# ON SCATTERED COUNTABLE METRIC SPACES

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Abstract. The compact countable metric spaces are topologically classified simply by the classical Mazurkiewicz-Sierpiński theorem. Our concern is non-compact case. After viewing the scattered countable metric spaces of length 2 and the locally compact countable metric spaces, we shall prove Theorem 2, the main theorem of the present paper. Theorem 2 presents a topological classification of a class of scattered countable metric spaces which is far from the class of locally compact countable metric spaces.

**1. Preliminaries.** Let $X$ be a topological space. The following is Cantor's well-known process of deriving which is done by transfinite induction. (cf. Kuratowski [1])

Let $X^{(0)} = X$ and $X_{(0)}$ the set of the isolated points of $X^{(0)}$. If $\beta$ is a non-limit ordinal, let $X^{(\beta)} = X^{(\beta-1)} - X_{(\beta-1)}$ and $X_{(\beta)}$ the set of the isolated points of $X^{(\beta)}$, where $\beta - 1$ means the ordinal preceding $\beta$. If $\beta$ is a limit ordinal, let $X^{(\beta)} = \cap_{\gamma<\beta} X^{(\gamma)}$ and $X_{(\beta)}$ the set of the isolated points of $X^{(\beta)}$.

Each $X^{(\beta)}$ is a closed subset of $X$, and each $X_{(\beta)}$ is a discrete open subset of $X^{(\beta)}$.

A space $X$ is called *scattered* if $X^{(\alpha)} = \emptyset$ for some $\alpha$. A scattered space is also characterized as a space in which every non-empty (closed) subspace has an isolated point. The first ordinal $\alpha$ for which $X^{(\alpha)}$ vanishes is called the *length* of the scattered space $X$ and is denoted by $\mathrm{leng}(X)$. For a point $x$ of $X$, we write $\mathrm{rank}\, x = \beta$ if $x \in X_{(\beta)}$. A scattered space $X$ has the following properties which will be used in this paper implicitly and frequently. Let $\beta$ be an ordinal and $U$ an open set of $X$.

(a) $X^{(\beta)} \cap U = U^{(\beta)}$ and $X_{(\beta)} \cap U = U_{(\beta)}$ (and hence we have the following two).
(b) $\mathrm{leng}(U) = \beta$ if and only if $U \cap X^{(\beta)} = \emptyset$ and $U \cap X^{(\gamma)} \neq \emptyset$ for every $\gamma < \beta$.
(c) $X_{(\beta)}$ is dense in $X^{(\beta)}$.

A scattered countable metric space $X$ of length $\alpha$ has in addition the following properties.

(d) The length $\alpha$ is a countable or finite ordinal. ( For compact case, $\alpha$ is in addition a non-limit ordinal )
(e) If $\beta + 1 < \alpha$ then $|X_{(\beta)}| = \omega$ with $\omega$ the first countable ordinal. If $\beta + 1 = \alpha$ then $|X^{(\beta)}| = |X_{(\beta)}| \leq \omega$. ( For compact case, $|X^{(\beta)}| = |X_{(\beta)}| < \omega$.)

If the length $\alpha > 0$ is a non-limit ordinal and $|X^{(\alpha-1)}| = \mathfrak{m}$, $1 \leq \mathfrak{m} \leq \omega$, we write $\mathrm{type}\, X = (\alpha, \mathfrak{m})$.

The following is the well-known Mazurkiewicz-Sierpiński theorem which caused and led us to write the present paper.

**Theorem 1.** (Mazurkiewicz-Sierpiński [2]) *A compact countable metric space $X$ of type $(\alpha, n)$ is homeomorphic to the ordinals $(0,\ \omega^{\alpha-1}n\,]$ with the order topology. Hence the*

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*topological type of a compact countable metric space is uniquely determined by its type* $(\alpha,\ n)$, *and the number of topological types of compact countable metric spaces is* $\aleph_1$.

The compact countable metric space of type $(\alpha,\ n)$ is denoted by $MS(\alpha,\ n)$.

## 2. Scattered countable metric spaces of length 2.

We start with type $(2, 1)$.

**Proposition 1.** *Let* $X$ *be a scattered countable metric space of type* $(2,\ 1)$ *with* $X^{(1)} = X_{(1)} = \{\,p\,\}$. *Then* $X$ *admits precisely three topological types. Each type is characterized by the existence of a clopen neighborhood base* $X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ *of* $p$ *satisfying*

(r) $|U_m - U_{m+1}| = 1$ *for every* $m$, *or*
(r') $|U_1 - U_2| = \omega$ *and* $|U_m - U_{m+1}| = 1$ *for every* $m \geq 2$, *or*
(s) $|U_m - U_{m+1}| = \omega$ *for every* $m$.

