ON FIBERING CERTAIN 3-MANIFOLDS OVER THE CIRCLE

Masami Urakubo and Mayumi Ohmiya

Received December 10, 2001

Abstract. The fibration of a certain 3-dimensional manifold over the circle is studied to generalize celebrated Tollefson's theorem. It is proved that if the 3-dimensional manifold admits the proper k-cyclic action, then it can be fibered over the circle. In addition, the fibration of the orbit space over the circle is obtained.

1 Introduction. In this paper, we study the fibration of certain 3-dimensional manifolds over the circle S^1. In [2], J. L. Tollefson proved that if the 3-manifold M^3 admits the proper $\mathbb{Z}_k$-action for some prime number k and $H_1(M^3/\mathbb{Z}_k;\mathbb{Z})$ is k-torsion free, then M^3 can be fibered over the circle S^1. The main goal of the present work is to relax the conditions on M^3 and k in Tollefson's theorem mentioned above. In addition we obtain the fibering the orbit space $M^3/\mathbb{Z}_k$ over S^1.

The contents of the present paper are as follows. In §2, we describe preliminary materials, and state Theorem 1 which is the one of our main results concerned with a simple criteria for the infinite cyclic covering of given CW-complex to be dominated by a finite CW-complex. In §3, we prove several lemmas which are necessary to prove Theorem 1. §4 is devoted to the proof of Theorem 1. In §5, applying Theorem 1, we clarify the condition for the existence of the fibering map $M^3 \to S^1$. In §6, applying the result in §5, we consider the fibration of the orbit 3-manifold $M^3/\mathbb{Z}_k$.

2 Preliminaries and results. Let X be the topological space. Suppose that $g : X \to S^1$ is the continuous map, and $P_k : S^1 \to S^1$ is the standard k-fold covering map defined by $P_k(t) = t^k$, where S^1 is the circle. W_k denotes the k-fold covering of X induced by the map g, that is,

$$W_k = \{(x,t) \in X \times S^1 \mid g(x) = P_k(t)\} \tag{1}$$

Then we have the commutative diagram

$$\begin{array}{ccc} W_k & \xrightarrow{\rho_2} & S^1 \\ {\scriptstyle\rho_1}\downarrow & & \downarrow{\scriptstyle P_k} \\ X & \xrightarrow[g]{} & S^1, \end{array} \tag{2}$$

where $\rho_1(x,t) = x$ and $\rho_2(x,t) = t$. We say that W_k admits the proper free $\mathbb{Z}_k$-action if a generator of the covering $\mathbb{Z}_k$-action on W_k is homotopic to the identity Id_{W_k}.

2000 *Mathematics Subject Classification.* 55R99, 57M10, 57S17 .
Key words and phrases. fibering, 3-manifold, proper free $\mathbb{Z}_k$-action.

Similarly to the above, we have the infinite cyclic covering $\overline{X}$ of X induced via g by the commutative diagram

$$\begin{array}{ccc} \overline{X} & \longrightarrow & \mathbb{R} \\ \Big\downarrow & & \Big\downarrow \text{ex} \\ X & \xrightarrow[g]{} & S^1, \end{array}$$

where ex; $\mathbb{R} \to S^1$ is the universal covering map defined by $\text{ex}(s) = \exp(2\pi i s)$.

We say that Y is dominated by Z if there exists a continuous maps $\phi : Y \to Z$ and $\psi : Z \to Y$ such that $\psi \circ \phi : Y \to Y$ is homotopic to the identity Id_Y.

Now we can state the first main result of the present paper.

Theorem 1. *Let X be an arcwise connected finite CW-complex and $g : X \to S^1$ be a continuous map such that $g_* : \pi_1(X) \to \pi_1(S^1)$ is an epimorphism. If W_k admits a proper free $\mathbb{Z}_k$-action for some integer $k \geq 2$, then $\overline{X}$ is dominated by a finite CW-complex.*

3 Lemmas.

In this section we prove five lemmas which are necessary to show Theorem1.

Firstly we define the following two spaces;

$$X[m,n] = \{(x,s) \in X \times [m,n] \mid g(x) = \exp(2\pi i s)\},$$

$$X[m,n]_* = X[m,n]/(x,m) \sim (x,n),$$

that is, $X[m,n]_*$ is the space in which (x,m) is identified with (x,n) for every $x \in X$, where $m, n \in \mathbb{Z}$ and $m < n$. Then we have the following.

