MULTISTAGE NON-ZERO-SUM ARBITRATION GAME

MINORU SAKAGUCHI*

Received February 10, 2003

Abstract. By introducing a specified definition of the equilibrium values of two-person two-choice games, a non-zero-sum multistage arbitration game is formulated and solved. At each random offer $X_i, i = 1, 2, \cdots, n$, comes up, two players must decide either to accept it terminating the game, or to reject it expecting that a larger random value may come up in the near future. Arbitration comes in when they choose different choices. Each player aims to maximize the expected reward he can get. It is shown that if X_i is unifomly distributed in $[0, 1]$, then even when arbitration stands 100 percent in favor of the accepting side, the advantage for the players is only one percent. It is also shown that players are more advantageous when arbitration favors the rejecting side than when it favors the accepting side.

1 Problem. Let $X_i, i = 1, 2, \cdots, n$, be *i.i.d.* random variables each with uniform distribution on $[0, 1]$. As each X_i comes up, each player I and II must choose simultaneously and independently of other player's choice, either to accept (A) or to reject (R) it. If the choice-pair is A-A, they get $\frac{1}{2}X_i$ each, and the game terminates. If the choice-pair is R-R, X_i is rejected and the next X_{i+1} is presented and the game continues. If the players choices are different, arbitration comes in and forces players to divide at $100p$ ($\bar{p}$) percent in favor of the accepting (rejecting) side, and the game terminates. If all of the first $n-1$ random values are rejected, both players must accept the n-th. Each player aims to maximize the expected reward he can get, and the problem is to find a reasonable solution to this two-person competitive n-stage game.

Let u_n be the CEV(common equilibrium value) of the game(*c.f.*, the game is symmetric for the players). The Optimality Equation is

$$(u_n, u_n) = E[\text{eq.val. } \mathbf{M}_n(X)] \quad \left(n \geq 1, u_1 = \frac{1}{4}\right) \tag{1.1}$$

where the payoff matrix is

$$\mathbf{M}_n(x) = \begin{array}{c|c|c|} & R & A \\ \hline R & u_{n-1}, u_{n-1} & \bar{p}x, px \\ \hline A & px, \bar{p}x & \frac{x}{2}, \frac{x}{2} \\ \hline \end{array} \tag{1.2}$$

As is well-known in the Nash theory of competitive games, the equilibrium is often undetermined even in the two-person two-choice games, which we investigate in the present article. So we present the following assumption.

Assumption A *If equilibrium consists of some corner and/or edge and a unique inner point, then the latter is adopted for the equilibrium. If eq. consists of a single point, either corner or inner point, this is adopted for the equilibrium.*

2000 *Mathematics Subject Classification.* 60G40, 90C39, 90D10.
Key words and phrases. Optimal stopping game, Nash equilibrium, two-person competitive game, arbitration.

When players choose different choices, and arbitration comes in, there are two scenes : (1) Arbitrator recognizes which player (I or II) chooses which (R or A), and, (2) arbitrator only knows that one player chooses R and the other player A. A remarkable feature involved in the game discussed in the present paper is the fact that arbitration comes in by (2) not by (1). The literatures [1, 2, 3, 4] in arbitration games in the past discusses about the arbitration by (1). Three-person arbitration games of "odd-man-wins" and "odd-man-out" [5] by the present author, is also by (2).

In Sections [2∼4] of the present paper, the solution of the problem under the Assumption A is derived. We discuss in Section 3 the case where arbitration stands in favor of the accepting side, *i.e.*, $\frac{1}{2} < p \leq 1$, and in Section 4, it stands in favor of the rejecting side, *i.e.*, $0 \leq p < \frac{1}{2}$.

2 Related Bimatrix Games. Rewriting (1.2) we have, if $p \neq \frac{1}{2}$,

$$\mathbf{M}_n(x) = \frac{x}{2}\mathbf{E} + \left(p - \frac{1}{2}\right) x \Big[\mathbf{M}(c)\Big]_{c=(x^{-1}u_{n-1}-1/2)/(p-1/2)}, \tag{2.1}$$

where

$$\mathbf{M}(c) = \begin{array}{c|c|c|} & \text{R} & \text{A} \\ \hline \text{R} & c, c & -1, 1 \\ \hline \text{A} & 1, -1 & 0, 0 \\ \hline \end{array} \quad \text{and} \quad \mathbf{E} = \begin{array}{|c|c|} \hline 1,1 & 1,1 \\ \hline 1,1 & 1,1 \\ \hline \end{array} \tag{2.2}$$