The $X$ 's which admit clopen neighborhood bases satisfying (r), (r'), (s) are respectively denoted by $\boldsymbol{r},\ \boldsymbol{r}',\ \boldsymbol{s}$. Typical spaces are as follows :

| $\boldsymbol{r}$ | $\boldsymbol{r}'$ | $\boldsymbol{s}$ |
|---|---|---|
| $MS(2,1)$ , $[\,0,\omega]$ | $MS(2,1) \oplus \mathbb{N}$ , $[\,0,\omega 2] - \{\,\omega\,\}$ | $MS(3,1) - MS(3,1)_{(1)}$ , $[\,0,\omega^2] - \{\,\omega n\,|\,1 \leq n < \omega\,\}$ |

, where $\oplus$ means the topological sum and $\mathbb{N}$ denotes the countable discrete space.

If $1 \leq n < \omega$, the space of the form

$$\overbrace{r \oplus r \oplus \cdots \oplus r}^{n} \quad (\text{resp} \ \ \overbrace{s \oplus s \oplus \cdots \oplus s}^{n})$$

is denoted by $n\boldsymbol{r}$ (resp $n\boldsymbol{s}$) and the space of the form

$$\overbrace{r \oplus r \oplus r \oplus \cdots}^{\omega} \quad (\text{resp} \ \ \overbrace{s \oplus s \oplus s \oplus \cdots}^{\omega})$$

by $\omega\boldsymbol{r}$ (resp $\omega\boldsymbol{s}$).

**Definition 1.** A space $X$ is said to *absorb* a space $Y$ if $X \approx X \oplus Y$, where $\approx$ means the left side is homeomorphic to the right side.

A finite points space is absorbed by $n\boldsymbol{r},\ \omega\boldsymbol{r},\ n\boldsymbol{s},\ \omega\boldsymbol{s}$. The countable discrete space $\mathbb{N}$ is absorbed by $\omega\boldsymbol{r},\ n\boldsymbol{s},\ \omega\boldsymbol{s}$ but not by $n\boldsymbol{r}$.

**Proposition 2.** *Let* $X$ *be a scattered countable metric space of type* $(2,\ n)$, $1 \leq n < \omega$. *Then* $X$ *admits precisely* $n+2$ *topological types as follows :*

$$n\boldsymbol{r},\ n\boldsymbol{s},\ k\boldsymbol{r} \oplus (n-k)\boldsymbol{s},\ 1 \leq k \leq n-1,\ n\boldsymbol{r} \oplus \mathbb{N}.$$

**Proposition 3.** *Let* $X$ *be a scattered countable metric space of type* $(2, \omega)$. *Then* $X$ *is homeomorphic to one and only one of the following spaces :*

$$\omega\boldsymbol{r},\ \omega\boldsymbol{s},\ k\boldsymbol{r} \oplus \omega\boldsymbol{s},\ 1 \leq k < \omega,\ k\boldsymbol{s} \oplus \omega\boldsymbol{r},\ 1 \leq k < \omega,\ \omega\boldsymbol{r} \oplus \omega\boldsymbol{s}.$$

Proof. Using 0-dimensionality we can take a discrete family $\{U_x : x \in X^{(1)}\}$ of clopen sets of $X$ so that $U_x \cap X^{(1)} = \{x\}$. We may assume $U_x \approx \boldsymbol{r}$ or $\boldsymbol{s}$. Put $U = \cup_{x \in X^{(1)}} U_x$ and $R = X - U$. Then $R$ is homeomorphic to $\emptyset$, a finite points space or $\mathbb{N}$. A finite points space is absorbed by $U$ and can be vanished. The residue $\mathbb{N}$ is not absorbed by $U$ only when $U \approx n\boldsymbol{r}$. This complets the proof.

**3. Non-compact locally compact countable metric spaces.** The non-compact locally compact countable metric spaces can be topologically classified easily by using Alexandrov's one-point compactification $X^* = X \cup \{p\}$.

**Proposition 4.** *If $\alpha$ is a limit ordinal with $\alpha < \omega_1$, then a locally compact countable metric space $X$ of length $\alpha$ has the unique topological type.*

Proof. In this case, type $X^* = (\alpha + 1, 1)$ and the point $p$ has the highest rank $\alpha$ in $X^*$. Theorem 1 says that $X^*$ has the unique topological type so that $X$ does because the rank of a point is preserved under homeomorphisms. This completes the proof.

**Proposition 5.** *If $\alpha$ is a non-limit ordinal with $0 < \alpha < \omega_1$, then a locally compact countable metric space $X$ of type $(\alpha, \omega)$ has the unique topological type.*

Proof. In this case, it is also true that type $X^* = (\alpha + 1, 1)$ and the point $p$ has the highest rank $\alpha$ in $X^*$ because if not, $X^{(\alpha-1)}$ would not have an accumulation point in the compact space $X^*$. Theorem 1 says again that $X^*$ has the unique topological type so that $X$ does. This completes the proof.

If $\alpha$ is a limit ordinal (resp a non-limit ordinal), $LC(\alpha)$ denotes the unique locally compact countable metric space of length $\alpha$ (resp of type $(\alpha, \omega)$ ) assured by the propositions above. $LC(0)$ denotes the empty set for convenience.