Lemma 1. *$X[0,k]_*$ is homeomorphic to W_k defined by* (1);

$$X[0,k]_* \simeq W_k.$$

Proof. Let us consider the maps

$$\phi_1 : W_k \longrightarrow X[0,k]_*$$

and

$$\psi_1 : X[0,k]_* \longrightarrow W_k$$

defined by

$$\phi_1(x,t) = (x, ks)$$

for $t = \exp(2\pi i s) \in S^1$, $(0 \leq s < 1)$, and

$$\psi_1(x,s) = (x, \exp\left(\frac{2\pi i s}{k}\right)), \quad (0 \leq s < k).$$

It is easy to see that the maps ϕ_1 and ψ_1 are well defined and continuous. Moreover we have

$$\psi_1 \circ \phi_1(x,t) = \psi_1(x,ks) = (x, \exp(2\pi i s)) = (x,t),$$

where $t = \exp(2\pi i s) \in S^1, (0 \leq s < 1)$, and

$$\phi_1 \circ \psi_1(x,s) = \phi_1(x, \exp\left(\frac{2\pi i s}{k}\right)) = (x,s)$$

for $0 \leq s < 1$. Hence ϕ_1 is the homeomorphism. □

Let h be a generator of the proper free $\mathbb{Z}_k$-action on W_k, that is, h is homotopic to Id_{W_k}. Let

$$H : W_k \times I \longrightarrow W_k$$

be a homotopy from Id_{W_k} to h, where $I = [0,1]$.

Put

$$H(x,t,j) = (h_j^{(1)}(x,t), h_j^{(2)}(x,t)),\ (x,t) \in W_k,\ j \in I,$$

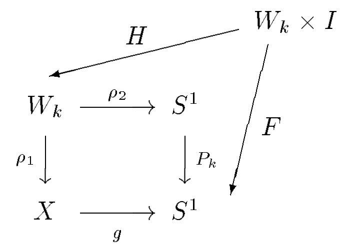
then we have

$$h_0^{(1)}(x,t) = x,\ h_0^{(2)}(x,t) = t,\ h_1^{(1)}(x,t) = x,\ \{h_1^{(2)}(x,t)\}^k = t^k. \tag{3}$$

Let us consider

$$F = P_k \circ \rho_2 \circ H : W_k \times I \longrightarrow S^1,$$

where P_k is the standard k-fold covering map of S^1 and $\rho_1(x,t) = x,\ \rho_2(x,t) = t$.



We have

$$\begin{aligned} F(x,t,j) &= P_k \circ \rho_2 \circ H(x,t,j) = P_k \circ \rho_2(h_j^{(1)}(x,t), h_j^{(2)}(x,t)) \\ &= P_k(h_j^{(2)}(x,t)) = h_j^{(2)}(x,t)^k = g(h_j^{(1)}(x,t)). \end{aligned} \tag{4}$$

From (3) and (4), we have

$$\begin{aligned} F(x,t,0) &= (h_0^{(2)}(x,t))^k = t^k \\ F(x,t,1) &= (h_1^{(2)}(x,t))^k = t^k. \end{aligned}$$

Hence, if we fix $(x,t) \in W_k$,

$$F(x,t,*) : I/\{0,1\} \simeq S^1 \longrightarrow S^1$$

defines the map from S^1 to S^1. On the one hand, note that W_k is connected. Thus $F(x,t,j)$ takes on only one non-zero degree c to every $(x,t) \in W_k$. Next we have

Lemma 2. *Define* $\tilde{h} : \overline{X} \to \overline{X}$ *by*

$$\tilde{h}(x, ks + km) = (x, ks + km + c),$$

where $0 \leq s < 1,\ m \in \mathbb{Z}$, *c is the degree of* $F(x,t,j)$ *mentioned above, and k is given in Theorem 1. Then the identity map* $Id_{\overline{X}}$ *is homotopic to* $\tilde{h}$.

