EXTENDING FROBENIUS' HIGHER CHARACTERS

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Abstract. Given a finite group G with irreducible character $\chi \in \mathrm{Irr}(G)$, $r \in \mathbb{N}$ and a partition λ of r, we define higher characters $\chi_\lambda^{(r)}$ of G, following Frobenius [2]. We interpret them as a generalization of Schur functions in noncommuting variables, as a multilinear invariant map, as the sum over $\mathfrak{S}_r$ of a character of the wreath product $G \wr \mathfrak{S}_r$, and as the trace of a $e_\lambda G_r e_\lambda$-module. Using these interpretations, we are able to compute $(\chi + \psi)_\lambda^{(r)}$ in terms of $\chi_\mu^{(a)}$ and $\psi_\nu^{(b)}$ where $a + b = r$ and μ and ν are partitions of a and b, respectively.

We show distinct higher characters are orthogonal in section 3.1.

These $\chi_\lambda^{(r)}$ have the property that they are constant on $\mathfrak{S}_r \times G$ orbits of G^r, i.e. invariant under diagonal conjugation and permutation of entries of an r-tuple. By decomposing $\mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda) = e_\lambda G_r e_\lambda$, we find an orthogonal family of functions with this property. In the case G is abelian, we show this family forms a basis for all such functions on G^r. This doesn't happen for general G, as shown on some examples in the appendix. However, in both cases, we show that it is only necessary to consider $\lambda = (r)$ (or $\lambda = (1^r)$) in theorems 8 and 9. The reader may skip straight to section 3.2 for these results.

1 Introduction This paper is a largely expository study of Frobenius' higher characters $\chi^{(r)}$ and illuminates three different contexts in which they arise. In [2] Frobenius gives the $\chi_i^{(r)}$ in terms of the irreducible characters χ_i, as in definition 2, as well as in terms of the group determinant.

Character theory grew out of Frobenius' study of the group determinant, initiated in correspondence with Dedekind in 1896. Given a finite group G of cardinality n, choose n indeterminants $\{x_g\}_{g\in G}$ and construct the $n \times n$ matrix $X_G = [x_{gh^{-1}}]$. The group determinant is defined as $\Theta_G = det(X_G)$. Frobenius factored $\Theta_G = \prod \phi_i^{d_i}$ over $\mathbb{C}$. The ϕ_i are irreducible homogeneous polynomials of degree d_i and correspond to the irreducible complex representations ρ_i with character χ_i. One can easily recover $\chi_i(g)$ as the coefficient of $x_1^{d_i-1}$ in $\frac{\partial \phi_i}{\partial x_g}$. Further, $\phi_i = \det(\sum_{g\in G} \rho_i(g) x_g)$. But what of the other coefficients? The higher characters $\chi_i^{(r)}$ we study below arise as other coefficients occurring in the ϕ_i and their derivatives.

Recently, there has been a renewed interest in the group determinant. In 1991, Formanek and Sibley [1] proved the group determinant determines the group. In fact, Hoehnke and Johnson show we only need the $\chi_i^{(1)}, \chi_i^{(2)}$, and $\chi_i^{(3)}$ to determine the group in [4], and a result of Mansfield's gives a more elementary proof of this in [9].

Frobenius gives the higher characters $\chi_i^{(r)}$ in terms of the χ_i as in definition 2 as well as in terms of the group determinant. In this study, we use the former and a natural generalization of it to prove properties (such as orthogonality and behavior under direct sum) of the higher characters and exhibit them as occurring in other contexts of math.

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(Orthogonality aslo follows from considering the higher characters as coming from characters of a wreath product. [5])

We assume the reader is familiar with the basics of representation theory of finite groups and in particular with that of the symmetric group and its connection to symmetric functions [8].

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2 χ_λ

2.1 Notation and some definitions Let $\mathfrak{S}_r$ denote the symmetric group on r letters. Given a permutation $\sigma \in \mathfrak{S}_r$, we can write it in cycle notation

$$\sigma = \big(m_1\ \sigma(m_1)\ \sigma^2(m_1)\cdots\sigma^{\mu_1}(m_1)\big)\big(m_2\ \ \sigma(m_2)\ \ \sigma^2(m_2)\cdots\sigma^{\mu_2}(m_2)\big)\cdots\big(m_l\ \ \cdots\sigma^{\mu_\ell}(m_l)\big).$$

