NON-ZERO-SUM BEST-CHOICE GAMES WHERE TWO STOPS ARE REQUIRED

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ABSTRACT. Suppose that players I and II want to jointly employ two secretaries successively one-by-one from a set of n applicants. Best ability of management (foreign language) is wanted by I (II). We assume that these two kinds of abilities are mutually independent for every applicant. Applicants present themselves one-by-one sequentially. Facing each applicant, each player chooses either to Accept or to Reject. The game ends either when the second time of choice-pair A–A happens getting the payoffs predetermined by the game rule, or when $n-2$ applicants except the last two are rejected. If choice-pair is A–R or R–A, then arbitration comes in and forces players to take the same choice as I's (II's) with probability p $(\bar{p})$, $\frac{1}{2} \leq p \leq 1$. Each player aims to maximize the expected payoff he can get. Explicit solutions are derived to this n-stage game, for the cases where abilities of each applicant are observed as bivariate random variables with full-information and with no-information. Some numerical results are presented.

In the beginnings of Sections 1 and 2, we present results on one-stop best-choice games and then proceed to two-stop games in Sections $(1a) \sim (2b)$.

1. A Non-zero-sum Full-information Best-choice Game.

Let $(X_i, Y_i), i = 1, 2, \cdots, n$, be *i.i.d.* random variables each with bivariate-independent uniform distribution on $[0, 1]^2$ interpreted here as the ability-pair of the i-th applicant jointly observed by I and II in stage i. As each (X_i, Y_i) comes up, players I and II must choose either to Accept (A) or to Reject(R) it expecting that better applicant may come up in the future.

If both players accept the i-th then the game terminates with payoffs X_i to I and Y_i to II. If both players reject the i-th, this is rejected and $(i+1)$-st applicant is presented and the game continues. If one player accepts the i-th and the other reject it, then arbitration comes in and forces players to take the same choice as I's (II's) with probability p $(\bar{p})$. Arbitration is fair (unfair) if $p = (\neq)\ \frac{1}{2}$. We consider that $\frac{1}{2} \leq p \leq 1$ throughout this paper, without losing generality. If $n-1$ applicants are rejected except the last, then players should accept the last one. Each player aims to maximize the expected payoff he can get.

Let (u_n, v_n) be the values of the game $\Gamma_n^{(1)}$, say. The Optimality Equation is

$$(u_n, v_n) = E[\text{eq.val. } M_n(X, Y)] \tag{1.1}$$

where

(1.2) $M_n(x, y) =$

	R	A
R	u_{n-1}, v_{n-1}	$pu_{n-1} + \bar{p}x, pv_{n-1} + \bar{p}y$
A	$px + \bar{p}u_{n-1}, py + \bar{p}v_{n-1}$	x, y

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$$(n \geq 1; u_0 = v_0 = 0)$$

Define state (x, y, n) to mean that n applicants remain to be observed and the first one has just been observed with values x and y.

Theorem 1 (i) *The equilibrium strategy-pair is : In state (x, y, n)*
I *chooses* A (R), *if* $x \geq (<)\ u_{n-1}$, *independently of y,*
II *chooses* A (R), *if* $y \geq (<)\ v_{n-1}$, *independently of x.*
The values of the game $\Gamma_n^{(1)}$ *satisfy the simultaneous recurrence relation*

$$u_n = T_1(u_{n-1}, v_{n-1}), \quad v_n = T_2(u_{n-1}, v_{n-1}) \tag{1.3}$$

where

$$T_1(u, v) = \frac{1}{2}\left\{pu^2 + \bar{p}(2u-1)v + 1\right\}, \quad T_2(u, v) = \frac{1}{2}\left\{\bar{p}v^2 + p(2v-1)u + 1\right\}. \tag{1.4}$$

(ii) $u_n \geq v_n, n \geq 1,$ *as* $n \to \infty, u_n \uparrow n_\infty$ *and* $v_n \uparrow v_\infty$, *where* (u_∞, v_∞) *is a unique root of the system*

$$u = T_1(u, v), \quad v = T_2(u, v)$$

or equivalently

$$pu^2 = (2u-1)(1-\bar{p}v), \quad \bar{p}v^2 = (2v-1)(1-pu) \tag{1.5}$$

For the proof see Ref. [2]. It also contains details for the cases $p = \frac{1}{2}$ and 1.

1a. Maximizing the Sum of Values of Two Accepted Applicants.

Now suppose that players must employ two secretaries from a set of n applicants. If $n-2$ applicants are rejected except the last two, then players should accept these two applicants. Each player aims to maximize the expected value of the sum of two r.v.s he accepts.

