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CONGRUENCES ON *BCC*-ALGEBRAS

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Abstract. Using fuzzy BCC-ideals, the quotient structure of BCC-algebras is discussed. We show that (1) If $\mathfrak{f} : G \to H$ is an onto homomorphism of BCC-algebras, and if $\bar{B}$ is a fuzzy BCC-ideal of H, then $G/\mathfrak{f}^{-1}(\bar{B})$ is isomorphic to $H/\bar{B}$; (2) If $\bar{A}$ and $\bar{B}$ are fuzzy BCC-ideals of BCC-algebras G and H, respectively, then $\frac{G\times H}{\bar{A}\times\bar{B}} \cong G/\bar{A} \times H/\bar{B}$; and (3) If $\bar{A}$ is a fuzzy BCC-ideal of G, and if J is a BCC-ideal of G such that $J/\bar{A}$ is a BCC-ideal of $G/\bar{A}$, then $\frac{G/\bar{A}}{J/\bar{A}} \cong G/J$.

1. Introduction

In 1966, Y. Imai and K. Iséki ([8]) defined a class of algebras of type (2,0) called *BCK-algebras* which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra ([9]). The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [11]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori ([10]) introduced a notion of BCC-algebras, and W. A. Dudek ([1, 2]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [6], W. A. Dudek and X. H. Zhang introduced a notion of BCC-ideals in BCC-algebras and described connections between such ideals and congruences. W. A. Dudek and Y. B. Jun ([3]) considered the fuzzification of BCC-ideals in BCC-algebras. They showed that every fuzzy BCC-ideal of a BCC-algebra is a fuzzy BCK-ideal, and showed that the converse is not true by providing an example. They also proved that in a BCC-algebra every fuzzy BCK-ideal is a fuzzy BCC-subalgebra, and in a BCK-algebra the notion of a fuzzy BCK-ideal and a fuzzy BCC-ideal coincide. W. A. Dudek, Y. B. Jun and Z. Stojaković ([5]) described several properties of fuzzy BCC-ideals in BCC-algebras, and discussed an extension of fuzzy BCC-ideals. In this paper we consider the quotient structure of BCC-algebras using fuzzy BCC-ideals. We show that (1) If $\mathfrak{f} : G \to H$ is an onto homomorphism of BCC-algebras, and if $\bar{B}$ is a fuzzy BCC-ideal of H, then $G/\mathfrak{f}^{-1}(\bar{B})$ is isomorphic to $H/\bar{B}$; (2) If $\bar{A}$ and $\bar{B}$ are fuzzy BCC-ideals of BCC-algebras G and H, respectively, then $\frac{G\times H}{\bar{A}\times\bar{B}} \cong G/\bar{A} \times H/\bar{B}$; and (3) If $\bar{A}$ is a fuzzy BCC-ideal of G, and if J is a BCC-ideal of G such that $J/\bar{A}$ is a BCC-ideal of $G/\bar{A}$, then $\frac{G/\bar{A}}{J/\bar{A}} \cong G/J$.

2. Preliminaries

Recall that a *BCC-algebra* is an algebra $(G, *, 0)$ of type (2,0) satisfying the following axioms:

(C1) $((x * y) * (z * y)) * (x * z) = 0$,
(C2) $0 * x = 0$,

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(C3) $x * 0 = x$,
(C4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

for every $x, y, z \in G$. For any BCC-algebra G, the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on G. In a BCC-algebra G, the following holds (see [7]).

(p1) $x \leq x$,
(p2) $x * y \leq x$,
(p3) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$

for all $x, y, z \in G$. Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (see [2]). Note that a BCC-algebra is a BCK-algebra if and only if it satisfies

- $(x * y) * z = (x * z) * y$, $\forall x, y, z \in G$.

A non-empty subset A of a BCC-algebra G is called a *BCC-ideal* of G if it satisfies

- $0 \in A$,
- $\forall x, y, z \in G$, $y \in A$, $(x * y) * z \in A \Rightarrow x * z \in A$.

Note that any BCC-ideal of a BCC-algebra is a BCC-subalgebra (see [6]).

Definition 2.1. [3] A fuzzy set $\bar{A}$ in a BCC-algebra G is called a *fuzzy BCC-ideal* of G if it satisfies

(F1) $\bar{A}(0) \geq \bar{A}(x)$, $\forall x \in G$,
(F2) $\bar{A}(x * y) \geq \min\{\bar{A}((x * a) * y), \bar{A}(a)\}$, $\forall a, x, y \in G$.