Therefore, if we define

$$V(c) = \text{CEV of } \mathbf{M}(c), \quad \widetilde{V}(c) = \text{CEV of } -\mathbf{M}(c) \tag{2.3}$$

then from (1.1) and (2.1) we have

$$u_n = \frac{1}{4} + \left(p - \frac{1}{2}\right) \int_0^1 xV\left(\frac{x^{-1}u_{n-1} - 1/2}{p - 1/2}\right) dx, \quad \text{if } \frac{1}{2} < p \leq 1; \tag{2.4}$$

$$u_n = \frac{1}{4} + \left(\frac{1}{2} - p\right) \int_0^1 x\widetilde{V}\left(\frac{x^{-1}u_{n-1} - 1/2}{p - 1/2}\right) dx, \quad \text{if } 0 \leq p < \frac{1}{2}. \tag{2.5}$$

We have, analogously, if $p = \frac{1}{2}$,

$$u_n = \frac{1}{4} + \int_0^1 x\left\{\text{CEV of } \left[\mathbf{M}^0(b)\right]_{b=x^{-1}u_{n-1}-1/2}\right\} dx \tag{2.6}$$

where

$$\mathbf{M}^0(b) = \begin{array}{c|c|c|} & \text{R} & \text{A} \\ \hline \text{R} & b, b & 0, 0 \\ \hline \text{A} & 0, 0 & 0, 0 \\ \hline \end{array} \tag{2.7}$$

In (2.4)∼(2.6) $u_n - 1/4$ represents players' merit which comes from our n-stage game $(n > 1)$.

Lemma 1 *The eq.val. of the bimatrix game* $\mathbf{M}(c)$ *:*
If $c < 1$, A-A *is a unique eq. point* (EP), *and* CEV *is zero ;*
If $c > 1$, *there are two* EPs, R-R, *and* A-A *and a unique common mixed strategy* (R, A ; $c^{-1}, 1 - c^{-1}$). *The value corresponding to the mixed-strategy is* c^{-1}. *:*
If $c = 1$, *there are two* EPs R-R *and* A-A, *and no mixed-strategy eq. The game is a coordination game, i.e., both players get profit by coordinating their choices to choose* R-R *not* A-A.

Lemma 2 *The eq. val. of the bimatrix game* $-\mathbf{M}(c)$ *:*
If $c<1$*,* R-R *is a unique eq. point, and* $\widetilde{V}(c)=-c.$ *;*
If $c>1$*,* A-R, R-A *and the common mixed strategy* (R, A ;$c^{-1}, 1-c^{-1}$) *are in eq. The value corresponding to the mixed strategy is* $-c^{-1}$ *;*
If $c=1$*,* R-R, R-A *and* A-R *are in eq. and no mixed-strategy eq. exists. The game is a coordination game, that is, one player gets profit by coordinating their choices to choose either* R-A *or* A-R.

Lemma 3 *The equilibrium of the matrix game* $\mathbf{M}^0(b)$ *given by* (2.6) *:*
If $b<0$*, there are three* EP*s* R-A, A-R *and* A-A*, all giving* $(0,0)$*, and no mixed-strategy eq. ;*
If $b>0$*, two* EP*s* R-R *and* A-A*, no mixed-strategy eq. The game is a coordination game, in which both players get profit by coordinating their choices to choose* R-R *not* A-A.

Proofs of these lemmas are easy and so omitted. By our Assumption A there is no eq. for $\mathbf{M}(c)$ if $c=1$, and no eq. for $\mathbf{M}^0(b)$ if $b>0$. Note that $V(c)$ is decreasing for $c>1$.

3 n-Stage Game where $\frac{1}{2}<p\le 1$.

Define state (n,x) to mean that the first random variable X_1 in the n-stage game turns out to be x.