Proof. Let us consider the map

$$\rho\ : \overline{X} \ni (x, ks + km) \longmapsto (x, ks) \in \overline{X}[0,k]_* \quad (0 \leq s < 1,\ m \in \mathbb{Z}).$$

By the definition of the space $\overline{X}[0,k]_*$, it is easy to see that the map ρ is well defined and continuous. Next we show that there exists a map $\widetilde{F}$ such that the following diagram is commutative.

$$
\begin{array}{ccccccc}
 & & & & & & \mathbb{R} \\
 & \nearrow^{\widetilde{F}} & & & & & \downarrow \\
\overline{X} \times I & \xrightarrow{\rho \times \mathrm{Id}} & \overline{X}[0,k]_* \times I & \xrightarrow{\phi_1^{-1} \times \mathrm{Id}} & W_k \times I & \xrightarrow{F} & S^1
\end{array}
$$

We have immediately

$$
\begin{aligned}
&F \circ (\phi_1^{-1} \times \mathrm{Id}) \circ (\rho \times \mathrm{Id})(x, ks+km, j) \\
&\quad = F \circ (\phi_1^{-1} \times \mathrm{Id})(x, ks, j) = F(x, \exp(2\pi i s), j) = (h_j^{(2)}(x, \exp(2\pi i s))^k.
\end{aligned}
$$

In particular, for $j = 0$, we have

$$
\begin{aligned}
&F \circ (\phi_1^{-1} \times \mathrm{Id}) \circ (\rho \times \mathrm{Id})(x, ks+km, 0) \\
&\quad = (h_0^{(2)}(x, \exp(2\pi i s))^k = (\exp(2\pi i s))^k = \exp(2\pi i k s) = \exp(2\pi i (ks + km)).
\end{aligned}
$$

Let us define the map $\widetilde{F}_0 : \overline{X} \times \{0\} \to \mathbb{R}$ by

$$
\widetilde{F}_0(x, ks+km, 0) = ks + km, \ 0 \leq s < 1, \ m \in \mathbb{Z},
$$

then $\widetilde{F}_0$ is the lifting of $F \circ (\phi_1^{-1} \times \mathrm{Id}) \circ (\rho \times \mathrm{Id}) |_{\overline{X} \times \{0\}}$. Hence, by the covering homotopy property, there exists a continuous map $\widetilde{F} : \overline{X} \times I \to \mathbb{R}$ which is the extension of $\widetilde{F}_0$ such that the above diagram is commutative. We have immediately

$$
\begin{aligned}
\exp(2\pi i(\widetilde{F}(x, ks+km, j)) &= h_j^{(2)}(x, \exp(2\pi i s)^k, \\
\widetilde{F}(x, ks+km, 0) &= ks + km, \\
\widetilde{F}(x, ks+km, 1) &= ks + km + c.
\end{aligned}
$$

Let us define the map $\widetilde{H} : \overline{X} \times I \to \overline{X}$ by

$$
\widetilde{H}(x, ks+km, j) = (h_j^{(1)}(x, \exp(2\pi i s)), \widetilde{F}(x, ks+km, j)).
$$

Then the map $\widetilde{H}$ turns out to be the homotopy from the identity $\mathrm{Id}_{\overline{X}}$ to $\tilde{h}$ by the following three facts (i), (ii) and (iii):
(i) $\widetilde{H}$ is well defined, since

$$
(h_j^{(1)}(x, \exp(2\pi i s)), \widetilde{F}(x, ks+km, j)) \in \overline{X}
$$

holds by

$$
g(h_j^{(1)}(x, \exp(2\pi i s)) = h_j^{(2)}(x, \exp(2\pi i s))^k = \exp(2\pi i \widetilde{F}(x, ks+km, j)).
$$

(ii) $\widetilde{H}(x, ks+km, 0) = ((h_0^{(1)}(x, \exp(2\pi i s)), \widetilde{F}(x, ks+km, 0)) = (x, ks+km)$.

(iii) $\widetilde{H}(x, ks+km, 1) = ((h_1^{(1)}(x, \exp(2\pi i s)), \widetilde{F}(x, ks+km, 1)) = (x, ks+km+c)$.

This completes the proof. □

Remark. If $c < 0$, we define $\tilde{h}' : \overline{X} \to \overline{X}$ by $\tilde{h}'(x, s) = (x, s - c)$. Then it follows that $\mathrm{Id}_{\overline{X}}$ is homotopic to $\tilde{h}'$. Hence, in what follows, we assume that the degree c ia a positive integer.

For the degree c, we consider the two spaces W_c and $\overline{W}_c$, where $\overline{W}_c$ is the infinite cyclic covering of W_c induced by the following $\overline{g}$;

$$\begin{array}{ccc} W_c & \xrightarrow{\overline{g}} & S^1 \\ \downarrow & & \downarrow P_c \\ X & \xrightarrow[g]{} & S^1. \end{array}$$

Next we have the following.