Definition 2.2. [3] A fuzzy set $\bar{A}$ in a BCC-algebra G is called a *fuzzy BCK-ideal* of G if it satisfies (F1) and

(F3) $\bar{A}(x) \geq \min\{\bar{A}(x * y), \bar{A}(y)\}$, $\forall x, y \in G$.

Lemma 2.3. [3, Theorem 4.3] *In a BCC-algebra, every fuzzy BCC-ideal is a fuzzy BCK-ideal.*

3. Congruence relations

In what follows, let G denote a BCC-algebra unless otherwise specified. Let $\bar{A}$ be a fuzzy BCC-ideal of G and $\alpha \in [0, 1)$. We consider a relation on G as follows:

$$\Re_{\bar{A},\alpha} := \{(x, y) \in G \times G \mid \bar{A}(x * y) > \alpha,\ \bar{A}(y * x) > \alpha\}.$$

Lemma 3.1. *Let $\bar{A}$ be a fuzzy BCC-ideal of G and $\alpha \in [0, 1)$. If $\Re_{\bar{A},\alpha} \neq \emptyset$, then $\bar{A}(0) > \alpha$.*

Proof. If $\Re_{\bar{A},\alpha} \neq \emptyset$, then there exists $(x, y) \in G \times G$ such that $\bar{A}(x * y) > \alpha$. It follows from (F1) that $\bar{A}(0) \geq \bar{A}(x * y) > \alpha$. This completes the proof. □

Proposition 3.2. *Let $\bar{A}$ be a fuzzy BCC-ideal of G and $\alpha \in [0, 1)$. If $\Re_{\bar{A},\alpha} \neq \emptyset$, then $\Re_{\bar{A},\alpha}$ is a congruence on G*

Proof. Note from (p1) and Lemma 3.1 that $\bar{A}(x * x) = \bar{A}(0) > \alpha$ for all $x \in G$. Hence $(x, x) \in \Re_{\bar{A},\alpha}$ for all $x \in G$, and so $\Re_{\bar{A},\alpha}$ is reflexive. Obviously, $\Re_{\bar{A},\alpha}$ is symmetric. Let $x, y, z \in G$ be such that $(x, y) \in \Re_{\bar{A},\alpha}$ and $(y, z) \in \Re_{\bar{A},\alpha}$. Then $\bar{A}(x * y) > \alpha$, $\bar{A}(y * x) > \alpha$, $\bar{A}(y * z) > \alpha$, and $\bar{A}(z * y) > \alpha$. Since $((x * z) * (y * z)) * (x * y) = 0$ by (C1), we have

$$\bar{A}(((x * z) * (y * z)) * (x * y)) = \bar{A}(0) > \alpha.$$

Since $\bar{A}$ is a fuzzy BCK-ideal by Lemma 2.3, it follows from (F3) that

$$\bar{A}((x * z) * (y * z)) \geq \min\{\bar{A}(((x * z) * (y * z)) * (x * y)),\ \bar{A}(x * y)\} > \alpha$$

so that $\bar{A}(x * z) \geq \min\{\bar{A}((x * z) * (y * z)),\ \bar{A}(y * z)\} > \alpha$. Similarly we have $\bar{A}(z * x) > \alpha$, and thus $(x, z) \in \Re_{\bar{A},\alpha}$. Therefore $\Re_{\bar{A},\alpha}$ is transitive, and hence $\Re_{\bar{A},\alpha}$ is an equivalence relation on G. Now, let $x, y, u, v \in G$ be such that $(x, u) \in \Re_{\bar{A},\alpha}$ and $(y, v) \in \Re_{\bar{A},\alpha}$. Then $\bar{A}(x*u) > \alpha$, $\bar{A}(u*x) > \alpha$, $\bar{A}(y*v) > \alpha$, and $\bar{A}(v*y) > \alpha$. Since $((x*y)*(u*y))*(x*u) = 0$, we have

$$\begin{aligned} \bar{A}((x*y)*(u*y)) &\geq \min\{\bar{A}(((x*y)*(u*y))*(x*u)),\ \bar{A}(x*u)\} \\ &= \min\{\bar{A}(0),\ \bar{A}(x*u)\} > \alpha. \end{aligned}$$

Similarly, $\bar{A}((u*y)*(x*y)) > \alpha$. Hence $(x*y, u*y) \in \Re_{\bar{A},\alpha}$. On the other hand, since $((u*y)*(v*y))*(u*v) = 0$, it follows from (F2) and Lemma 3.1 that

$$\begin{aligned} \bar{A}((u*y)*(u*v)) &\geq \min\{\bar{A}(((u*y)*(v*y))*(u*v)),\ \bar{A}(v*y)\} \\ &= \min\{\bar{A}(0),\ \bar{A}(v*y)\} > \alpha. \end{aligned}$$