Let (U_n, V_n) be the values of the game $\Gamma_n^{(2a)}$, say. The Optimality Equation is

$$(U_n, V_n) = E[\text{eq.val. } M_n(X, Y)] \tag{1.6}$$

where

$$M_n(x, y) = \begin{array}{c|c|c|} & \text{R} & \text{A} \\ \hline \text{R} & U_{n-1}, V_{n-1} & \begin{array}{l} \bar{p}(x+u_{n-1}) + pU_{n-1}, \\ \quad \bar{p}(y+v_{n-1}) + pV_{n-1} \end{array} \\ \hline \text{A} & \begin{array}{l} p(x+u_{n-1}) + \bar{p}U_{n-1}, \\ \quad p(y+v_{n-1}) + \bar{p}V_{n-1} \end{array} & x+u_{n-1}, y+v_{n-1} \\ \hline \end{array} \tag{1.7}$$

$$\left(n \geq 2; U_2 = V_2 = 1, \ \ u_1 = v_1 = \frac{1}{2}\right)$$

and u_{n-1}, v_{n-1} are the eq. values of game $\Gamma_{n-1}^{(1)}$.

Define state $[x, y, n]$ to mean that n applicants remain to be observed in the game $\Gamma_n^{(2a)}$ and the first applicant has just been observed with values x and y.

Theorem 2 *The equilibrium strategy-pair is : In state* $[x, y, n]$
I *chooses* A (R), *if* $x > (\le)\ U_{n-1} - u_{n-1}$, *independently of* y,
II *chooses* A (R), *if* $y > (\le)\ V_{n-1} - v_{n-1}$, *independently of* x.
Values of the game satisfy the simultaneous upward recursion

$$U_n = u_{n-1} + T_1(U_{n-1} - u_{n-1}, V_{n-1} - v_{n-1}), \tag{1.8}$$

$$V_n = v_{n-1} + T_2(U_{n-1} - u_{n-1}, V_{n-1} - v_{n-1}), \tag{1.9}$$

where $T_i(u, v), i = 1, 2$, *are defined by* (1.4).

Proof. The following assertion [A] commonly holds true for the beginning part in the proofs of Theorems 2,3,5 and 6.

Assertion A For $p = 1$, an eq. strategy-pair in state $[x, y, n]$ is :
I chooses A (R), if $x > (\le)\ U_{n-1} - u_{n-1}$,
II always chooses R, either until the earliest A by I happens, or until I rejects all applicants except the last two.

The rest of the proof is for $\frac{1}{2} \le p < 1$. It is easy to find that if $\frac{1}{2} \le p < 1$, then the bimatrix game (1.7), for each $(x, y) \in [0, 1]^2$ has the unique pure-strategy eq. such that

(1.10)

	If $y \le V_{n-1} - v_{n-1}$	$y > V_{n-1} - v_{n-1}$
If $x \le U_{n-1} - u_{n-1}$	R–R U, V	R–A $\bar{p}(x+u)+pU, \bar{p}(y+v)+pV$
$> U_{n-1} - u_{n-1}$	A–R $p(x+u)+\bar{p}U, p(y+v)+\bar{p}V$	A–A $x + u, y + v$

Here the second row in each cell shows payoffs to I and II, where $u_{n-1}, v_{n-1}, U_{n-1}, V_{n-1}$ are abbreviated by those without subscripts.

The first component matrix in (1.10) is

$$U \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (x - U + u) \begin{bmatrix} 0 & \bar{p} \\ p & 1 \end{bmatrix},$$

which, when $E_{(x,y)}$ is taken, becomes

$$\begin{aligned} U \quad + \quad & E_{(x,y)}[(x - U + u)\{p\ I(x > U - u, y \le V - v) + I(x > U - u, y > V - v) \\ & \qquad + \bar{p}\ I(x \le U - u, y > V - v)\}] \\ = \quad & U + p(V - v) \int_{U-u}^{1} (x - U + u)dx + (\bar{V} + v) \int_{U-u}^{1} (x - U + u)dx \\ & \qquad - \bar{p}(\bar{V} + v) \int_{0}^{U-u} (U - u - x)dx. \end{aligned}$$

This is found, after some algebra, to be equal to $U + T_1(U - u, V - v) - (U - u)$ *i.e.*, Eq. (1.8). Eq. (1.9) is analogously derived by starting from the fact that second component matrix in (1.10) is

$$V \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (y - V + v) \begin{bmatrix} 0 & \bar{p} \\ p & 1 \end{bmatrix},$$

and proceeding in the same way. □

1b. Maximizing the Minimum Value of Two Accepted Applicants.

We here consider the case where each player aims to maximize the expected value of the minimum of two r.v.s he accepts.