**Proposition 6.** *Let $\alpha$ be a non-limit ordinal with $0 < \alpha < \omega_1$ and let $X$ be a non-compact locally compact countable metric space $X$ of type $(\alpha, n)$, $1 \leq n < \omega$. Then the topological type of $X$ is uniquely determined by the rank of $p$ in $X^*$ and is homeomorphic to*

$$MS(\alpha, n) \oplus LC(\beta), \ \beta < \alpha,$$

*with $\beta$ the rank of $p$.*

Proof. Taking a clopen set $U$ of $X^*$ so that $U \cap (X^*)^{(\beta)} = \{p\}$ we have

$$X^* = (X^* - U) \cup U \approx MS(\alpha, n) \oplus MS(\beta + 1, 1)$$

which implies $X \approx X^* - \{p\} \approx MS(\alpha, n) \oplus LC(\beta)$. This completes the proof.

**Remark.** Though a detailed description as above was not given in [2], it was proved there that the number of topological types of locally compact countable metric spaces is $\aleph_1$ because every locally compact countable metric space is obtained from $MS(\alpha, n)$, $\alpha < \omega_1$, $1 \leq n < \omega$, by removing a point.

**4. Main theorem.**

**Definition 2.** Let $X$ be a scattered countable metric space. A non-isolated point $x$ of $X$ whose rank $\beta$ is a non-limit ordinal is called a *regular* point if $x$ has a clopen neighborhood base $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ satisfying

$$|(U_m - U_{m+1}) \cap X_{(\beta-1)}| < \omega \quad \text{for every } m.$$

If not, $x$ is called a *singular* point.

A point $x$ of $X$ whose rank $\beta$ is a limit ordinal is called a *regular* point if $x$ has a clopen neighborhood base $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ in $X$ satisfying

$$\mathrm{leng}(U_m - U_{m+1}) < \beta \quad \text{for every } m.$$

If not, $x$ is called a *singular* point.

**Remark.** If a point $x$, whose rank $\beta$ is a non-limit ordinal, is a regular point, then using 0-dimensionality we can choose $U_m$ so that

$$|(U_m - U_{m+1}) \cap X_{(\beta-1)}| = 1 \quad \text{for every } m.$$

If a point $x$, whose rank $\beta$ is a non-limit ordinal, is a singular point, then we can also choose $U_m$ so that

$$|(U_m - U_{m+1}) \cap X_{(\beta-1)}| = \omega \quad \text{for every } m.$$

If a point $x$, whose rank $\beta$ is a limit ordinal, is a singular point, then we can choose $U_m$ so that

$$\mathrm{leng}(U_m - U_{m+1}) = \beta \quad \text{for every } m.$$

The term 'regular' comes from the following fact.

**Proposition 7.** *Every non-isolated point of a locally compact countable metric space $X$ is a regular point.*

Proof. Let $x$ be a non-isolated point of $X$. If $\mathrm{rank}\, x = \beta$ is a non-limit ordinal, take a compact clopen set $U$ so that $U \cap X^{(\beta)} = \{x\}$ and take a clopen neighborhood base $U = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ of $x$. Then $|(U_m - U_{m+1}) \cap X_{(\beta-1)}| < \omega$ for every $m$, because if $|(U_m - U_{m+1}) \cap X_{(\beta-1)}| = \omega$ for some $m$ then $(U_m - U_{m+1}) \cap X_{(\beta-1)}$ would not have an accumulation point. If $\beta$ is a limit ordinal, take $U$ and $U_m$, $m = 1, 2, 3, \ldots$, as above. Then each $U_m - U_{m+1}$ is compact so that $\mathrm{leng}(U_m - U_{m+1})$ is a non-limit ordinal. Thus $\mathrm{leng}(U_m - U_{m+1}) < \beta$ for every $m$. This completes the proof.

**Definition 3.** Let $X$ be a scattered countable metric space of length $\alpha \geq 2$. Let $\Phi$ be a function , with no continuity assumed, of the interval $(0, \alpha)$ to the two points set $\{\mathrm{r}, \mathrm{s}\}$. We define $X$ to have *rankwise uniform type $\Phi$* if

$$\begin{cases} \text{every point of } X_{(\beta)} \text{ is a regular point} & \text{if } \Phi(\beta) = \mathrm{r}, \\ \text{every point of } X_{(\beta)} \text{ is a singular point} & \text{if } \Phi(\beta) = \mathrm{s}. \end{cases}$$

Let $X$ be a scattered countable metric space of length $\alpha$ having rankwise uniform type $\Phi$. Propositins 1, 2 and 3 tell us the following : For each non-limit ordinal $0 < \beta < \alpha$,