Similarly, we get $\bar{A}((u*v)*(u*y)) > \alpha$. Therefore $(u*y, u*v) \in \Re_{\bar{A},\alpha}$. Using the transitivity of $\Re_{\bar{A},\alpha}$, we conclude that $(x*y, u*v) \in \Re_{\bar{A},\alpha}$. Consequently, $\Re_{\bar{A},\alpha}$ is a congruence on G. □

Corollary 3.3. *Let $\bar{A}$ be a fuzzy BCC-ideal of G and $\alpha \in [0,1)$. If $\bar{A}(0) > \alpha$, then $\Re_{\bar{A},\alpha}$ is a congruence on G.*

Let $\bar{A}$ be a fuzzy BCC-ideal of G and let $\alpha \in [0,1)$. Denote by $[x]_\alpha^{\bar{A}}$ the set $\{y \in G \mid (x, y) \in \Re_{\bar{A},\alpha}\}$ and by $G/\bar{A}$ the set $\{[x]_\alpha^{\bar{A}} \mid x \in G\}$. Define a binary operation $\ominus$ on $G/\bar{A}$ by

$$[x]_\alpha^{\bar{A}} \ominus [y]_\alpha^{\bar{A}} = [x*y]_\alpha^{\bar{A}}$$

for all $x, y \in G$. First we shall verify that the operation $\ominus$ is well-defined. Assume that $[x]_\alpha^{\bar{A}} = [u]_\alpha^{\bar{A}}$ and $[y]_\alpha^{\bar{A}} = [v]_\alpha^{\bar{A}}$, i.e., $(x, u) \in \Re_{\bar{A},\alpha}$ and $(y, v) \in \Re_{\bar{A},\alpha}$. Then $(x*y, u*v) \in \Re_{\bar{A},\alpha}$ since $\Re_{\bar{A},\alpha}$ is a congruence on G. Let $w \in [x]_\alpha^{\bar{A}} \ominus [y]_\alpha^{\bar{A}}$. Then $(w, x*y) \in \Re_{\bar{A},\alpha}$, and so $(w, u*v) \in \Re_{\bar{A},\alpha}$. Hence $w \in [u]_\alpha^{\bar{A}} \ominus [v]_\alpha^{\bar{A}}$, and therefore $[x]_\alpha^{\bar{A}} \ominus [y]_\alpha^{\bar{A}} = [u]_\alpha^{\bar{A}} \ominus [v]_\alpha^{\bar{A}}$. Consequently, the operation $\ominus$ is well-defined. Next we show that $G/\bar{A}$ is a BCC-algebra with respect to the operation $\ominus$. Let $[x]_\alpha^{\bar{A}}, [y]_\alpha^{\bar{A}}, [z]_\alpha^{\bar{A}} \in G/\bar{A}$. Then

$$\begin{aligned} &(([x]_\alpha^{\bar{A}} \ominus [y]_\alpha^{\bar{A}}) \ominus ([z]_\alpha^{\bar{A}} \ominus [y]_\alpha^{\bar{A}})) \ominus ([x]_\alpha^{\bar{A}} \ominus [z]_\alpha^{\bar{A}}) \\ &= ([x*y]_\alpha^{\bar{A}} \ominus [z*y]_\alpha^{\bar{A}}) \ominus [x*z]_\alpha^{\bar{A}} \\ &= [(x*y)*(z*y)]_\alpha^{\bar{A}} \ominus [x*z]_\alpha^{\bar{A}} \\ &= [((x*y)*(z*y))*(x*z)]_\alpha^{\bar{A}} \\ &= [0]_\alpha^{\bar{A}}, \end{aligned}$$

which shows that (C1) is true. Similarly, we obtain (C2) and (C3). Suppose that $[x]_\alpha^{\bar{A}} \ominus [y]_\alpha^{\bar{A}} = [0]_\alpha^{\bar{A}}$ and $[y]_\alpha^{\bar{A}} \ominus [x]_\alpha^{\bar{A}} = [0]_\alpha^{\bar{A}}$. Then $[x*y]_\alpha^{\bar{A}} = [0]_\alpha^{\bar{A}} = [y*x]_\alpha^{\bar{A}}$, which implies that $\bar{A}(x*y) = \bar{A}((x*y)*0) > \alpha$ and $\bar{A}(y*x) = \bar{A}((y*x)*0) > \alpha$. Hence $(x, y) \in \Re_{\bar{A},\alpha}$, and so $[x]_\alpha^{\bar{A}} = [y]_\alpha^{\bar{A}}$. Therefore we have the following theorem.