(U_n, V_n) be the eq. values of the game $\Gamma_n^{(2b)}$, say. The Optimality Equation is

$$(U_n, V_n) = E[\text{eq.val. } M_n(X, Y)], \tag{1.11}$$

where

(1.12) $M_n(x, y) =$

	R	A
R	U_{n-1}, V_{n-1}	$\bar{p}(x \wedge u_{n-1}) + pU_{n-1}$, $\bar{p}(y \wedge v_{n-1}) + pV_{n-1}$
A	$p(x \wedge u_{n-1}) + \bar{p}U_{n-1}$, $p(y \wedge v_{n-1}) + \bar{p}V_{n-1}$	$x \wedge u_{n-1}, y \wedge v_{n-1}$

$$\left(n \geq 3, U_2 = V_2 = E(X_1 \wedge X_2) = \frac{1}{3}, u_2 = \frac{1}{8}p + \frac{1}{2}, v_2 = \frac{1}{8}\bar{p} + \frac{1}{2} \right)$$

and (u_n, v_n) is the values of the game $\Gamma_n^{(1)}$ discussed in Section 1.

Define state $[x, y, n]$ to mean that n applicants remain to be observed in the game $\Gamma_n^{(2b)}$ and the first one has just been observed with values x and y.

Theorem 3 *The eq. strategy-pair is : In state* $[x, y, n]$
I *chooses* A (R)*, if* $x > (\leq) U_{n-1}$, *indep.of* y
II *chooses* A (R)*, if* $y > (\leq) V_{n-1}$, *indep.of* x.
The values (U_n, V_n) *of the game* $\Gamma_n^{(2b)}$ *satisfy the simultaneous upward recursion*

$$U_n = R_1(U_{n-1}, V_{n-1}), \quad V_n = R_2(U_{n-1}, V_{n-1}) \tag{1.13}$$

where

$$R_1(U, V) = \frac{1}{2}pU^2 + \bar{p}UV + (u_{n-1} - \frac{1}{2}u_{n-1}^2)(1 - \bar{p}V), \tag{1.14}$$

$$R_2(U, V) = \frac{1}{2}\bar{p}V^2 + pUV + (v_{n-1} - \frac{1}{2}v_{n-1}^2)(1 - pU), \tag{1.15}$$

Proof. Assertion [A] is true with $U_{n-1} - u_{n-1}$ replaced by U_{n-1} (see proof of Theorem 2). It is easy to find that the bimatrix game (1.12) with $\frac{1}{2} \leq p < 1$, for each $(x, y) \in [0, 1]^2$ has the unique pure-strategy eq. such that

(1.16)

	If $y \leq V_{n-1}$	$y > V_{n-1}$
If $x \leq U_{n-1}$	R–R U, V	R–A $\bar{p}(x \wedge u) + pU, \bar{p}(y \wedge v) + pV$
$x > U_{n-1}$	A–R $p(x \wedge u) + \bar{p}U, p(y \wedge v) + \bar{p}V$	A–A $x \wedge u, y \wedge v$

The first component matrix in (1.16) is

$$U\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + (x \wedge u - U)\begin{bmatrix}0 & \bar{p}\\ p & 1\end{bmatrix},$$

which, when $E_{(x,y)}$ is taken, become

$$\begin{aligned} U \quad &+ \quad E_{(x,y)}[(x \wedge u - U)\{p\ I(x > U) + \bar{p}\ I(y > V)\}] \\ &= \quad U + (p + \bar{p}\bar{V}) \int_U^1 (x \wedge u - U)dx - \bar{p}\bar{V} \int_0^U (U - x)dx \end{aligned}$$

and finally this becomes $R_1(U, V)$ defined by (1.14).

The second component matrix in (1.16) is

$$V\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + (y \wedge v - V)\begin{bmatrix}0 & \bar{p}\\ p & 1\end{bmatrix},$$

which, when $E_{(x,y)}$ is taken, turns out to be equal to $R_2(U, V)$ defined by (1.15). Thus the proof is complete. $\square$

In Remark 3 in Section 3 a numerical example is presented.

2. A Non-zero-sum Best-choice Game Related to Secretary Problem.

Suppose that player I (Vice-president) and II (another Vice-president) want to jointly employ one secretary from a set of n applicants. The nice ability of management (foreign language) is wanted by I (II). We assume that these two kinds of abilities are mutually independent for each applicant. Players observe an independent sequence $\{(Y_i, Z_i)\}_{i=1}^n$ of bivariate r.v.s., one-by-one sequentially, which obeys probability distribution

$$Pr(Y_i = y, Z_i = z) = i^{-2}, \ \ \forall y, z \in \{1, 2, \cdots, i\}.$$

For a case where Y_i and Z_i are dependent, see Section 3 in Ref. [4].

After observing $(Y_i, Z_i) = (y, z)$ jointly by I and II, for the i-th applicant, each player chooses either A or R for this applicant. The game is played as described in Section 1, but with a difference that the "losses" to the players are $Q(i, y), Q(i, z)$, when the choice-pair is A–A for the i-th applicant. [Notice that for each kind of applicant's ability $Q(i, y) \equiv \frac{n+1}{i+1}y$ is the expected absolute rank for the i-th among n, under the condition that her (or his) relative rank relative to those who have already seen is y.] If all applicants except the last have been rejected, then A–A should be chosen for the last applicant. Each player aims to minimize the expected loss he can get. [*c.f.* The best(worst) among n has rank $1(n)$]

Define state (i, y, z) to mean that (1) the first $i-1$ applicants have been rejected and players face the i-th applicant, and (2) players jointly observe $Y_i = y$ and $Z_i = z$.