$\Phi(\beta) = \mathrm{r}$ is equivalent to

$$\begin{cases} X_{(\beta)} \cup X_{(\beta-1)} \approx n\boldsymbol{r} \ \text{ or } \ n\boldsymbol{r} \oplus \mathbb{N} & \text{if } \beta + 1 = \alpha \text{ and } |X^{(\beta)}| = n \ , \ 1 \leq n < \omega \,, \\ X_{(\beta)} \cup X_{(\beta-1)} \approx \omega\boldsymbol{r} & \text{if otherwise,} \end{cases}$$

and $\Phi(\beta) = \mathrm{s}$ is equivalent to

$$\begin{cases} X_{(\beta)} \cup X_{(\beta-1)} \approx n\boldsymbol{s} & \text{if } \beta + 1 = \alpha \text{ and } |X^{(\beta)}| = n \ , \ 1 \leq n < \omega \,, \\ X_{(\beta)} \cup X_{(\beta-1)} \approx \omega\boldsymbol{s} & \text{if otherwise.} \end{cases}$$

**Remark.** By Proposition 7 every locally compact countable metric space has the rankwise uniform type $\Phi$ taking the value r constantly. To go beyond the length 2, one might expect that this kind of rankwise uniform type is easy to deal with. Unfortunately this is not so. Indeed, a scattered countable metric space of type (3, 1) with the rankwise uniform type $\Phi(2) = \Phi(1) = \mathrm{r}$ admits just five topological types (see [3, Table 1]), three of which are locally compact and the other non-locally compact two are described as below :

$$T = [\,0,\,\omega^2] - \{\omega(2n-1) \,|\, 1 \le n < \omega\,\},$$

$$T' = [\,0,\,\omega^2 2) - \{\omega(2n-1) \,|\, 1 \le n < \omega\,\},$$

with the topologies induced from the order topology of $[\,0,\,\omega_1)$. Furthermore a scattered countable metric space of type (4, 1) with the rankwise uniform type $\Phi(3) = \Phi(2) = \Phi(1) = \mathrm{r}$ admits infinitely many topological types. In fact, $MS(4,\,1) \oplus nT,\ 1 \le n < \omega$, give, with $n$ varying, countably many topological types.

Being far away from locally compact spaces, we have the following results.

**Theorem 2.** *Let $X$ be a scattered countable metric space of type $(\alpha,\,\mathfrak{m})$, $2 \le \alpha < \omega_1$, $1 \le \mathfrak{m} \le \omega$, having a rankwise uniform type $\Phi$. Assume*

$$(*)\quad \Phi(\beta) = \Phi(\beta+1) = r \quad \textit{does not occur.}$$

(1) *In the exceptional case where $\alpha$ is a non-limit ordinal, $\mathfrak{m} < \omega$ and $\Phi(\alpha - 1) = r$, then $X$ has precisely two topological types.*

(2) *If otherwise, the topological type of $X$ is uniquely determined.*

**Corollary 1.** *Let $X$ be a scattered countable metric space of type $(\alpha,\,\mathfrak{m})$, $2 \le \alpha$, $1 \le \mathfrak{m} \le \omega$, with the rankwise uniform type $\Phi$ taking the value $s$ constantly. Then the topological type of $X$ is uniquely determined.*

**Remark.** Let $X$ be a scattered countable metric space with a rankwise uniform type $\Phi$ satisfying (*) and $U$ a clopen set of $X$ of length $\beta \ge 2$. It follows from (a) in Preliminaries that $U$ has the rankwise uniform type $\Phi|(0,\,\beta)$, the restriction to $(0,\,\beta)$ of $\Phi$, and $\Phi|(0,\,\beta)$ satisfies (*) as well. This fact will be used here and there in the proof without explicit mention.

Proof of Theorem 2. We shall prove the theorem by the transfinite induction on length $\alpha$. If $\alpha = 2$ the theorem is assured by Propositions 1,2 and 3. Let $\gamma$ be an ordinal. Assume that the theorem has been proved for every $\alpha < \gamma$.

**In case $\gamma$ and $\gamma - 1$ are both non-limit ordinals and $\Phi(\gamma - 1) = \mathrm{s}$ :** Let $X,\ Y$ be scattered countable metric spaces of type $(\gamma,\,\mathfrak{m})$ with a common rankwise uniform type $\Phi$ satisfying $\Phi(\gamma-1) = \mathrm{s}$ and (*). To show that $X \approx Y$, first consider the case $\mathfrak{m} = 1$. Let $a,\,b$ be the points of $X,\ Y$ respectively having the highest rank $\gamma - 1$ and let

$$X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \quad \text{and} \quad Y = V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$$

be clopen neighborhood bases of $a,\,b$ respectively satisfying

$$|(U_m - U_{m+1}) \cap X_{(\gamma-2)}| = \omega \quad \text{and} \quad |(V_m - V_{m+1}) \cap Y_{(\gamma-2)}| = \omega$$

for every $m$. Note that

$$\mathrm{type}\,(U_m - U_{m+1}) = \mathrm{type}\,(V_m - V_{m+1}) = (\gamma - 1,\,\omega),$$

which implies that these sets are outside the exceptional case (1) even if $\Phi(\gamma-2)=$ r. The induction hypothesis is now applied to give $U_m - U_{m+1} \approx V_m - V_{m+1}$ for every $m$. Taking a homeomorphism $h_m : U_m - U_{m+1} \longrightarrow V_m - V_{m+1}$ we can define a homeomorphism $h : X \longrightarrow Y$ by

$$h(x) = \begin{cases} h_m(x) & \text{if } x \in U_m - U_{m+1} \\ b & \text{if } x = a. \end{cases}$$

The continuity of $h$ at $a$ (or of $h^{-1}$ at $b$) is because $\{U_1, U_2, U_3, \dots\}$ and $\{V_1, V_2, V_3, \dots\}$ are open neighborhood bases of $a$, $b$ respectively.