Theorem 3.4. *If $\bar{A}$ is a fuzzy BCC-ideal of G and $\alpha \in [0,1)$, then $(G/\bar{A}, \ominus, [0]_\alpha^{\bar{A}})$ is a BCC-algebra.*

Using a BCC-ideal, Dudek and Zhang gave a congruence relation on G as follows: Let J be a BCC-ideal of G and let $x, y \in G$. The relation $\sim$ on G defined by

$$x \sim y \text{ if and only if } x*y \in J \text{ and } y*x \in J$$

is a congruence on G (see [6]). We denote the equivalence class containing x by $||x||_J$, i.e.,

$$||x||_J := \{y \in G \mid x \sim y\}.$$

Note that $x \sim y$ if and only if $||x||_J = ||y||_J$. Denote the set of all equivalence classes of G by G/J, i.e., $G/J := \{||x||_J \mid x \in G\}$. Then $(G/J, *, ||0||_J)$ is a BCC-algebra.

Let $\mathfrak{f}$ be a mapping defined on G. If $\bar{B}$ is a fuzzy set in $\mathfrak{f}(G)$, then the fuzzy set $\mathfrak{f}^{-1}(\bar{B}) := \bar{B} \circ \mathfrak{f}$ in G, i.e., the fuzzy set defined by $\mathfrak{f}^{-1}(\bar{B})(x) = \bar{B}(\mathfrak{f}(x))$ for all $x \in G$, is called the *preimage* of $\bar{B}$ under $\mathfrak{f}$.

Lemma 3.5. *Let $\mathfrak{f} : G \to H$ be an onto homomorphism of BCC-algebras. If $\bar{B}$ is a fuzzy BCC-ideal of H, then $\mathfrak{f}^{-1}(\bar{B})$ is a fuzzy BCC-ideal of G.*

Proof. Assume that $\bar{B}$ is a fuzzy BCC-ideal of H. Taking "min" instead of a t-norm "T" in [4, Proposition 3], we know that $\mathfrak{f}^{-1}(\bar{B})$ is a fuzzy BCC-ideal of G. □

Theorem 3.6. *Let $\mathfrak{f} : G \to H$ be an onto homomorphism of BCC-algebras. If $\bar{B}$ is a fuzzy BCC-ideal of H, then $G/\mathfrak{f}^{-1}(\bar{B})$ is isomorphic to $H/\bar{B}$.*

Proof. Let $\alpha \in [0, 1)$. Define a mapping $\mathfrak{h} : G/\mathfrak{f}^{-1}(\bar{B}) \to H/\bar{B}$ by

$$\mathfrak{h}([x]_\alpha^{\mathfrak{f}^{-1}(\bar{B})}) = [\mathfrak{f}(x)]_\alpha^{\bar{B}}, \ \ \forall [x]_\alpha^{\mathfrak{f}^{-1}(\bar{B})} \in G/\mathfrak{f}^{-1}(\bar{B}).$$

Assume that $[x]_\alpha^{\mathfrak{f}^{-1}(\bar{B})} = [y]_\alpha^{\mathfrak{f}^{-1}(\bar{B})}$. Then $(x, y) \in \Re_{\mathfrak{f}^{-1}(\bar{B}),\alpha}$, and so

$$\bar{B}(\mathfrak{f}(x) * \mathfrak{f}(y)) = \bar{B}(\mathfrak{f}(x * y)) = \mathfrak{f}^{-1}(\bar{B})(x * y) > \alpha$$

and

$$\bar{B}(\mathfrak{f}(y) * \mathfrak{f}(x)) = \bar{B}(\mathfrak{f}(y * x)) = \mathfrak{f}^{-1}(\bar{B})(y * x) > \alpha.$$