If we use the convention that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell$, then we say σ has *shape* $\mathrm{sh}(\sigma) = (\mu_1\ \mu_2 \cdots \mu_\ell) = \mu$. Then μ is a partition of r, written $\mu \vdash r$, of length $\ell(\mu) = l$. The partition μ will also be used to denote the conjugacy class of σ which has size $\frac{r!}{z_\mu}$, where $z_\mu = |C_{\mathfrak{S}_r}(\sigma)|$. (In general, $C_G(g)$ is the centralizer of $g \in G$.) Furthermore, a partition $\lambda \vdash r$ will also be used to denote an irreducible character $\lambda \in \mathrm{Irr}(\mathfrak{S}_r)$. Let

$$e_\lambda = \frac{\lambda(1)}{r!} \sum_{\sigma\in\mathfrak{S}_r} \lambda(\sigma^{-1})\sigma = \frac{\lambda(1)}{r!} \sum_{\sigma\in\mathfrak{S}_r} \lambda(\sigma)\sigma$$

denote the central idempotent of $\mathbb{C}[\mathfrak{S}_r]$, which we can also view as a projection. Then $E_\lambda = \mathbb{C}[\mathfrak{S}_r]e_\lambda = e_\lambda\mathbb{C}[\mathfrak{S}_r]$ has character $\lambda(1)\lambda$. We can further decompose $e_\lambda = e_\lambda^{(1)} + e_\lambda^{(2)} + \cdots + e_\lambda^{(\lambda(1))}$ into a sum of orthogonal idempotents, obtained, for instance, by multiplying the row and column stabilizers of a standard Young tableaux of shape λ. Then $\mathbb{C}[\mathfrak{S}_r]e_\lambda^{(i)}$ is an irreducible representation of $\mathfrak{S}_r$ with character λ and its isomorphism type is independent of i. Let $L_\lambda = \mathbb{C}[\mathfrak{S}_r]e_\lambda^{(1)}$. Throughout this paper, we will use L_λ when it is most convenient to use an irreducible representation, but E_λ when we want to make use of the fact that e_λ is central.

Definition 1 *Given χ a character of G, and $\sigma \in \mathfrak{S}_r$ of shape $\mathrm{sh}(\sigma) = \mu$ in cycle notation as above, define*

$$\chi_\sigma(g_1, \ldots, g_r) = \chi(g_{m_1} g_{\sigma(m_1)} \cdots g_{\sigma^{\mu_1}(m_1)}) \cdots \chi(g_{m_l} \cdots g_{\sigma^{\mu_\ell}(m_l)}).$$

Example 1 *If $\sigma = (124)(3)$ then $\chi_\sigma(g_1, g_2, g_3, g_4) = \chi(g_1g_2g_4)\chi(g_3)$.*

Let sgn denote the sign character of $\mathfrak{S}_r$ that is -1 on odd permutations and 1 on even permutations.

Definition 2 *If χ is a character of G then the r-character is*

$$\chi^{(r)}(g_1, g_2, \ldots, g_r) = \sum_{\sigma\in\mathfrak{S}_r} sgn(\sigma)\chi_\sigma(g_1, g_2, \ldots, g_r).$$

A perfectly natural generalization of these higher characters is obtained by replacing $\mathrm{sgn}(\sigma)$ with any irreducible character λ of $\mathfrak{S}_r$. In fact, in [6], Johnson denotes by $\chi^{(r,+)}$ the r-character one gets using the trivial representation of $\mathfrak{S}_r$, and shows that these and the $\chi^{(r)}$ are orthogonal for fixed r, as χ ranges over $\mathrm{Irr}(G)$, the irreducible characters of G. Making this generalization, one could interpret these new r-characters as coefficients of a "group immanant". See [7].

Definition 3 *For χ an irreducible character of G and $\lambda \in \mathrm{Irr}(\mathfrak{S}_r)$ define*

$$\chi_\lambda(g_1, g_2, \dots, g_r) = \chi_\lambda^{(r)}(g_1, g_2, \dots, g_r) = \sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma)\chi_\sigma(g_1, g_2, \dots, g_r).$$

In section 3.1 we'll show that distinct higher characters are orthogonal.

If G has a representation V with character χ, we can interpret χ_λ as coming from the action of the wreath product of G and $\mathfrak{S}_r$ on $V^{\otimes r}$.

Let $G^r = \underbrace{G \times G \times \cdots \times G}_{r}$ and let G_r denote the *wreath product* $G \wr \mathfrak{S}_r = \mathfrak{S}_r \ltimes G^r$ where

$$(g_1, g_2, \dots, g_r)^\sigma = \sigma^{-1}(g_1, g_2, \dots, g_r)\sigma = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \dots, g_{\sigma^{-1}(r)}).$$

Let $\Delta(G)$ denote the copy of G that sits in G^r diagonally.