Lemma 3. *$\overline{X}$ is homeomorphic to $\overline{W}_c$.*

Proof. Let us consider the following commutative diagram;

$$\begin{array}{ccc} \overline{W}_c & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \mathrm{ex} \\ W_c & \xrightarrow{\overline{g}} & S^1 \\ \downarrow & & \downarrow P_c \\ X & \xrightarrow[g]{} & S^1. \end{array}$$

We have

$$\begin{aligned} \overline{W}_c &= \{(x,t,s) \in W_c \times \mid \overline{g}(x,t) = \exp(2\pi i s)\} \\ &= \{(x,t,s) \in X \times S^1 \times \mathbb{R} \mid g(x) = t^c,\ t = \exp(2\pi i s)\}. \end{aligned}$$

Hence $\overline{W}_c$ is homeomorphic to the space

$$\{(x,s) \in X \times \mathbb{R} \mid g(x) = \exp(2\pi i c s)\}.$$

On the other hand, we have

$$\overline{X} = \{(x,s) \in X \times \mathbb{R} \mid g(x) = \exp(2\pi i s)\}$$

Let us define the maps ϕ_2 and ψ_2 by $\phi_2(x,s) = (x, s/c)$ and $\psi_2(x,s) = (x, cs)$ respectively. It is easily seen that the maps ϕ_2 and ψ_2 are well-defined and continuous. Moreover, it follows that $\psi_2 \circ \phi_2(x,s) = (x,s)$ and $\phi_2 \circ \psi_2(x,s) = (x,s)$. Hence $\phi_2 : \overline{X} \to \overline{W}_c$ is the homeomorphism. □

Next we have the following.

Lemma 4. *The map $\tau : \overline{W}_c \to \overline{W}_c$ defined by $\tau(x,s) = (x, s+1)$ is homotopic to $Id_{\overline{W}_c}$.*

Proof. Let us define the map $\widetilde{L}$ by the commutative diagram

$$\begin{array}{ccc} \overline{X} \times I & \xrightarrow{\widetilde{H}} & \overline{X} \\ \phi_2 \times \mathrm{Id} \downarrow & & \downarrow \phi_2 \\ \overline{W}_c \times I & \xrightarrow[\widetilde{L}]{} & \overline{W}_c. \end{array}$$

We have

$$\begin{aligned} \widetilde{L}(x,s,0) &= \phi_2 \circ \widetilde{H} \circ (\phi_2 \times \mathrm{Id})^{-1}(x,s,0) \\ &= \phi_2 \circ \widetilde{H}(x, cs, 0) \\ &= \phi_2(x, cs) = (x,s), \end{aligned}$$

and

$$\begin{aligned}\widetilde{L}(x,s,1) &= \phi_2 \circ \widetilde{H} \circ (\phi_2 \times \mathrm{Id})^{-1}(x,s,1) \\ &= \phi_2 \circ \widetilde{H}(x,cs,1) \\ &= \phi_2(x,cs+c) = (x,s+1).\end{aligned}$$

Hence the map $\widetilde{L}$ is the homotopy from $\mathrm{Id}_{\overline{W}_c}$ to τ. □

Next we have the following.

Lemma 5. *$\overline{W}_c$ is dominated by $\overline{W}_c \underset{\mathbb{Z}}{\times} \mathbb{R}$.*

Proof. Put

$$\overline{W}_c \underset{\mathbb{Z}}{\times} \mathbb{R} = \overline{W}_c \times \mathbb{R}/(x,s,t) \sim (x,s+m,t-m), \quad m \in \mathbb{Z}.$$

Now let us define the map $\phi : \overline{W}_c \to \overline{W}_c \underset{\mathbb{Z}}{\times} \mathbb{R}$ by $\phi(x,s) = [x,s,0] \in \overline{W}_c \underset{\mathbb{Z}}{\times} \mathbb{R}$. Then ϕ is well-defined continuous map. In addition, let us define the map $\psi' : \overline{W}_c \times \mathbb{R} \to \overline{W}_c$ by