It follows that $(\mathfrak{f}(x), \mathfrak{f}(y)) \in \Re_{\bar{B},\alpha}$ so that $[\mathfrak{f}(x)]_\alpha^{\bar{B}} = [\mathfrak{f}(y)]_\alpha^{\bar{B}}$. Hence $\mathfrak{h}$ is well-defined. We claim that $\mathfrak{h}$ is one-one. For any $[x]_\alpha^{\mathfrak{f}^{-1}(\bar{B})}$, $[y]_\alpha^{\mathfrak{f}^{-1}(\bar{B})} \in G/\mathfrak{f}^{-1}(\bar{B})$, if $\mathfrak{h}([x]_\alpha^{\mathfrak{f}^{-1}(\bar{B})}) = \mathfrak{h}([y]_\alpha^{\mathfrak{f}^{-1}(\bar{B})})$ then $[\mathfrak{f}(x)]_\alpha^{\bar{B}} = [\mathfrak{f}(y)]_\alpha^{\bar{B}}$ and hence $(\mathfrak{f}(x), \mathfrak{f}(y)) \in \Re_{\bar{B},\alpha}$. Thus

$$\mathfrak{f}^{-1}(\bar{B})(x * y) = \bar{B}(\mathfrak{f}(x * y)) = \bar{B}(\mathfrak{f}(x) * \mathfrak{f}(y)) > \alpha$$

and

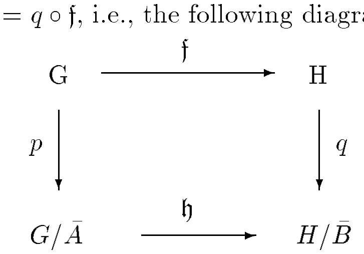
$$\mathfrak{f}^{-1}(\bar{B})(y * x) = \bar{B}(\mathfrak{f}(y * x)) = \bar{B}(\mathfrak{f}(y) * \mathfrak{f}(x)) > \alpha.$$

Therefore $(x, y) \in \Re_{\mathfrak{f}^{-1}(\bar{B}),\alpha}$, that is, $[x]_\alpha^{\mathfrak{f}^{-1}(\bar{B})} = [y]_\alpha^{\mathfrak{f}^{-1}(\bar{B})}$. Obviously, $\mathfrak{h}$ is onto. Finally, we show that $\mathfrak{h}$ is a homomorphism. Let $[x]_\alpha^{\mathfrak{f}^{-1}(\bar{B})}$, $[y]_\alpha^{\mathfrak{f}^{-1}(\bar{B})} \in G/\mathfrak{f}^{-1}(\bar{B})$. Then

$$\begin{aligned} \mathfrak{h}([x]_\alpha^{\mathfrak{f}^{-1}(\bar{B})} \ominus [y]_\alpha^{\mathfrak{f}^{-1}(\bar{B})}) &= \mathfrak{h}([x * y]_\alpha^{\mathfrak{f}^{-1}(\bar{B})}) \\ &= [\mathfrak{f}(x * y)]_\alpha^{\bar{B}} = [\mathfrak{f}(x) * \mathfrak{f}(y)]_\alpha^{\bar{B}} \\ &= [\mathfrak{f}(x)]_\alpha^{\bar{B}} \ominus [\mathfrak{f}(y)]_\alpha^{\bar{B}} \\ &= \mathfrak{h}([x]_\alpha^{\mathfrak{f}^{-1}(\bar{B})}) \ominus \mathfrak{h}([y]_\alpha^{\mathfrak{f}^{-1}(\bar{B})}). \end{aligned}$$

This proves the theorem. □

Given a fuzzy BCC-ideal of G and $\alpha \in [0, 1)$, the BCC-homomorphism $\pi : G \to G/\bar{A}$, $x \mapsto [x]_\alpha^{\bar{A}}$, is called the *natural* (or *canonical*) *homomorphism* of G onto $G/\bar{A}$. In the above Theorem 3.6, if we define canonical homomorphisms $p : G \to G/\bar{A}$ and $q : H \to H/\bar{B}$ then it is easy to show that $\mathfrak{h} \circ p = q \circ \mathfrak{f}$, i.e., the following diagram commutes:



The *fundamental homomorphism theorem* for BCC-algebras is well-known, i.e., if $\mathfrak{f}: G \to H$ is an onto homomorphism of BCC-algebras, then $G/Ker\mathfrak{f} \cong H$. Given BCC-algebras G_1 and G_2 define a binary operation "$\odot$" on $G_1 \times G_2$ by

$$(x_1, x_2) \odot (y_1, y_2) := (x_1 * y_1, x_2 * y_2),\ \forall (x_1, x_2), (y_1, y_2) \in G_1 \times G_2.$$

Then it can be easily seen that $(G_1 \times G_2; \odot, (0, 0))$ is a BCC-algebra. Now we discuss a fuzzy BCC-ideal in a BCC-algebra $G_1 \times G_2$.