Let u_i, v_i be the equilibrium values for the n-stage game $G_n^{(1)}$, say, after the first i applicants have been rejected. The game horizon n is at present omitted in our notation for simplicity. Then it is clear that the Optimality Equation is given by

$$(u_{i-1}, v_{i-1}) = i^{-2} \sum_{y,z=1}^{i} \text{eq.val. } M_i(y, z) \tag{2.1}$$

where

$$(2.2)\qquad M_i(y,z) = \begin{array}{c|c|c|} & \text{R} & \text{A} \\ \hline \text{R} & u_i, v_i & \bar{p}Q(i,y)+pu_i, \bar{p}Q(i,z)+pv_i \\ \hline \text{A} & pQ(i,y)+\bar{p}u_i, pQ(i,z)+\bar{p}v_i & Q(i,y), Q(i,z) \\ \hline \end{array}$$

$$\left(i = n-1, \cdots, 2, 1\ ;\ u_{n-1} = v_{n-1} = n^{-1} \sum_{y=1}^{n} y = \frac{n+1}{2} \right)$$

Theorem 4 *The eq. strategy-pair is : In state* (i, y, z)*,*
I *chooses* A (R)*, if* $Q(i,y) \leq (>)\ u_i$ *indep. of* z*,*
II *chooses* A (R)*, if* $Q(i,z) \leq (>)\ v_i$ *indep. of* y*.*
The values u_i *and* v_i *satisfy the simultaneous downward recursion*

$$(2.3)\qquad u_{i-1} = pE[Q(i,Y_i) \wedge u_i] + \bar{p}E\left[\frac{n+1}{2} I(Q(i,Z_i) \leq v_i) + u_i I(Q(i,Z_i) > v_i)\right]$$

$$(2.4)\qquad v_{i-1} = \bar{p}E[Q(i,Z_i) \wedge v_i] + pE\left[\frac{n+1}{2} I(Q(i,Y_i) \leq u_i) + v_i I(Q(i,Y_i) > u_i)\right].$$

The eq. values of the game $G_n^{(1)}$ *are* $u_0, v_0 (\equiv u^{(n)}, v^{(n)}$*, say).*

For the proof and a numerical example see Ref. [4].

2a. Minimizing the Sum of Losses for Two Accepted Applicants.

Now suppose that players want to employ two secretaries from a set of n applicants.

If $n-2$ applicants are rejected except the last two, then players must accept these two. Each player aims to minimize the sum of the expected losses by the two r.v.s he accepts.

Define state $[i, y, z]$ to mean that (1) the first $i-1$ applicants have been rejected and players face the i-th applicant, and (2) players jointly observe $Y_i = y$ and $Z_i = z$. Let U_i, V_i be the eq. values for the game (denoted by $G_n^{(2a)}$) after the first i applicants have been rejected. Optimality Equation is evidently

$$(2.5)\qquad (U_{i-1}, V_{i-1}) = i^{-2} \sum_{y,z=1}^{i} \text{eq.val. } M(y,z),$$

$$(2.6)\qquad M_i(y,z) = \begin{array}{c|c|c|} & \text{R} & \text{A} \\ \hline \text{R} & U_i, V_i & \begin{array}{l} pU_i + \bar{p}(Q(i,y)+u_i), \\ \qquad pV_i + \bar{p}(Q(i,z)+v_i) \end{array} \\ \hline \text{A} & \begin{array}{l} p(Q(i,y)+u_i) + \bar{p}U_i, \\ \qquad p(Q(i,z)+v_i) + \bar{p}V_i \end{array} & Q(i,y)+u_i, Q(i,z)+v_i \\ \hline \end{array}$$

$$(i = n-2, \cdots, 2, 1; U_{n-2} = V_{n-2} = n+1)$$

and u_i and v_i are the conditional values of the game $G_n^{(1)}$, given that the first i applicants have been rejected. Here notice that

$$U_{n-2} = E[Q(n-1, Y_{n-1}) + Q(n, Y_n)] = \frac{n+1}{2} + \frac{n+1}{2} = n+1.$$

