq m$, such that

$$r_n(\cdots(r_2(r_1(e(e(e(x)*e(y)))*e(a_1))*e(a_2))*\cdots)*e(a_n))=0, \tag{4}$$

$$s_m(\cdots(s_2(s_1(e(y)*e(b_1))*e(b_2))*\cdots)*e(b_m))=0. \tag{5}$$

Note that (4) is equivalent to the following:

$$r_n(\cdots(r_2(r_1((e(x)*e(y))*e(a_1))*e(a_2))*\cdots)*e(a_n))=0,$$

which implies that

$$r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))*r_n\cdots r_2r_1e(y)=0,$$

that is, $r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n)) \leq r_n\cdots r_2r_1e(y)$. Left "$\cdot$"-multiplying both sides of the above inequality by s_1, we have

$$\begin{aligned} & s_1(r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))) \\ & \quad \leq s_1r_n\cdots r_2r_1e(y) = r_n\cdots r_2r_1s_1e(y). \end{aligned}$$

Right "$*$"-multiplying both sides of the above inequality by $s_1r_n\cdots r_1e(b_1)$, we get

$$\begin{aligned} & s_1(r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n)))*s_1r_n\cdots r_1e(b_1) \\ & \quad \leq r_n\cdots r_2r_1s_1e(y)*s_1r_n\cdots r_1e(b_1) \\ & \quad = r_n\cdots r_2r_1s_1(e(y)*e(b_1)), \end{aligned}$$

and so

$$s_1(r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))*r_n\cdots r_2r_1e(b_1)) \\ \leq r_n\cdots r_2r_1s_1(e(y)*e(b_1)).$$

Left "$\cdot$"-multiplying both sides of the above inequality by s_2, we obtain

$$s_2(s_1(r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))*r_n\cdots r_2r_1e(b_1))) \\ \leq s_2r_n\cdots r_2r_1s_1(e(y)*e(b_1)).$$

Right "$*$"-multiplying both sides of the above inequality by $s_2r_n\cdots r_1e(b_2)$, we get

$$s_2(s_1(r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))* \\ r_n\cdots r_2r_1e(b_1))*r_n\cdots r_1e(b_2)) \\ \leq r_n\cdots r_2r_1s_2(s_1(e(y)*e(b_1))*e(b_2)).$$

Repeating the above argument m-times, we conclude that

$$s_m(\cdots(s_1(r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))* \\ r_n\cdots r_1e(b_1))*\cdots)*r_n\cdots r_1e(b_m)) \\ \leq r_n\cdots r_1(s_m(\cdots(s_2(s_1(e(y)*e(b_1))*e(b_2))*\cdots)*e(b_m)))=0.$$

Consequently,

$$\begin{aligned} 0 &= s_m(\cdots(s_1(r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))* \\ &\qquad r_n\cdots r_1e(b_1))*\cdots)*r_n\cdots r_1e(b_m) \\ &= s_m(\cdots(s_1(r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))*e(b_1))*\cdots)*e(b_m)), \end{aligned}$$

which implies $x\in\Omega_2$. Let $k\in X$ and $x\in\Omega_2$. Then there exist $a_1,a_2,\cdots,a_n\in F$ and $r_1,r_2,\cdots,r_n\in X\setminus\{0\}$ such that

$$r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))=0.$$

Since F is stable, $ka_1,ka_2,\cdots,ka_n\in F$. Hence

$$\begin{aligned} & r_n(\cdots(r_2(r_1(e(kx)*e(ka_1))*e(ka_2))*\cdots)*e(ka_n)) \\ &\quad = r_n(\cdots(r_2(r_1(ke(x)*ke(a_1))*ke(a_2))*\cdots)*ke(a_n)) \\ &\quad = k(r_n(\cdots(r_2(r_1(e(x)*e(a_1))*e(a_2))*\cdots)*e(a_n))) \\ &\quad = k0=0, \end{aligned}$$

and so $kx\in\Omega_2$, i.e., Ω_2 is stable. Obviously, $F\subseteq\Omega_2$. Summarizing the above facts we know that Ω_2 is a KS-filter of X containing F. This completes the proof. □

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