If $2 \leq \mathfrak{m} \leq \omega$ , we can decompose $X$, using 0-dimensionality, as $X \approx \oplus_{\lambda \in \Lambda} X_\lambda$, where $|\Lambda| = \mathfrak{m}$ and each $X_\lambda$ is of type $(\gamma, 1)$. Since, as proved above, each $X_\lambda$ admits the unique topology, so does $X$. This completes the proof.

**In case $\gamma$ and $\gamma-1$ are both non-limit ordinals and $\Phi(\gamma-1) =$ r :** Let $X$ be a scattered countable metric space of type $(\gamma, 1)$ with a rankwise uniform type $\Phi$ satisfying $\Phi(\gamma-1) =$ r and (*). Since $\Phi(\gamma-1) =$ r, one and only one of the following two cases occurs :

$$X_{(\gamma-1)} \cup X_{(\gamma-2)} \approx \ \mathbf{r} \quad \text{or} \quad \mathbf{r'}$$

Let us denote, for the time being, by $R$ the $X$ in which the former happens and by $R'$ the $X$ in which the latter happens. To show the uniqueness of $R$ let $X,\ Y$ be scattered countable metric spaces of type $(\gamma, 1)$ with the rankwise uniform type $\Phi$ satisfying

$$X_{(\gamma-1)} \cup X_{(\gamma-2)} \approx \mathbf{r} \approx Y_{(\gamma-1)} \cup Y_{(\gamma-2)}.$$

Let $a$, $b$ be the points of $X,\ Y$ respectively having the highest rank $\gamma-1$ and let

$$X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots \quad \text{and} \quad Y = V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$$

be clopen neighborhood bases of $a$, $b$ respectively satisfying

$$|(U_m - U_{m+1}) \cap X_{(\gamma-2)}| = |(V_m - V_{m+1}) \cap Y_{(\gamma-2)}| = 1$$

for every $m$. Note that

$$\text{type}\,(U_m - U_{m+1}) = \text{type}\,(V_m - V_{m+1}) = (\gamma-1,\ 1)$$

and $\Phi(\gamma-2) =$ s by condition (*). This implies that these sets are outside the exceptional case (1) so that $U_m - U_{m+1} \approx V_m - V_{m+1}$ by the induction hypothesis. Now in a simlar way to that in the preceding case, we can define a homeomorphism $h : X \longrightarrow Y$. The uniqueness of $R$ has been thus assured.

To show the uniqueness of $R'$, decompose $R'$ as $R' \approx R \oplus J$ so that

$$\text{type}\, R = (\gamma,\ 1),\ \ R_{(\gamma-1)} \cup R_{(\gamma-2)} \approx \boldsymbol{r} \text{ and type}\, J = (\gamma-1,\ \omega).$$

Then $R$ is unique as proved above and $J$ is unique by the induction hypothesis. Thus $R'$ is unique as desired.

As for $X$ of type $(\gamma, n)$, $1 \leq n < \omega$, write $X = \oplus_{i=1}^{n} X_i$ with $X_i$ of type $(\gamma, 1)$. Then, as proved above, each $X_i$ is homeomorhic to $R$ or $R \oplus J$. If $X_i \approx R$ for every $i$ then $X \approx nR$. If $X_i \approx R \oplus J$ for some $i$, note that the topological sum of finitely (or countably) many $J$'s is homeomorphic to $J$ because of its uniqueness. Thus $X \approx nR \oplus J$. Consequently $X$ has precisely two topological types because $nR$ can not absorb $J$.

To show the uniquness of $X$ of type $(\gamma, \omega)$, write $X = \oplus_{i=1}^{\infty} X_i$ with $X_i \approx R$ or $R \oplus J$ so that $X \approx \omega R$ or $\omega R \oplus J$. To hide $J$ into $\omega R$, write $J = \oplus_{i=1}^{\infty} J_i$ with $J_i$ of type $(\gamma-1,\ 1)$.

Since type$(R \oplus J_i) = (\gamma, 1)$ and $(R \oplus J_i)_{(\gamma-1)} \cup (R \oplus J_i)_{(\gamma-2)} \approx \boldsymbol{r}$ it follows from the uniqueness of R that $R \oplus J_i \approx R$ which yields

$$\omega R \oplus J \approx \overbrace{(R \oplus J_1) \oplus (R \oplus J_2) \oplus (R \oplus J_3) \oplus \cdots}^{\omega} \approx \omega R$$

as desired. This completes the proof.