$$\psi'(x,s,t) = \widetilde{L}(x,s+m,r)$$

if $t = m + r$ for $0 \leq r < 1$ and $m \in \mathbb{Z}$. Then ψ' is well-defined. If ψ' is continuous at $(x,s,0) \in \overline{W}_c \times \mathbb{R}$, then ψ' turns out to be continuous at every point $(x,s,t) \in \overline{W}_c \times \mathbb{R}$ by the definition of ψ'. Therefore it suffices to show that ψ' is continuous at $(x,s,0)$. Now suppose that $(x_\alpha, s_\alpha, \varepsilon_\alpha)$ tends to $(x,s,0)$ in $\overline{W}_c \times \mathbb{R}$. We have

$$\begin{aligned}\psi'(x_\alpha, s_\alpha, \varepsilon_\alpha) &= \widetilde{L}(x_\alpha, s_\alpha, \varepsilon_\alpha) \\ &\longrightarrow \widetilde{L}(x,s,0) = (x,s)\end{aligned}$$

as $\varepsilon_\alpha \downarrow 0$. On the other hand, we have

$$\begin{aligned}\psi'(x_\alpha, s_\alpha, \varepsilon_\alpha) &= \widetilde{L}(x_\alpha, s_\alpha, \varepsilon_\alpha) \\ &= \widetilde{L}(x_\alpha, s_\alpha - 1, \varepsilon_\alpha + 1) \\ &= \widetilde{L}(x, s-1, 1) = (x,s)\end{aligned}$$

as $\varepsilon_\alpha \uparrow 0$. Since

$$\psi'(x,s,0) = \widetilde{L}(x,s,0) = (x,s),$$

it is shown that ψ' is continuous at $(x,s,0)$. Hence the map ψ' is continuous. Since

$$\psi'(x,s,t) = \psi'(x,s+m,s-m) \quad m \in \mathbb{Z}$$

by the definition, ψ' can be extended to the continuous map $\psi : \overline{W}_c \underset{\mathbb{Z}}{\times} \mathbb{R} \to \overline{W}_c$. Since

$$\psi \circ \phi(x,s) = \psi(x,s,0) = \widetilde{L}(x,s,0) = (x,s),$$

$\psi \circ \phi$ is the identity map of $\overline{W}_c$. This completes the proof. □

4 Proof of Theorem 1. By using the above five lemmas, we can prove Theorem 1 immediately.

In fact, first of all, we have

$$\overline{X} \simeq \overline{W}_c$$

by lemma 3. Next let us regard the infinite covering $\overline{W}_c$ as the principal $\mathbb{Z}$-bundle. Then there exists the associated principal $\mathbb{R}$-bundle

$$\mathbb{R} \longrightarrow \overline{W}_c \underset{\mathbb{Z}}{\times} \mathbb{R} \longrightarrow W_c.$$

On the other hand, by lemma 5, $\overline{W}_c$ turns out to be dominated by $\overline{W}_c \times_{\mathbb{Z}} \mathbb{R}$. Since $\mathbb{R}$ is ∞-connected and W_c is the CW-complex, the associated principal $\mathbb{R}$-bundle is trivial. Hence

$\overline{W}_c \times_{\mathbb{Z}} \mathbb{R}$ is homotopy equivalent to W_c. Hence $\overline{X}$ is dominated by the finite CW-complex W_c. This completes the proof of Theorem 1.

5 Fibering 3-manifold. In this section, we clarify the condition for the existence of the fibering map $M^3 \to S^1$ for the 3-manifold M^3.

Let us define two classes C_0 and C_1 as follows. $(X, g) \in C_0$ if and only if X is the arcwise connected finite CW-complex and $g : X \to S^1$ is the continuous map such that $g_* : \pi_1(X) \to \pi_1(S^1)$ is the epimorphism. On the one hand, $(X, g) \in C_1$ if and only if $(X, g) \in C_0$ and there exists an integer $k \geq 2$ such that a generator of covering $\mathbb{Z}_k$-action h is homotopic to the identity map of W_k, where W_k is the k-fold covering over X which is induced from the standard k-fold covering over S^1 via g.

Here we briefly mention the celebrated theorem due to Stallings concerned with the fibration of 3-manifold; Suppose that the topological 3-manifold M^3 is compact and irreducible. If there exists the exact sequence

$$0 \longrightarrow G \longrightarrow \pi_1(M^3) \xrightarrow[\phi]{} \pi_1(S^1) \longrightarrow 0$$

such that G is finitely generated and $G \neq \mathbb{Z}/2\mathbb{Z}$. Then there exists the fibering map $g : M^3 \to S^1$ such that $g_* = \phi$, and the fiber T is the connected 2-manifold with $\pi_1(T) \cong G$.