Theorem 1 *Let* $\frac{1}{2}<p\le 1$*. The* CES *(common eq. strategy) in state* (n,x) *is :*
Employ the mixed strategy $\left(\text{R, A };\dfrac{(p-1/2)\,x}{u_{n-1}-x/2}, \dfrac{u_{n-1}-px}{u_{n-1}-x/2}\right)$*, if* $x<p^{-1}u_{n-1}$ *:*
and choose A*, if* $x>p^{-1}u_{n-1}$*.*
The sequence $\{u_n\}$ *is determined by the recursion*

$$u_n = k(p)u_{n-1}^2+\frac{1}{4} \qquad \left(n\ge 2; u_1=\frac{1}{4}\right), \tag{3.1}$$

where

$$k(p)=\left(p-\frac{1}{2}\right)^2\left[-8\log\left(1-\frac{1}{2p}\right)-\frac{4p+1}{p^2}\right]. \tag{3.2}$$

$k(p), \frac{1}{2}<p\le 1$*, is increasing with values* $k\left(\frac{1}{2}\right)=0$ *and* $k(1)=2\log 2-\frac{5}{4}\cong 0.1363$*. Moreover, as* $n\to\infty, u_n\uparrow u_\infty(p)=\frac{1}{2}\left(1+\sqrt{1-k(p)}\right)^{-1}$*.*

Proof. First we have

$$c\equiv\frac{x^{-1}u_{n-1}-1/2}{p-1/2}\left\{\begin{array}{c}<\\>\end{array}\right\}1, \quad \text{if } x\left\{\begin{array}{c}>\\<\end{array}\right\}p^{-1}u_{n-1}$$

Hence, from (2.4) and Lemma 1, the second term in the r.h.s. of (2.4) is

$$\left(p-\frac{1}{2}\right)\int_0^1 x\left\{c^{-1}I\left(x<p^{-1}u_{n-1}\right)+0\cdot I\left(x>p^{-1}u_{n-1}\right)\right\}dx$$

$$=\begin{cases}\left(p-\frac{1}{2}\right)^2\displaystyle\int_0^{p^{-1}u_{n-1}}\frac{x^2}{u_{n-1}-x/2}dx=\left(p-\frac{1}{2}\right)^2\left\{-8\log\left(1-\frac{1}{2p}\right)-\frac{4p+1}{p^2}\right\}u_{n-1}^2, & \text{if } u_{n-1}<p;\\ \left(p-\frac{1}{2}\right)^2\displaystyle\int_0^1\frac{x^2}{u_{n-1}-x/2}dx=\left(p-\frac{1}{2}\right)^2\left\{-8\log\left(1-\frac{1}{2u_{n-1}}\right)-\frac{4u_{n-1}+1}{u_{n-1}^2}\right\}u_{n-1}^2, & \text{if } u_{n-1}>p.\end{cases}$$

That is, if we define

$$(3.3)\qquad T(u)=\begin{cases} k(p)u^2+\frac{1}{4}, & \text{if } u<p \\ \left(\dfrac{p-1/2}{u-1/2}\right)^2 k(u)u^2+\dfrac{1}{4}, & \text{if } u>p, \end{cases}$$

where $k(p)$ is given by (3.2), then $u_n=T(u_{n-1})$.

Next we show that $k(p)$ is positive and increasing in $p\in(\frac{1}{2},1]$. We have $k\left(\frac{1}{2}+0\right)=0, (c.f.\ 0\log 0=0)$. $K(1)=2\log 2-\frac{5}{4}\cong 0.15$ and differentiation gives

$$\begin{aligned}\left(p-\frac{1}{2}\right)^{-1}k'(p) &= 2\left\{-8\log\left(1-\frac{1}{2p}\right)-\frac{4p+1}{p^2}\right\}-p^{-3} \\ &> 16\left\{\frac{1}{2}p^{-1}+\frac{1}{8}p^{-2}+\frac{1}{24}p^{-3}+\frac{1}{64}p^{-4}+\frac{1}{160}p^{-5}\right\}-\frac{8p+2}{p^2}-p^{-3} \\ &= \frac{1}{60}p^{-5}\left(6+15p-20p^2\right)>0, \qquad \forall p\in(\tfrac{1}{2},1].\end{aligned}$$

implying that $k(p)>0$, for $\frac{1}{2}<p\le 1$.
So, $T(u)$, defined by (3.1), is increasing for $u<p$.
Since

$$u_{n-1}<\frac{1}{2}<p\Rightarrow u_n=k(p)u_{n-1}^2+\frac{1}{4}<\frac{1}{4}(k(p)+1)<\frac{1}{4}(k(1)+1)<\frac{1}{2},$$

and $u_1=\frac{1}{4}<\frac{1}{2}$, we have $u_n<\frac{1}{2},\forall n\ge 1$.
Therefore, by (3.3),

$$u_n=T(u_{n-1})=k(p)u_{n-1}^2+\frac{1}{4},\qquad \forall n\ge 1,\ \frac{1}{2}<\forall p\le 1,$$

which is (3.1)-(3.2).