**In case $\gamma$ is a limit ordinal :** Let $X,\ Y$ be scattered countable metric spaces of length $\gamma$ with a common rankwise uniform type $\varPhi$ satisfying (*) . Write for each $\beta < \gamma$

$$X - X^{(\beta)} = \cup_{m=1}^{\infty} U_m^{\beta},\ Y - Y^{(\beta)} = \cup_{m=1}^{\infty} V_m^{\beta}$$

by clopen sets $U_m^{\beta}$ , $m = 1, 2, 3, \ldots$, of $X$ and $V_m^{\beta}$ , $m = 1, 2, 3, \ldots$ , of $Y$. Rewrite

$$\{ U_m^{\beta} \,|\, \beta < \gamma\ ,\ m = 1, 2, 3, \ldots \} = \{ U_1, U_2, U_3, \ldots \},$$

$$\{ V_m^{\beta} \,|\, \beta < \gamma\ ,\ m = 1, 2, 3, \ldots \} = \{ V_1, V_2, V_3, \ldots \}.$$

Then $\cup_{m=1}^{\infty} U_m = X$, $\cup_{m=1}^{\infty} V_m = Y$ and $\text{leng}(U_m) < \gamma$ and $\text{leng}(V_m) < \gamma$ for every $m$.

Put $\Sigma = \{ \sigma \in (0, \gamma) \,|\, \varPhi(\sigma) = \text{s}\}$. Note that $\Sigma$ is cofinal in the interval $(0, \gamma)$ by the condition (*). We can thus take a function $\rho : \{1, 2, 3, \ldots\} \longrightarrow \Sigma$ satisfying

$$\rho(m) > \max\{\text{leng}(U_m), \text{leng}(V_m)\} \text{ and } \rho(m+1) > \rho(m) \text{ for every } m.$$

For each $\sigma \in \Sigma$, fix $x_\sigma \in X_{(\sigma)}$, $y_\sigma \in Y_{(\sigma)}$, a clopen set $A_\sigma$ of $X$ and a clopen set $B_\sigma$ of $Y$ so that

$$A_\sigma \cap X^{(\sigma)} = \{x_\sigma\} \text{ and } B_\sigma \cap Y^{(\sigma)} = \{y_\sigma\}.$$

Now define

$$E_1 = U_1 \cup A_{\rho(1)},\ E_m = (U_m \cup A_{\rho(m)}) - \cup_{i=1}^{m-1} (U_i \cup A_{\rho(i)}),$$

$$F_1 = V_1 \cup B_{\rho(1)},\ F_m = (V_m \cup B_{\rho(m)}) - \cup_{i=1}^{m-1} (V_i \cup B_{\rho(i)}),$$

for each $m = 2, 3, 4, \ldots$. Then $\{E_1, E_2, E_3, \ldots\}$ and $\{F_1, F_2, F_3, \ldots\}$ are disjoint clopen covers of $X$ and $Y$ respectively. Note that $\text{leng}(E_m) = \text{leng}(F_m) = \rho(m) + 1 < \gamma$ and that $x_{\rho(m)}$ is the only point having the highest rank $\rho(m)$ in the space $E_m$ and so is $y_{\rho(m)}$ in $F_m$. Thus we have type $E_m$ = type $F_m = (\rho(m)+1, 1)$. Since $\varPhi(\rho(m)) = \text{s}$, $E_m$ and $F_m$ are outside the exceptional case. The induction hypothesis is now applied to obtain $E_m \approx F_m$ for every $m$, which yields $X \approx Y$. This completes the proof.

**In case $\gamma$ is a non-limit ordinal, $\gamma - 1$ is a limit ordinal and $\varPhi(\gamma-1) = \mathbf{r}$** Let $X$ be a scattered countable metric space of type $(\gamma, 1)$ with a rankwise uniform type $\varPhi$ satisfying $\varPhi(\gamma - 1) = \text{r}$ and (*). Let $a$ be the point of $X$ having the highest rank $\gamma - 1$. Since $\gamma - 1$ is a limit ordinal and $\varPhi(\gamma - 1) = \text{r}$, one and only one of the following two cases occurs :

(r) $a$ has a clopen neighborhood base $X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ satisfying

$$\text{leng}(U_m - U_{m+1}) < \gamma - 1 \text{ for every } m \text{ or}$$

(r') $a$ has a clopen neighborhood base $X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ satisfying

$$\text{leng}(U_1 - U_2) = \gamma - 1 \text{ and } \text{leng}(U_m - U_{m+1}) < \gamma - 1 \text{ for every } m \geq 2.$$

Let us denote, for the time being, by $R$ the $X$ in which the former happens and by $R'$ the $X$ in which the latter happens. To show the uniqueness of $R$ let $X,\ Y$ be scattered countable metric spaces of type $(\gamma,\,1)$ with the rankwise uniform type $\Phi$ and with $a,\,b$ the points of $X,\ Y$ respectively having the highest rank $\gamma-1$, and let

$$X=U_1\supseteq U_2\supseteq U_3\supseteq\cdots\quad\text{and}\quad Y=V_1\supseteq V_2\supseteq V_3\supseteq\cdots$$

be clopen neighborhood bases of $a,\,b$ respectively satisfying

$$\operatorname{leng}(U_m-U_{m+1})<\gamma-1\quad\text{and}\quad\operatorname{leng}(V_m-V_{m+1})<\gamma-1$$

for every $m$.