Proposition 3.7. *Let $\bar{A}$ and $\bar{B}$ be fuzzy BCC-ideals of BCC-algebras G_1 and G_2 respectively. Define a mapping $\bar{A} \times \bar{B} : G_1 \times G_2 \to [0, 1]$ by*

$$(\bar{A} \times \bar{B})(x, y) = \min\{\bar{A}(x), \bar{B}(y)\},\ \forall (x, y) \in G_1 \times G_2.$$

Then $\bar{A} \times \bar{B}$ is a fuzzy BCC-ideal of $G_1 \times G_2$.

Proof. For any $(x, y) \in G_1 \times G_2$ we have

$$(\bar{A} \times \bar{B})(0, 0) = \min\{\bar{A}(0), \bar{B}(0)\} \geq \min\{\bar{A}(x), \bar{B}(y)\} = (\bar{A} \times \bar{B})(x, y).$$

Let $(x_1, x_2), (y_1, y_2), (a_1, a_2) \in G_1 \times G_2$. Then

$$\begin{aligned} &(\bar{A} \times \bar{B})(((x_1, x_2) \odot (a_1, a_2)) \odot (y_1, y_2)) \\ &= (\bar{A} \times \bar{B})((x_1 * a_1) * y_1,\ (x_2 * a_2) * y_2) \\ &= \min\{\bar{A}((x_1 * a_1) * y_1),\ \bar{B}((x_2 * a_2) * y_2)\} \end{aligned}$$

and $(\bar{A} \times \bar{B})(a_1, a_2) = \min\{\bar{A}(a_1), \bar{B}(a_2)\}$. Hence

$$\begin{aligned} &(\bar{A} \times \bar{B})((x_1, x_2) \odot (y_1, y_2)) \\ &= (\bar{A} \times \bar{B})(x_1 * y_1, x_2 * y_2) = \min\{\bar{A}(x_1 * y_1),\ \bar{B}(x_2 * y_2)\} \\ &\geq \min\{\min\{\bar{A}((x_1 * a_1) * y_1), \bar{A}(a_1)\},\ \min\{\bar{B}((x_2 * a_2) * y_2), \bar{B}(a_2)\}\} \\ &= \min\{\min\{\bar{A}((x_1 * a_1) * y_1), \bar{B}((x_2 * a_2) * y_2)\},\ \min\{\bar{A}(a_1), \bar{B}(a_2)\}\} \\ &= \min\{(\bar{A} \times \bar{B})(((x_1, x_2) \odot (a_1, a_2)) \odot (y_1, y_2)), (\bar{A} \times \bar{B})(a_1, a_2)\}. \end{aligned}$$

This shows that $\bar{A} \times \bar{B}$ is a fuzzy BCC-ideal of $G_1 \times G_2$. □

Theorem 3.8. *If $\bar{A}$ and $\bar{B}$ are fuzzy BCC-ideals of BCC-algebras G and H, respectively, then $\frac{G \times H}{\bar{A} \times \bar{B}} \cong G/\bar{A} \times H/\bar{B}$.*

Proof. Let $\alpha \in [0, 1)$. If we define $\Psi : G \times H \to G/\bar{A} \times H/\bar{B}$ by $\Psi(x, y) = ([x]_\alpha^{\bar{A}}, [y]_\alpha^{\bar{B}})$, then it is easy to verify that Ψ is an onto homomorphism. By the fundamental homomorphism theorem, we obtain $\frac{G \times H}{Ker\Psi} \cong G/\bar{A} \times H/\bar{B}$. We now claim that $||(x, y)||_{Ker\Psi} = [(x, y)]_\alpha^{\bar{A} \times \bar{B}}$. Indeed,

$$\begin{aligned} &(a, b) \in ||(x, y)||_{Ker\Psi} \\ &\Leftrightarrow (a, b) * (x, y) \in Ker\Psi,\ (x, y) * (a, b) \in Ker\Psi \\ &\Leftrightarrow (a * x, b * y) \in Ker\Psi,\ (x * a, y * b) \in Ker\Psi \\ &\Leftrightarrow \Psi(a * x, b * y) = ([0]_\alpha^{\bar{A}}, [0]_\alpha^{\bar{B}}) = \Psi(x * a, y * b) \\ &\Leftrightarrow ([a * x]_\alpha^{\bar{A}}, [b * y]_\alpha^{\bar{B}}) = ([0]_\alpha^{\bar{A}}, [0]_\alpha^{\bar{B}}) = ([x * a]_\alpha^{\bar{A}}, [y * b]_\alpha^{\bar{B}}) \\ &\Leftrightarrow [a]_\alpha^{\bar{A}} \ominus [x]_\alpha^{\bar{A}} = [0]_\alpha^{\bar{A}} = [x]_\alpha^{\bar{A}} \ominus [a]_\alpha^{\bar{A}},\ [b]_\alpha^{\bar{B}} \ominus [y]_\alpha^{\bar{B}} = [0]_\alpha^{\bar{B}} = [y]_\alpha^{\bar{B}} \ominus [b]_\alpha^{\bar{B}} \\ &\Leftrightarrow [a]_\alpha^{\bar{A}} = [x]_\alpha^{\bar{A}},\ [b]_\alpha^{\bar{B}} = [y]_\alpha^{\bar{B}} \\ &\Leftrightarrow (a, x) \in \Re_\alpha^{\bar{A}},\ (b, y) \in \Re_\alpha^{\bar{B}} \end{aligned}$$