Then it follows that

$$u_n>u_{n-1}\Rightarrow u_{n+1}=T(u_n)>T(u_{n-1})=u_n,$$

which, together with $u_2=\frac{1}{4}+\frac{1}{16}k(p)>u_1$, gives convergence $u_n\uparrow u_\infty\in\left(0,\frac{1}{2}\right)$. Evidently $u_\infty\in\left(0,\frac{1}{2}\right)$ satisfies $T(u)=u$, that is, $u_\infty(p)=\frac{1}{2}\left(1\pm\sqrt{1-k(p)}\right)^{-1}$. Here the larger root is $\frac{1}{2}\left(1-\sqrt{1-k(p)}\right)^{-1}>\frac{1}{2}$ and the smaller root is $\frac{1}{2}\left(1+\sqrt{1-k(p)}\right)^{-1}<\frac{1}{2}$ (see Figure 1 in Section 5).

This completes the proof of the theorem. □

Computation gives

Case	$k(p)$	$u_\infty(p)$
$p=1$	$2\log 2-5/4\cong 0.1363$	$\left(2+\sqrt{9-8\log 2}\right)^{-1}\cong 0.2592$
3/4	0.1049	0.2569
0.6	0.0489	0.2531

Convergence of $\{u_n\}$ is very fast. Even for $p=1$,

$$u_n=0.25852, 0.25911, 0.25915, 0.259154,\cdots,\text{for }\ p=2,3,4,5,\cdots,\text{resp.}$$

We observe from Theorem 1 that even if arbitration stands perfectly in favor of the accepting side *i.e.*, $p = 1$, players' merit is only one percent. It is shown in the next section that players are more advantageous when arbitration mildly favors the rejection side. For example if $p = 0.3$, players' merit is five percent (See Theorem 2).

4 n-Stage Game where $\mathbf{0 \le p < \frac{1}{2}}$.

Theorem 2 **(i)** *Let* $\frac{1}{4} \le p < \frac{1}{2}$. *The CES in state* (n, x) *is : Choose* R, *if* $x < p^{-1}u_{n-1} < 1$ *;*
employ the mixed strategy $\left(\mathrm{R},\ \mathrm{A};\ \dfrac{(1/2-p)\,x}{x/2-u_{n-1}}, \dfrac{px-u_{n-1}}{x/2-u_{n-1}}\right)$, *if* $p^{-1}u_{n-1} < x$ *;*
and choose R, *for* $\forall x \in (0,1)$, *if* $p^{-1}u_{n-1} > 1$.

The sequence $\{u_n\}$ *is determined by the recursion* $u_n = f(u_{n-1}|p), (n \ge 1, u_1 = 1/4)$, *where*

$$(4.1) \quad f(u|p) = p\bar{p} - (1-2p)^2 u + \left[4p - 3 + p^{-1} - 2(1-2p)^2 \log\left(\frac{2p}{1-2p} \Big/ \frac{2u}{1-2u}\right)\right] u^2, \qquad \textit{for } u < p.$$

As $n \to \infty, u_n \uparrow p, \quad \forall p \in [1/4, 1/2)$.
(ii). *Let* $0 \le p < \frac{1}{4}$. *The CES in state* (n, x) *is : Choose* R $n-1$ *times repeatedly, and* A *at the* n*-th, independently of* $\forall x \in (0,1)$. *The CEV is* $1/4$.

Proof. We have

$$c \equiv \frac{1/2 - x^{-1}u_{n-1}}{1/2-p} \left\{ \begin{matrix} < \\ > \end{matrix} \right\} 1, \quad \text{if } x \left\{ \begin{matrix} < \\ > \end{matrix} \right\} p^{-1}u_{n-1}.$$