Hereafter the proof goes a way somewhat similar to that in the preceding case. Put $\Sigma=\{\,\sigma\in(0,\,\gamma-1)\,|\,\Phi(\sigma)=\mathrm{s}\}$ and take a function $\rho:\{1,\,2,\,3,\,\ldots\}\longrightarrow\Sigma$ satisfying

$$\rho(m)>\max\{\operatorname{leng}(U_m-U_{m+1}),\,\operatorname{leng}(V_m-V_{m+1})\}\quad\text{and}\quad\rho(m+1)>\rho(m)\quad\text{for every } m.$$

For each $\sigma\in\Sigma$, fix $x_\sigma\in X_{(\sigma)},\ y_\sigma\in Y_{(\sigma)}$, a clopen set $A_\sigma$ of $X$ and a clopen set $B_\sigma$ of $Y$ so that

$$A_\sigma\cap X^{(\sigma)}=\{x_\sigma\}\quad\text{and}\quad B_\sigma\cap Y^{(\sigma)}=\{y_\sigma\}.$$

Putting $U'_m=U_m-U_{m+1}$ and $V'_m=V_m-V_{m+1}$, define

$$E_1=U'_1\cup A_{\rho(1)}\ ,\ \ E_m=(U'_m\cup A_{\rho(m)})-\cup_{i=1}^{m-1}\,(U'_i\cup A_{\rho(i)}),$$

$$F_1=V'_1\cup B_{\rho(1)},\ \ F_m=(V'_m\cup B_{\rho(m)})-\cup_{i=1}^{m-1}\,(V'_i\cup B_{\rho(i)}),$$

for each $m=2,\,3,\,4,\,\ldots$. Then $\{E_1,\,E_2,\,E_3,\,\ldots\}$ and $\{F_1,\,F_2,\,F_3,\,\ldots\}$ are disjoint families of clopen sets of $X,\ Y$ respectively, and

$$\{X-\cup_{i=1}^{m}E_i\,|\,m=1,\,2,\,3,\,\ldots\}\quad\text{and}\quad\{Y-\cup_{i=1}^{m}F_i\,|\,m=1,\,2,\,3,\,\ldots\}$$

are clopen neighborhood bases of $a,\,b$ respectively. Since type $E_m=$ type $F_m=(\rho(m)+1,\,1)$ and $\Phi(\rho(m))=\mathrm{s}$ it follows from the induction hypothesis that $E_m\approx F_m$ for every $m$. Taking a homeomorphism $h_m:E_m\longrightarrow F_m$ we can define a homeomorphism $h:X\longrightarrow Y$ by

$$h(x)=\begin{cases} h_m(x) & \text{if } x\in E_m\\ b & \text{if } x=a.\end{cases}$$

Thus $X\approx Y$. The uniqueness of $R$ has been verified.

To show the uniqueness of $R'$ recall that, in (r') above, $\operatorname{leng}(U_1-U_2)=\gamma-1$ and $\gamma-1$ is a limit ordinal. Thus $U_1-U_2$ has the unique topological type by the induction hypothesis. Denoting this type of space by $J$, we have $R'\approx R\oplus J$ which assures the uniqueness of $R'$.

As for $X$ of type $(\gamma,\,n),\ 1\le n<\omega$, write $X=\oplus_{i=1}^{n}X_i$ with $X_i$ of type $(\gamma,\,1)$. Then, as proved above, each $X_i$ is homeomorhic to $R$ or $R\oplus J$. If $X_i\approx R$ for every $i$ then $X\approx nR$. If $X_i\approx R\oplus J$ for some $i$, note that the topological sum of finitely (or countably) many $J$'s is homeomorphic to $J$ because of its uniqueness. We thus have $X\approx nR\oplus J$. Consequently $X$ has precisely two topological types because $nR$ can not absorb $J$.

To show the uniquness of $X$ of type $(\gamma,\,\omega)$, write $X=\oplus_{i=1}^{\infty}X_i$ with $X_i\approx R$ or $R\oplus J$ so that $X\approx\omega R$ or $\omega R\oplus J$. To vanish $J$, write $J=\oplus_{i=1}^{\infty}J_i$ with $\operatorname{leng}(J_i)<\gamma-1$ (; as in the top of the preceding case with $\gamma$ replaced by $\gamma-1$, write $J=\cup_{m=1}^{\infty}U_m$ by clopen sets $U_m$ with leng $(U_m)<\gamma-1$ and put $J_1=U_1$ and $J_i=U_i-\cup_{m=1}^{i-1}U_m$ for $i\ge 2$). Since type $(R\oplus J_i)=(\gamma,\,1)$ and the case (r) happens in $R\oplus J_i$ it follows from the uniqueness of $R$ that $R\oplus J_i\approx R$ so that

$$\omega R\oplus J\approx\overbrace{(R\oplus J_1)\oplus(R\oplus J_2)\oplus(R\oplus J_3)\oplus\cdots}^{\omega}\approx\omega R$$

as desired.