Theorem 5 *The equilibrium strategy-pair is : In state* $[i, y, z]$,
I *chooses* A (R), *if* $Q(i,y) \le (>)\ U_i - u_i$ *indep. of* z,
II *chooses* A (R), *iff* $Q(i,z) \le (>)\ V_i - v_i$ *indep. of* y.
The values U_i *and* V_i *satisfy the simultaneous downward recursion*

$$(2.7)\quad \begin{aligned} U_{i-1} &= pE[(Q(i,Y_i)+u_i)\wedge U_i] \\ &+ \bar{p}E\left[\left(\frac{n+1}{2}+u_i\right) I(Q(i,Z_i)+v_i \le V_i) + U_i I(Q(i,Z_i)+v_i > V_i)\right] \end{aligned}$$

$$(2.8)\quad \begin{aligned} V_{i-1} &= \bar{p}E[(Q(i,Z_i)+v_i)\wedge V_i] \\ &+ pE\left[\left(\frac{n+1}{2}+v_i\right) I(Q(i,Y_i)+u_i \le U_i) + V_i I(Q(i,Y_i)+u_i > U_i)\right] \end{aligned}$$

The equilibrium values of the game $G_n^{(2a)}$ *are* $U_0, V_0 (= U^{(n)}, V^{(n)}$, *say).*

Proof. Assertion [A] (in the proof of Theorem 2) is true, if the condition $x > (\le) U_{n-1} - u_{n-1}$ is replaced by $Q(i,y) \le (>) U_i - u_i$ and the state $[x, y, n]$ replaced by $[i, y, z]$.

It is easy to find that, for $\frac{1}{2} \le p < 1$ the bimatrix game (2.6) for each $(y, z) \in \{1, \cdots, i\} \times \{1, 2, \cdots, i\}$ has the unique pure-strategy eq. such that

(2.9)

	If $Q(i,z) \le V_i - v_i$	$Q(i,z) > V_i - v_i$
If $Q(i,y) \le U_i - u_i$	A–A $Q(i,y)+u, Q(i,z)+x$	A–R $p(Q(i,y)+u)+\bar{p}U$, $p(Q(i,z)+v)+\bar{p}V$
$> U_i - u_i$	R–A $pU + \bar{p}(Q(i,y)+u)$, $pV+\bar{p}(Q(i,z)+v)$	R–R U, V

Here the second row in each cell shows the losses to I and II, where subscripts in u_i, v_i, U_i, V_i are omitted for simplicity.

The first component matrix in (2.9) is

$$(Q(i,y)+u)\begin{bmatrix} 1 & p \\ \bar{p} & 0 \end{bmatrix} + U \begin{bmatrix} 0 & \bar{p} \\ p & 1 \end{bmatrix},$$

which, when $i^{-2}\sum_{y,z=1}^{i}$ is taken, becomes

$$(2.10)\qquad i^{-2}\sum_{y,z=1}^{i}(Q(i,y)+u)\{p\ I(Q(i,y)+u \le U) + \bar{p}\ I(Q(i,z)+v \le V)\}$$

$$+i^{-2}U\sum_{y,z=1}^{i}\{p\ I(Q(i,y)+u > U)\ \bar{p}\ I(Q(i,z)+v > V)\}$$

The first sum is equal to

(2.11)

$$p\ i^{-1}\sum_{y=1}^{i}(Q(i,y)+u)\ I(Q(i,y)+u \le U) + \bar{p}\left(\frac{n+1}{2}+u\right) i^{-1}\sum_{z=1}^{i} I(Q(i,z)+v \le V)$$

since $i^{-1}\sum_{y=1}^{i} Q(i,y) = \frac{n+1}{2}$. The second sum in (2.10) is equal to

$$i^{-1}U\left[p\sum_{y=1}^{i} I(Q(i,y)+u>U)+\bar{p}\sum_{z=1}^{i} I(Q(i,z)+v>V)\right]. \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.10), we obtain (2.7).

By starting from the fact that the second component matrix in (2.9) is

$$(Q(i,z)+v)\begin{bmatrix} 1 & p \\ \bar{p} & 0 \end{bmatrix} + V\begin{bmatrix} 0 & \bar{p} \\ p & 1 \end{bmatrix},$$

and proceeding in the same way as above, we obtain (2.8).

This completes the proof of the theorem. □

2b. Minimizing Maximum Loss of Two Accepted Applicants.

Let us consider the case where each player aims to minimize the maximum loss of two r.v.s he accepts. Let state $[i,y,z]$ and values u_i, v_i are defined as the same as in Section 2.

Let U_i, V_i be the eq. values for the game(denoted by $G_n^{(2b)}$, say) after the first i applicants have been rejected. Then

$$(U_{i-1}, V_{i-1}) = i^{-2}\sum_{y,z=1}^{i} \text{eq. val. } M_i(y,z) \tag{2.13}$$

where

$$M_i(y,z) = \begin{array}{c|c|c} & \text{R} & \text{A} \\ \hline \text{R} & U_i, V_i & \begin{array}{r} pU_i+\bar{p}(Q(i,y)\vee u_i), \\ pV_i+\bar{p}(Q(i,z)\vee v_i) \end{array} \\ \hline \text{A} & \begin{array}{r} p(Q(i,y)\vee u_i)+\bar{p}U_i, \\ p(Q(i,z)\vee v_i)+\bar{p}V_i \end{array} & Q(i,y)\vee u_i, Q(i,z)\vee v_i \end{array} \tag{2.14}$$