and

$$\begin{aligned}
&(a,b) \in [(x,y)]_\alpha^{\bar{A}\times\bar{B}} \\
&\quad \Leftrightarrow \quad ((a,b),(x,y)) \in \Re_\alpha^{\bar{A}\times\bar{B}} \\
&\quad \Leftrightarrow \quad (\bar{A}\times\bar{B})((a,b)\odot(x,y)) > \alpha,\ (\bar{A}\times\bar{B})((x,y)\odot(a,b)) > \alpha \\
&\quad \Leftrightarrow \quad (\bar{A}\times\bar{B})(a*x, b*y) > \alpha,\ (\bar{A}\times\bar{B})(x*a, y*b) > \alpha \\
&\quad \Leftrightarrow \quad \min\{\bar{A}(a*x), \bar{B}(b*y)\} > \alpha,\ \min\{\bar{A}(x*a), \bar{B}(y*b)\} > \alpha \\
&\quad \Leftrightarrow \quad (a,x) \in \Re_\alpha^{\bar{A}},\ (b,y) \in \Re_\alpha^{\bar{B}},
\end{aligned}$$

which shows that $||(x,y)||_{Ker\Psi} = [(x,y)]_\alpha^{\bar{A}\times\bar{B}}$. Hence

$$\frac{G\times H}{\bar{A}\times\bar{B}} = \frac{G\times H}{Ker\Psi} \cong G/\bar{A}\times H/\bar{B},$$

proving the proof. □

Theorem 3.9. *Let $\bar{A}$ be a fuzzy BCC-ideal of G and let $\alpha \in [0,1)$. If J^* is a BCC-ideal of $G/\bar{A}$, then there exists a BCC-ideal*

$$J := \cup\{[x]_\alpha^{\bar{A}} \mid [x]_\alpha^{\bar{A}} \in J^*\}$$

in G such that $J/\bar{A} = J^*$.

Proof. If J^* is a BCC-ideal of $G/\bar{A}$, then $[0]_\alpha^{\bar{A}} \in J^*$ and so $0 \in J$. Let $x,y,z \in G$ be such that $y \in J$ and $(x*y)*z \in J$. Then $y \in [a]_\alpha^{\bar{A}}$ and $(x*y)*z \in [b]_\alpha^{\bar{A}}$ for some $[a]_\alpha^{\bar{A}}, [b]_\alpha^{\bar{A}} \in J^*$. It follows that $[y]_\alpha^{\bar{A}} = [a]_\alpha^{\bar{A}}$ and

$$[b]_\alpha^{\bar{A}} = [(x*y)*z]_\alpha^{\bar{A}} = ([x]_\alpha^{\bar{A}} \ominus [y]_\alpha^{\bar{A}}) \ominus [z]_\alpha^{\bar{A}} = ([x]_\alpha^{\bar{A}} \ominus [a]_\alpha^{\bar{A}}) \ominus [z]_\alpha^{\bar{A}}$$

so that $[x*z]_\alpha^{\bar{A}} = [x]_\alpha^{\bar{A}} \ominus [z]_\alpha^{\bar{A}} \in J^*$ since J^* is a BCC-ideal. Thus $x*z \in J$, and so J is a BCC-ideal of G. Moreover,

$$\begin{aligned}
J/\bar{A} &= \{[u]_\alpha^{\bar{A}} \mid u \in J\} \\
&= \{[u]_\alpha^{\bar{A}} \mid \exists [x]_\alpha^{\bar{A}} \in J^* \text{ such that } u \in [x]_\alpha^{\bar{A}}\} \\
&= \{[u]_\alpha^{\bar{A}} \mid \exists [x]_\alpha^{\bar{A}} \in J^* \text{ such that } [u]_\alpha^{\bar{A}} = [x]_\alpha^{\bar{A}}\} \\
&= \{[u]_\alpha^{\bar{A}} \mid [u]_\alpha^{\bar{A}} \in J^*\} \\
&= J^*,
\end{aligned}$$

proving the proof. □

Theorem 3.10. *Let $\bar{A}$ be a fuzzy BCC-ideal of G. If J is a BCC-ideal of G such that $J/\bar{A}$ is a BCC-ideal of $G/\bar{A}$, then $\frac{G/\bar{A}}{J/\bar{A}} \cong G/J$.*