**In case $\gamma$ is a non-limit ordinal, $\gamma - 1$ is a limit ordinal and $\Phi(\gamma - 1) = \mathrm{s}$ :** This is the easiest case. Let $X$ be a scattered countable metric space of type $(\gamma,\, 1)$ with a rankwise uniform type $\Phi$ satisfying $\Phi(\gamma - 1) = \mathrm{s}$ and (*). Let $\{a\} = X^{(\gamma-1)}$ and take a clopen neighborhood base $X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ of $a$ satisfying $\mathrm{leng}(U_m - U_{m+1}) = \gamma - 1$ for every $m$. Then by the induction hypothesis, each $U_m - U_{m+1}$ has the unique topological type so that $X$ does.

As for the type $(\gamma,\, \mathfrak{m})$, $2 \le \mathfrak{m} \le \omega$, we have only to decompose $X$ as $X \approx \oplus_{\lambda\in\Lambda} X_\lambda$, where $|\Lambda| = \mathfrak{m}$ and each $X_\lambda$ is of type $(\gamma,\, 1)$. The uniqueness of $X_\lambda$ implies the uniqueness of $X$, which completes the proof.

**Existence.** Let $\alpha$ be an ordinal with $2 \le \alpha < \omega_1$ and let $\Phi : (0,\, \alpha) \longrightarrow \{\mathrm{r},\, \mathrm{s}\}$ be a function (which does not necessarily satisfy the condition (*)). A scattered countable metric space of length $\alpha$ with the rankwise uniform type $\Phi$ is given in the following way.

Let $\overline{\alpha}$ denote the first limit ordinal not smaller than $\alpha$ and let $\varphi : [\,0,\, \alpha) \longrightarrow [\,0,\, \overline{\alpha})$ be a function satisfying

(1) $\varphi(0) = 0$,

(2) if $\beta$ is a non-limit ordinal, then

$$\varphi(\beta) = \begin{cases} \varphi(\beta - 1) + 1 & \text{if } \ \Phi(\beta) = \mathrm{r} \\ \varphi(\beta - 1) + 2 & \text{if } \ \Phi(\beta) = \mathrm{s}, \end{cases}$$

(3) if $\beta$ is a limit ordinal, then

$$\varphi(\beta) = \begin{cases} \beta & \text{if } \ \Phi(\beta) = \mathrm{r} \\ \beta + 1 & \text{if } \ \Phi(\beta) = \mathrm{s}. \end{cases}$$

It is easily proved that such a function $\varphi$ exists and is unique. Putting $K = MS(\overline{\alpha} + 1,\, 1)$, define

$$X = \cup_{\beta\in[\,0,\,\alpha)}\, K_{(\varphi(\beta))}$$

with the topology induced from $K$. Then $X$ has the rankwise uniform type $\Phi$, $\mathrm{leng}X = \alpha$, and $\mathrm{type}\, X = (\alpha,\, \omega)$ if $\alpha$ is a non-limit ordinal.

To obtain $X'$ of type $(\alpha,\, m)$ with $\alpha$ a non-limit ordinal and $1 \le m < \omega$, take $m$ many points $x_1,\, x_2,\, \ldots\, ,\, x_m$ in $K_{(\varphi(\alpha-1))}$ and take a clopen set $U$ of $K$ so that

$$U \cap K_{(\varphi(\alpha-1))} = \{x_1,\, x_2,\, \ldots\, ,\, x_m\}.$$

Put $X' = U \cap X$.

As for our exceptional case, the spaces $R$ in the proof of Theorem 2 can be obtained in this way. If one requires the spaces $R \oplus J$, define

$$X'' = X' \oplus (\cup_{\beta\in[\,0,\,\alpha-1)}\, K_{(\varphi(\beta))}).$$

We have thus finished the proof of Theorem 2.

**Remark.** Mazurkiewicz and Sierpiński constructed in [2] $2^{\aleph_0}$ many distinct scattered countable metric spaces of length $\omega$ by using the notion of 'lacune'. However the spaces constructed there do not have rankwise uniform types in our terminology.

## References

[1] K. Kuratowski, *Topology vol. II*, Academic Press (1968).

[2] S. Mazurkiewicz and W. Sierpiński, *Contribution à la topologie des ensembles dénombrables*, Fund. Math. **1** (1920), 17-27.

[3] S. Oka, *Topological classification of the scattered countable metric spaces of length* 3, to appear.

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