$$(i = n-2,\cdots,2,1; U_{n-2} = V_{n-2})$$

instead of (2.6), and

$$\begin{aligned} U_{n-2} &= E[Q(n-1,Y_{n-1})\vee Q(n,Y_n)] = E\left[\left(\frac{n+1}{n}Y_{n-1}\right)\vee Y_n\right] \\ &= \frac{1}{n(n-1)}\left\{\sum_{y=1}^{n} y\left[\frac{ny}{n+1}\right] + \frac{n+1}{n}\sum_{z=1}^{n-1} z\left[\frac{n+1}{n}z\right]\right\}. \end{aligned} \tag{2.15}$$

We shall prove

Theorem 6 *The eq. strategy-pair is : In state* $[i,y,z]$
I *chooses* A (R), *if* $Q(i,y)\le(>)\ U_i$, *indep. of* z
II *chooses* A (R), *if* $Q(i,z)\le(>)\ V_i$, *indep. of* y.
The values U_i *and* V_i *satisfy the simultaneous downward recursion*

$$\begin{aligned} U_{i-1} &= pE[(Q(i,Y_i)\wedge U_i)+(u_i-Q(i,Y_i))\ I(Q(i,Y_i)\le u_i)] \\ &+ \bar{p}E[E(Q(i,Y_i)\vee u_i)\ I(Q(i,Z_i)\le V_i)+U_iI(Q(i,Z_i)>V_i)], \end{aligned} \tag{2.16}$$

$$
\begin{aligned}
(2.17)\qquad V_{i-1} &= \bar{p}E[(Q(i,Z_i)\wedge V_i)+(v_i-Q(i,Z_i))\ I(Q(i,Z_i)\le v_i)] \\
&+ pE[E(Q(i,Z_i)\vee v_i)\ I(Q(i,Y_i)\le U_i)+V_iI(Q(i,Y_i)>U_i)].
\end{aligned}
$$

The eq. values for the game $G_n^{(2b)}$ *are equal to* $U_0, V_0(\equiv U^{(n)}, V^{(n)}$*, say)*

Proof. Assertion [A] is true, if the condition $x\ge(<)U_{n-1}-u_{n-1}$ is replaced by $Q(i,y)\le(>)U_i$, and the state $[x,y,n]$ replaced by $[i,y,z]$. Notice that $1\le u_i<U_i$ and $1\le v_i<V_i,\forall i$.

The rest of the proof is for the case $\frac{1}{2}\le p<1$. It is easy to find that the bimatrix game (2.14) for each (y,z), has the unique pure-strategy eq. such that

(2.18)

	If $Q(i,z)\le V_i$	$Q(i,z)>V_i$
If $Q(i,y)\le U_i$	A–A $Q(i,y)\vee u, Q(i,z)\vee v$	A–R $p(Q(i,y)\vee u)+\bar{p}U$, $p(Q(i,z)\vee v)+\bar{p}V$
$Q(i,y)>U_i$	R–A $pU+\bar{p}(Q(i,y)\vee u)$, $pV+\bar{p}(Q(i,z)\vee v)$	R–R U,V

($c.f.$ Subscripts in $u_i, v_i, U_i, V_i$are omitted)

The first component matrix in (2.19) is

$$
(Q(i,y)\vee u)\begin{bmatrix}1 & p\\ \bar{p} & 0\end{bmatrix}+U\begin{bmatrix}0 & \bar{p}\\ p & 1\end{bmatrix},
$$

which, when $i^{-2}\sum_{y,z=1}^{i}$ is taken, becomes

$$
\begin{aligned}
& i^{-2}\sum_{y,z=1}^{i}(Q(i,y)\vee u)\{I(Q(i,y)\le U,Q(i,z)\le V) \\
&\qquad\qquad +p\ I(Q(i,y)\le U,Q(i,z)>V)+\bar{p}\ I(Q(i,y)>U,Q(i,z)\le V)\} \\
+\ & i^{-2}\sum_{y,z=1}^{i}U\{\bar{p}\ I(Q(i,y)\le U,Q(i,z)>V) \\
&\qquad\qquad +pI(Q(i,y)>U,Q(i,z)\le V)+I(Q(i,y)>U,Q(i,z)>V)\}.
\end{aligned}
$$