Therefore from (2.5) and Lemma 2, the second term in the r.h.s. of (2.5), except the minus sign is

$$\left(\frac{1}{2}-p\right)\int_0^1 x\left\{cI\left(x < p^{-1}u_{n-1}\right) + c^{-1}I\left(x > p^{-1}u_{n-1}\right)\right\}dx$$

$$= \begin{cases} \left(\dfrac{1}{2}-p\right)\left[\displaystyle\int_0^{p^{-1}u_{n-1}} \frac{x/2-u_{n-1}}{1/2-p}dx + \int_{p^{-1}u_{n-1}}^1 \frac{(1/2-p)\,x^2}{x/2-u_{n-1}}dx\right], & \text{if } p^{-1}u_{n-1} < 1; \\ \displaystyle\int_0^1 \left(\frac{x}{2} - u_{n-1}\right)dx, & \text{if } p^{-1}u_{n-1} > 1 \end{cases}$$

$$= \begin{cases} \dfrac{1-4p}{4p^2}u_{n-1}^2 + \left(\dfrac{1}{2}-p\right)^2 \displaystyle\int_{p^{-1}u_{n-1}}^1 \frac{x^2}{x/2-u_{n-1}}dx, & \text{if } p^{-1}u_{n-1} < 1; \\ \dfrac{1}{4} - u_{n-1}, & \text{if } p^{-1}u_{n-1} > 1. \end{cases}$$

Since

$$\int_{p^{-1}u_{n-1}}^1 \frac{x^2}{x/2-u_{n-1}}dx = 1 + 4u + \left\{8\log\left(\frac{2p}{1-2p} \Big/ \frac{2u}{1-2u}\right) - \frac{4p+1}{p^2}\right\}u^2,$$

we obtain

$$(4.2) \qquad u_n = \begin{cases} f(u_{n-1}|p), & \text{if } u_{n-1} < p \\ u_{n-1}, & \text{if } u_{n-1} > p, \end{cases}$$

where $f(u|p)$ is given by (4.1).

(i) : Let $\frac{1}{4} < p < \frac{1}{2}$. After some elementary and tedious calculations we find that $f(u|p)$ satisfies

$$\begin{aligned} f(0|p) &= p\bar{p} < \frac{1}{4} \\ &\le f\left(\frac{1}{4}\,\Big|\, p\right) = \frac{1}{16}\left[p^{-1} + (4p-1)(7-8p) + 2(1-2p)^2 \log \frac{1-2p}{2p}\right] \\ &\le f(p|p) = p, \end{aligned}$$

and furthermore $f'(0+0|p) = -(1-2p)^2, f'\left(\frac{1}{4}|p\right) = 2p - \frac{3}{2} + \frac{1}{2p} + (1-2p)^2 \log \frac{1-2p}{2p} > 0$, and $f'(p|p) = 1$. Therefore it follows that

$$u_n > u_{n-1} \Rightarrow u_{n+1} = f(u_n|p) > f(u_{n-1}|p) = u_n,$$

which, together with

$$u_2 = f(u_1|p) = f\left(\frac{1}{4}\,\Big|\, p\right) > \frac{1}{4} \qquad \text{(See Figure 2 in Section 5)}$$

gives the convergence of $\{u_n\}$. The limit is $u_\infty = p$.

(ii): Let $0 \le p < \frac{1}{4}$. Then $u_n > p, \forall n \ge 1$. Because, by (4.2),

$$u_{n-1} > p \Rightarrow u_n = u_{n-1} > p, \text{ and } u_1 = \frac{1}{4} > p.$$

Hence $u_n \equiv \frac{1}{4}, \forall n \ge 1$, follows.

This completes the proof of Theorem 2. □

Some computed values of u_n s are shown below.

u_n	$n=2$	3	4	5	6	7	$\cdots$
$p=0.3$	0.25722	0.26329	0.26754	0.27088	0.27355	0.27578	$\cdots$
$p=0.4$	0.29182	0.31486	0.32968	0.34006	0.34776	0.35367	$\cdots$

5 Remarks. 1. Figures 1 and 2 show the functions $T(u) = k(p)u^2 + \frac{1}{4}$, with $p \in \left(\frac{1}{2}, 1\right)$ and $f(u|p)$ with $p \in \left(\frac{1}{4}, \frac{1}{2}\right)$, respectively. The two graphs approach the different ones, that is,

$$u \equiv \frac{1}{4} \text{ in Fig. 1, as } p \to \frac{1}{2} + 0; \quad \text{and} \quad u^2 + \frac{1}{4} \text{ in Fig. 2, as } p \to \frac{1}{2} - 0.$$