Then we have the following

Theorem 2. *Let the topological 3-manifold M^3 be connected,compact and irreducible. Suppose that there exists $g : M^3 \to S^1$ such that $(M^3, g) \in C_1$. Moreover assume that $H_1(M^3; \mathbb{Z})$ has no element of order 2. Then there exists the fibering map $M^3 \to S^1$ which is homotopic to g.*

Proof. By the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
\overline{M^3} & \longrightarrow & \mathbb{R} \\
\downarrow & & \downarrow \text{ex} \\
M^3 & \xrightarrow[g]{} & S^1,
\end{array}$$

where $\overline{M^3}$ is the infinite cyclic covering of M^3 induced via g, we obtain the exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(\mathbb{R}) & \longrightarrow & \pi_1(S^1) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow g_* & & \| & & \\
0 & \longrightarrow & \pi_1(\overline{M^3}) & \longrightarrow & \pi_1(M^3) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0,
\end{array}$$

where g_* is the epimorphism. Then we have the exact sequence

$$0 \longrightarrow \pi_1(\overline{M^3}) \longrightarrow \pi_1(M^3) \xrightarrow[g_*]{} \pi_1(S^1) \longrightarrow 0.$$

Here $\pi_1(\overline{M^3})$ corresponds to the group G in the fibering theorem due to Stallings mentioned above. Hence it suffices for the proof to show that $\pi_1(\overline{M^3})$ is finitely generated and $\pi_1(\overline{M^3}) \neq \mathbb{Z}/2\mathbb{Z}$. By triangulating M^3 as a finite complex, $\overline{M^3}$ turns out to be dominated by the finite CW-complex from Theorem 1. By [3], $\pi_1(\overline{M^3})$ is finitely generated. On the other hand, since $H_1(M^3; \mathbb{Z})$ is assumed to have no elements of order 2, we can coclude that $\pi_1(M^3)$ has no elements of order 2 from Hurewicz homomorphism. Since $\pi_1(\overline{M^3}) \to \pi_1(M^3)$ is the monomorphism, $\pi_1(\overline{M^3})$ has also no elements of order 2. Thus we have shown that

$\pi_1(\overline{M^3})$ satisfies the condition of the fibering theorem due to Stallings. This completes the proof. ☐

6 Fibering the orbit 3-manifold. In this section, applying Theorem 2, we consider the fibration of the orbit 3-manifold $M/\mathbb{Z}_k$. We have the following.

Theorem 3. *Let the topological 3-manifold M^3 be connected, compact and irreducible. Suppose that there exists the free $\mathbb{Z}_k$-action ($k \geq 2$) on M^3 such that there exists a generator h of the $\mathbb{Z}_k$-action which is homotopic to Id_{M^3}. If $H_1(M^3/\mathbb{Z}_k;\mathbb{Z})$ has no elements of order 2 and order k, then both $M^3/\mathbb{Z}_k$ and M^3 can be fibered over the circle.*

For the proof of Theorem 3, it is necessary to show the following algebraic fact.

Lemma 6. *Let F be the finitely generated free module and $p : F \to \mathbb{Z}_k$ be the epimorphism. Then there exists $v \in F$ such that $p(v) \in \mathbb{Z}_k$ is the generator of $\mathbb{Z}_k$ and there exists the basis B of F such that $v \in B$ and $B \setminus \{v\} \subset p^{-1}(0)$.*

Note that if we fix the isomorphism $F \cong \mathbb{Z}^l$, lemma 6 is equivalent to the following lemma 7 which is almost obvious.

Lemma 7. *Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_l\}$ be the standard basis of $\mathbb{Z}^l$. If the greatest common measure of integers $m_1, m_2, \cdots, m_l$ is 1, then there exists the basis of $\mathbb{Z}^l$ which contains $\sum_{j=1}^{l} m_j \boldsymbol{e}_j$.*

Now we can prove Theorem 3.