The first and second double sums become

$$
i^{-1}\sum_{y=1}^{i}(Q(i,y)\vee u)\left\{p\ I(Q(i,y)\le U)+\bar{p}i^{-1}\sum_{z=1}^{i}I(Q(i,z)\le V)\right\}
$$

and

$$
i^{-1}U\left\{p\sum_{y=1}^{i}I(Q(i,y)>U)+\bar{p}\sum_{z=1}^{i}I(Q(i,z)>V)\right\},
$$

respectively. Therefore the above equations and (2.5) give

$$
\begin{aligned}
U_{i-1} &= pi^{-1}\sum_{y=1}^{i}\{(Q(i,y)\vee u)\ I(Q(i,y)\le U)+UI(Q(i,y)>U)\} \\
&+ \bar{p}i^{-1}\sum_{y=1}^{i}\left\{i^{-1}\sum_{y=1}^{i}(Q(i,y)\vee u)\ I(Q(i,z)\le V)+UI(Q(i,z)>V)\right\}
\end{aligned}
$$

The inside of $\{\cdots\}$ in the first sum in the r.h.s. is

$$\begin{aligned} & \{Q(i,y)+(u-Q(i,y))^+\}I[Q(i,y)\le U)+UI(Q(i,y)>U) \\ = \; & Q(i,y)\wedge U+(u-Q(i,y))^+\{I(Q(i,y)\le u)+I(u<Q(i,y)\le U)\} \\ = \; & Q(i,y)\wedge U+(u-Q(i,y))I(Q(i,y)\le u) \end{aligned}$$

and thus we finally have Eq. (2.16).

By starting from the fact the second component matrix in (2.18) is

$$(Q(i,z)\vee v)\begin{bmatrix} 1 & p \\ \bar{p} & 0 \end{bmatrix}+V\begin{bmatrix} 0 & \bar{p} \\ p & 1 \end{bmatrix},$$

and proceeding in the same way as above we can obtain (2.17). This completes the proof of the theorem. □

A computational result is given in Remark 4 in Section 3.

3. Remarks.

Remark 1 The problem in this paper is a model of the secretary problem combined with the best-choice sequential game. One of the earliest and fundamental literature on secretary problem is Ref. [1]. In Ref. [2, 3] the full-information best-choice games are investigated.

Remark 2 Theorems 1 and 4 are fundamental to the arguments in Section 1 and Section 2, respectively. $\left\{\begin{array}{l}\text{Eq. (1.8) in Theorem 2}\\ \text{Eq. (1.13)-(1.14) in Theorem 3}\end{array}\right\}$ goes back to (1.3)-(1.4) in Theorem 1, if we take $\left\{\begin{array}{l}u_n=v_n=0\\ u_n=v_n=1\end{array}\right\}$, an easy-to-understand result.

Also $\left\{\begin{array}{l}\text{Eq. (2.8)-(2.9) in Theorem 5}\\ \text{Eq. (2.16)-(2.17) in theorem 6}\end{array}\right\}$ reduces to (2.3)-(2.4) in Theorem 4, if we take $\left\{\begin{array}{l}u_i=v_i=0\\ u_i=v_i=1\end{array}\right\}$.

Remark 3 We present some numerical results related to Theorem 3 in Section 1b, and Theorem 6 in Section 2b. Tables 1 and 2 are computed from Eq. (1.3)-(1.4) in Theorem 1, and Eq. (1.13)-(1.14) in Theorem 3, respectively.

Table 1. Eq. values of the game $\Gamma_n^{(1)}$ for various p and n

	$p=0.5$	0.6		1
n	$u_n=v_n$	u_n	v_n	$u_n(v_n=\frac{1}{2})$
1	0.5	0.5	0.5	0.5
2	0.5625	0.575	0.55	0.625
3	0.5967	0.6157	0.5778	0.6953
4	0.6179	0.6405	0.5955	0.7417
5	0.6319	0.6565	0.6076	0.7751
6	0.6415	0.6673	0.6162	0.8004
7	0.6483	0.6748	0.6225	0.8203
8	0.6531	0.6801	0.6271	0.8365
9	0.6566	0.6839	0.6305	0,8498
10	0.6592	0.6867	0.6331	0.8611
Limit	$\frac{2}{3}$	0.6946	06408	1

(Reproduced from Table 3 in Ref. [2])

Table 2. Eq. values of the game $\Gamma_n^{(2b)}$ in Theorem 3.

	$p=0.5$	$p=0.6$		$p=1$	
n	$U_n=V_n$	U_n	V_n	U_n	V_n
2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
3	0.4203	0.4328	0.4079	0.4852	0.3611
4	0.4632	0.4835	0.4434	0.5713	0.3683
5	0.4890	0.5140	0.4648	0.6298	0.3712
6	0.5059	0.5338	0.4791	0.6731	0.3726
7	0.5175	0.5472	0.4892	0.7066	0.3734
8	0.5256	0.5566	0.4964	0.7335	0.3739
9	0.5315	0.5631	0.5018	0,7556	0.3742
10	0.5357	0.5679	0.5058	0.7742	0.3744
Limit	0.5477**			1	$\frac{3}{8}(=0.375)$

** Smaller root of the equation $\frac{27}{4}U^2 - 11U + 4 = 0$.

From Table 2, the eq. play when $n = 10$ and $p = 0.6$ is : In state $[x, y, 10]$
I chooses A (R), if $x > (\leq)\ U_9 = 0.5631$
II chooses A (R), if $y > (\leq)\ V_9 = 0.5018$

If A–A (R–R) happens in state $[x, y, 10]$ either by their choice-pair itself, or by the outcome of arbitration, then players follow the eq. strategies in state $(x', y', 9)$ ($[x', y', 9]$) if the next r.v is x', y'. The eq. values are $U_{10} = 0.5679, V_{10} = 0.5058$.