Proof. Define $\phi : \frac{G/\bar{A}}{J/\bar{A}} \to G/J$ by $\phi(||[x]_\alpha^{\bar{A}}||_{J/\bar{A}}) = ||x||_J$ for all $||[x]_\alpha^{\bar{A}}||_{J/\bar{A}} \in \frac{G/\bar{A}}{J/\bar{A}}$. Suppose that $||[x]_\alpha^{\bar{A}}||_{J/\bar{A}} = ||[y]_\alpha^{\bar{A}}||_{J/\bar{A}}$ in $\frac{G/\bar{A}}{J/\bar{A}}$. Then $[x]_\alpha^{\bar{A}} \sim [y]_\alpha^{\bar{A}}$, and so $[x*y]_\alpha^{\bar{A}} = [x]_\alpha^{\bar{A}} \ominus [y]_\alpha^{\bar{A}} \in J/\bar{A}$ and $[y*x]_\alpha^{\bar{A}} = [y]_\alpha^{\bar{A}} \ominus [x]_\alpha^{\bar{A}} \in J/\bar{A}$. This means that $x*y \in J$ and $y*x \in J$, i.e., $x \sim y$. Thus

$$\phi(||[x]_\alpha^{\bar{A}}||_{J/\bar{A}}) = ||x||_J = ||y||_J = \phi(||[y]_\alpha^{\bar{A}}||_{J/\bar{A}}),$$

and so ϕ is well defined. For every $||[x]_\alpha^{\bar{A}}||_{J/\bar{A}}, ||[y]_\alpha^{\bar{A}}||_{J/\bar{A}} \in \frac{G/\bar{A}}{J/\bar{A}}$, we have

$$\begin{aligned}
&\phi(||[x]_\alpha^{\bar{A}}||_{J/\bar{A}} * ||[y]_\alpha^{\bar{A}}||_{J/\bar{A}}) = \phi(||[x]_\alpha^{\bar{A}} \ominus [y]_\alpha^{\bar{A}}||_{J/\bar{A}}) \\
&\quad = \phi(||[x*y]_\alpha^{\bar{A}}||_{J/\bar{A}}) = ||x*y||_J = ||x||_J * ||y||_J \\
&\quad = \phi(||[x]_\alpha^{\bar{A}}||_{J/\bar{A}}) * \phi(||[y]_\alpha^{\bar{A}}||_{J/\bar{A}}).
\end{aligned}$$

Hence ϕ is a homomorphism. Obviously, ϕ is onto. Finally, we show that ϕ is one-one. If $\phi(||[x]_\alpha^{\bar{A}}||_{J/\bar{A}}) = \phi(||[y]_\alpha^{\bar{A}}||_{J/\bar{A}})$, then $||x||_J = ||y||_J$ and hence $x \sim y$. If $[a]_\alpha^{\bar{A}} \in ||[x]_\alpha^{\bar{A}}||_{J/\bar{A}}$ then $[a]_\alpha^{\bar{A}} \sim [x]_\alpha^{\bar{A}}$ and hence $[a*x]_\alpha^{\bar{A}} = [a]_\alpha^{\bar{A}} \ominus [x]_\alpha^{\bar{A}} \in J/\bar{A}$ and $[x*a]_\alpha^{\bar{A}} = [x]_\alpha^{\bar{A}} \ominus [a]_\alpha^{\bar{A}} \in J/\bar{A}$. It follows that $a*x, x*a \in J$, i.e., $a \sim x$ so that $a \sim y$. Hence $[a]_\alpha^{\bar{A}} \in ||[y]_\alpha^{\bar{A}}||_{J/\bar{A}}$, which shows that $||[x]_\alpha^{\bar{A}}||_{J/\bar{A}} \subseteq ||[y]_\alpha^{\bar{A}}||_{J/\bar{A}}$. Similarly, we obtain $||[y]_\alpha^{\bar{A}}||_{J/\bar{A}} \subseteq ||[x]_\alpha^{\bar{A}}||_{J/\bar{A}}$. Therefore $\frac{G/\bar{A}}{J/\bar{A}} \cong G/J$, proving the proof. □

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