Proof. Since M^3 is compact and irreducible, $M^3/\mathbb{Z}_k$ is also compact and irreducible. Let us consider the following diagram.

$$\begin{array}{ccc}
\mathbb{Z}_k & \longrightarrow & \mathbb{Z}_k \\
\downarrow & & \downarrow \\
M^3 & \longrightarrow & E \\
\downarrow & & \downarrow \\
M^3/\mathbb{Z}_k & \underset{f}{\longrightarrow} & K(\mathbb{Z}_k, 1),
\end{array}$$

where $\mathbb{Z}_k \to E \to K(\mathbb{Z}_k, 1)$ is the universal $\mathbb{Z}_k$-bundle. Hence there exists the bundle map $f : M^3/\mathbb{Z}_k \to K(\mathbb{Z}_k, 1)$. Then we obtain the exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \| & & & & & & \\
0 & \longrightarrow & \pi_1(E) & \longrightarrow & \mathbb{Z}_k & \longrightarrow & \mathbb{Z}_k & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow\ f_* & & \| & & \\
0 & \longrightarrow & \pi_1(M^3) & \longrightarrow & \pi_1(M^3/\mathbb{Z}_k) & \longrightarrow & \mathbb{Z}_k & \longrightarrow & 0.
\end{array}$$

By this diagram, it turns out that $f_* : \pi_1(M^3/\mathbb{Z}_k) \to \mathbb{Z}_k$ is the epimorphism. Since $\mathbb{Z}_k$ is the abelian group, f_* can be factored as $f_* = \theta \cdot \beta$, where θ and β are defined by

$$\begin{array}{ccc}
\pi_1(M^3/\mathbb{Z}_k) & \xrightarrow{f_*} & \mathbb{Z}_k \\
{\scriptstyle\beta}\downarrow & & \uparrow{\scriptstyle\theta} \\
H_1(M^3/\mathbb{Z}_k;\mathbb{Z}) & \underset{\mathrm{Id}}{\longrightarrow} & H_1(M^3/\mathbb{Z}_k;\mathbb{Z}),
\end{array}$$

where β is Hurewicz homomorphism. Note that β and θ are the epimorphisms, and $H_1(M^3/\mathbb{Z}_k;\mathbb{Z})$ is k-torsion free. By lemma 6 and lemma 7, we obtain the following diagram from the above one.

$$\begin{array}{ccc} \pi_1(M^3/\mathbb{Z}_k) & \xrightarrow{f_*} & \mathbb{Z}_k \\ \downarrow & & \uparrow \text{mod } k\text{-reduction} \\ F \cong \oplus_l \mathbb{Z} & \xrightarrow[\text{Projection}]{} & \mathbb{Z}. \end{array}$$

Hence we have the following diagram.

$$\begin{array}{ccc} \pi_1(M^3/\mathbb{Z}_k) & \xrightarrow[f_*]{} & \mathbb{Z}_k \\ \downarrow & & \uparrow \\ \mathbb{Z} & \xrightarrow[\text{Id}]{} & \mathbb{Z}. \end{array}$$

Since S^1 and $K(\mathbb{Z}_k, 1)$ are Eilenberg-Maclane spaces, there exist the bundle maps

$$\begin{array}{ccccc} M^3 & \longrightarrow & S^1 & \longrightarrow & E \\ {\scriptstyle p}\downarrow & & {\scriptstyle P_k}\downarrow & & \downarrow \\ M^3/\mathbb{Z}_k & \xrightarrow[g]{} & S^1 & \longrightarrow & K(\mathbb{Z}_k, 1), \end{array}$$

where P_k is the standard k-fold covering and p is induced from P_k. Hence $(M^3/\mathbb{Z}_k, \mathbb{Z})$ belongs to the class C_1 and $H_1(M^3/\mathbb{Z}_k;\mathbb{Z})$ has no elements of order 2 by the assumption. Therefore, by Theorem 2, there exists the fibering map which is homotopic to g. This completes the proof. □

References

[1] J. Stallings, On fibering certain 3-manifolds, Topology of 3-manifolds and related topics (Ed. M. K. Fort Jr.) 1962, Prentice Hall, Englewood Cliffs, pp. 95–100.

[2] J. L. Tollefson, Fibering 3-manifolds that admit free $\mathbb{Z}_k$-actions, Trans. American Math. Soc., **147** (1970), pp.279–289.

[3] C. T. C. Wall, Finiteness conditions for CW-complex I, Ann. Math. **81** (1965), pp. 56–69

M. Urakubo, 1856-8, Ikoma, 630-0201, Japan
E-mail address: `matm@m4.kcn.ne.jp`

M. Ohmiya, Department of Electrical Engineering, Faculty of Engineering, Doshisha University, 1-3 Tatara Miyakodani, Kyotanabe, 610-0394, Japan
E-mail address: `momiya@mail.doshisha.ac.jp`