Remark 4 A numerical result is presented related with Theorems 4 and 6. Table 3 gives solutions to the 7-stage game $G_7^{(1)}$ for various p computed from (2.3)-(2.4) in Theorem 4. Table 4 shows solutions to $G_7^{(2b)}$ for various p, where computation is based on Eq. (2.17)-(2.18) in Theorem 6.

Table 3 Solutions to the 7-stage game $G_7^{(1)}$ for various p.

	$p=0.5$	$p=0.75$		$p=1$	
$u^{(7)}, v^{(7)}$	3.257	2.874	3.608	2.276	4
stage	Each player accepts iff $Y_i, Z_i =$	I accepts iff $Y_i =$	II accepts iff $Z_i =$	II always rejects. I accepts iff $Y_i =$	
$i = 1$	none	none	none	none	
2	1	1	1	none	
3	1	1	1	1	
4	1, 2	1	1, 2	1	
5	1, 2	1, 2	1, 2	1, 2	
6	1, 2, 3	1, 2, 3	1, 2, 3	1, 2, 3	

(Reproduced from Table in Ref. [4])

Table 4. Solutions to the 7-stage game $G_7^{(2b)}$ for various p.

	p=0.5			p=0.75					
$U^{(7)}, V^{(7)}$	3.907			3.537			4.243		
	$\frac{i+1}{8}u_i$	μ_i	$\frac{i+1}{8}U_i$	$\frac{i+1}{8}u_i$	$\frac{i+1}{8}v_i$	μ_i	ν_i	$\frac{i+1}{8}U_i$	$\frac{i+1}{8}V_i$
i =6	3.5	4.86		3.5	3.5	4.86	4.86		
5	2.68	4.63	3.88	2.52	2.84	4.54	4.72	3.857	3.857
4	2.09	4.47	2.85	1.88	2.29	4.35	4.63	2.717	2.974
3	1.69	4.43	2.12	1.44	1.79	4.29	4.53	1.941	2.287
2	1.19	4.25	1.54	1.03	1.33	4.04	4.44	1.385	1.662
1	0.81	4	0.98	0.72	0.90	4	4	0.846	1.081
	Each player accepts iff Y_i, Z_i =			I accepts iff Y_i =				II accepts iff Z_i =	
i =1	none			none				1	
2	1			1				1	
3	1, 2			1				1, 2	
4	1, 2			1, 2				1, 2	
5	1, 2, 3			1, 2, 3				1, 2, 3	

	p=1				
	2.913			4.734	
	$\frac{i+1}{8}u_i$	$\frac{i+1}{8}v_i$	ν_i	$\frac{i+1}{8}U_i$	$\frac{i+1}{8}V_i$
i =6	3.5	3.5	4.857		
5	2.36	3	4.8	3.857	3.857
4	1.68	2.5	4.8	2.572	3.086
3	1.21	2	4.667	1.765	2.434
2	0.85	1.5	4.667	1.185	1.801
1	0.57	1	4	0.728	1.184
	II always rejects. I accepts iff Y_i =				
i =1	none				
2	1				
3	1				
4	1, 2				
5	1, 2, 3				

Here $\mu_i \equiv E[Q(i, Y_i) \vee u_i]$, $\nu_i \equiv E[Q(i, Z_i) \vee v_i]$ and $U_5 = V_5 = 5.143$ by computing (2.15) for $n = 7$. u_i and v_i are computed from (2.3)-(2.4).

From Table 4 we observe the following. The eq. play in $G_7^{(2b)}$ for $p = 0.75$ is :
In state $[1, 1, 1]$ *i.e.*, at the beginning, I rejects and II accepts, and so arbitration forces players either to go to the second stage with prob. 0.75, facing the next state $[2, Y_2, Z_2]$, or to accept $Y_1 = Z_1 = 1$, with prob. 0.25, continuing to the next state $(2, Y_2, Z_2)$. The eq. values of the game are 3.537, 4.243.

Remark 5 The order relations and asymptotic behavior (as are mentioned in part (ii) of Theorem 1) of ① U_n, V_n in Theorems 2 and 3, and ② $u^{(n)}, v^{(n)}, U^{(n)}, V^{(n)}$ in Theorems 4, 5 and 6 remain to be investigated. It is not easy especially for ②.

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Each of which contains further references.

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