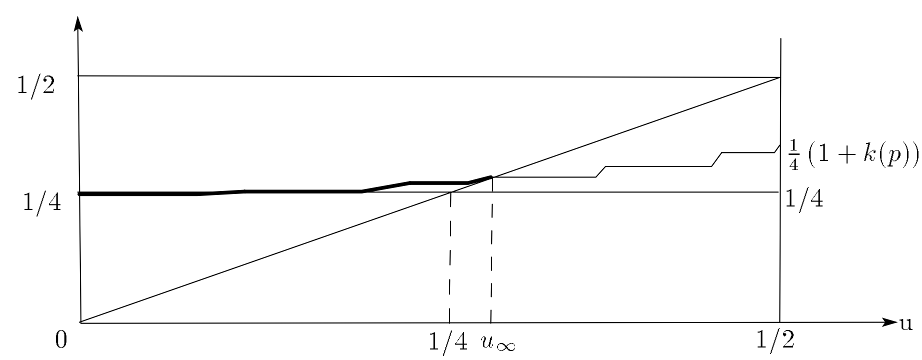

Figure 1. $T(u) = k(p)u^2 + \frac{1}{4}$ with $p \in \left(\frac{1}{2}, 1\right)$

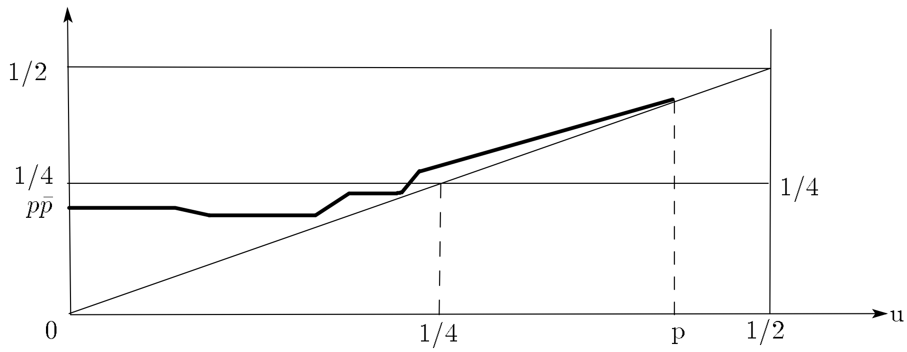

Figure 2. $f(u|p)$, with $p \in \left(\frac{1}{4}, \frac{1}{2}\right)$.

2. Our n-stage game has no equilibrium by our Assumption A for $p = \frac{1}{2}$. However if players are admitted to coordinate, we have

Theorem 3 *Let $p = \frac{1}{2}$, and suppose that players are admitted to coordinate. Then* CES *in state (n, x) is :*
If $x < 2u_{n-1}$, players coordinate to choose R, *not* A *;*
If $x > 2u_{n-1}$, choose A.
The sequence $\{u_n\}$ is determined by the recursion

$$u_n = u_{n-1}^2 + \frac{1}{4} \qquad \left(u_n \geq 1, u_1 = \frac{1}{4}\right).$$

and $u_n \uparrow 1/2$ as $n \to \infty$.

Proof. We have $b = x^{-1}u_{n-1} - \frac{1}{2} < (>) 0$, if $x > (<) 2u_{n-1}$ and hence from (2.6)-(2.7) and Lemma 3,

$$u_n - \frac{1}{4} = \int_0^1 x\left\{bI(x < 2u_{n-1}) + 0 \cdot I(x > 2u_{n-1})\right\} dx = \int_0^{2u_{n-1}} \left(u_{n-1} - \frac{x}{2}\right) dx.$$

Hence the result follows. □

3. Our n-stage game (1.1)-(1.2) has quite different solutions for the two extreme cases $p = 1$ and $p = 0$ as seen in Theorems 1 and 2, although they seemingly look similar. Furthermore the two particular cases $p = 1/2$ and $p = 1/4$ give somewhat abnormal phases to the solution of the problem.

References

[1] Mazalov, V.V., Sakaguchi, M. and Zabelin, A.A., *Multistage arbitration game with random offers*, Game Th. Appl. **8**(2002), 75-83.

[2] Sakaguchi, M., *Optimal stopping games where players have weighted privilege*, Game Th. Appl.**6**(2001), 116-131.

[3] Sakaguchi M., *Three-person stopping games under winning probability maximization and players unequally weighted privilege*, Sci. Math. Japonicae,**53**(2001), 417-433.

[4] Sakaguchi M., *Best-choice games where arbitration comes in*, to appear in Game Th. Appl., **9**(2003).

[5] Sakaguchi M., *Three-person games of odd-man-wins*, to appear.

Each of which contains further references.

*3-26-4 Midorigaoka, Toyonaka, Osaka, 560-0002, Japan,
Fax: +81-6-6856-2314 E-mail: smf@mc.kcom.ne.jp