If V is an irreducible representation of G with character χ of degree n, then $V^{\otimes r}$ is an irreducible representation of G^r, with action defined by

$$(g_1, g_2, \dots, g_r)v_1 \otimes \cdots \otimes v_r = g_1 v_1 \otimes \cdots \otimes g_r v_r,$$

and with character denoted $\chi^{\otimes r}$. The natural action of $\mathfrak{S}_r$ on $V^{\otimes r}$ via

$$\sigma\,(w_1 \otimes w_2 \otimes \cdots \otimes w_r) = w_{\sigma^{-1}(1)} \otimes w_{\sigma^{-1}(2)} \otimes \cdots \otimes w_{\sigma^{-1}(r)}$$

makes $V^{\otimes r}$ into an irreducible representation of the wreath product G_r. Let us denote its character by $\widetilde{\chi}$, noticing $\mathrm{Res}_{G^r}^{G_r}\widetilde{\chi} = \chi^{\otimes r}$ and $\mathrm{Res}_{\Delta(G)}^{G_r}\widetilde{\chi} = \chi^r$.

In the notation of [5], if χ is the character of the representation V, then our $\chi_\sigma(\underline{g}) = \widetilde{\chi}(\underline{g}\sigma)$ is denoted $\chi^{(\#^n V)\widetilde{}}(\underline{g};\sigma)$ (having identified the r-tuple $\underline{g}$ with the function $\{1, \dots, r\} \to G$ sending i to g_i). Further, $\lambda(\sigma)\chi_\sigma(\underline{g})$ is the character of $\overline{(\#^n V)}\widetilde{}\otimes \lambda$ evaluated at $(\underline{g};\sigma)$. Hence the orthogonality of these extended higher characters follows from the orthogonality of the irreducible characters of the wreath product G_r.

As a way of understanding $\chi_\lambda^{(r)}$ and its connection to $\widetilde{\chi}$, let's first consider χ_λ evaluated on $\Delta(G)$. We show below that $\chi_\lambda^{(r)}(g, g, \dots, g) = r!\chi_{S_\lambda V}(g)$, where we denote $S_\lambda V = \mathrm{Hom}_G(\mathrm{Res}_G^{G_r}\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} L_\lambda, V^{\otimes r})$. Schur-Weyl duality gives us that $V^{\otimes r} \simeq \bigoplus_{\lambda \vdash r}\lambda(1)S_\lambda V$ as a $\Delta(G)$ representation. See [8] or [3]. Notice $\lambda(1)S_\lambda V \simeq e_\lambda V^{\otimes r}$. It is immediate that $S_\lambda V$ is a representation of G, and for the convenience of the reader, we compute its character.

Claim 1 *$S_\lambda V$ is a representation of G with character*

$$\chi_{S_\lambda V}(g) = \frac{1}{r!}\chi_\lambda^{(r)}(g, g, \dots, g).$$

Proof: Recall that if $\{x_1, \ldots, x_n\}$ are the eigenvalues of g acting on V, then $\chi_{S_\lambda V}(g) = s_\lambda(x_1, \ldots, x_n)$, where s_λ denotes the Schur function. (See [8] or [3].) Notice also that g^m will have eigenvalues $\{x_1^m, \ldots, x_n^m\}$ and so

$$\chi(g^m) = x_1^m + x_2^m + \cdots + x_n^m = p_m(x_1, \ldots, x_n)$$

where p_m denotes the m^{th}-power sum symmetric function. Recall $p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_\ell}$. Then the well-known expansion of the Schur functions in terms of power sum symmetric functions gives us:

$$\begin{aligned}
\chi_{S_\lambda V}(g) &= s_\lambda(x_1, \ldots, x_n) \\
&= \sum_{\mu \vdash r} \frac{1}{z_\mu} \lambda(\mu) p_\mu(x_1, \ldots, x_n) \\
&= \frac{1}{r!} \sum_{\mu \vdash r} \frac{r!}{z_\mu} \lambda(\mu) \chi(g^{\mu_1}) \chi(g^{\mu_2}) \cdots \chi(g^{\mu_\ell}) \\
&= \frac{1}{r!} \chi_\lambda^{(r)}(g, g, \ldots, g)
\end{aligned}$$

□

This suggests we can interpret the $\chi_\lambda^{(r)}(g_1, g_2, \ldots, g_r)$ as coming from the action of the corner ring $e_\lambda \mathbb{C}[G_r] e_\lambda \simeq \mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda)$ on $e_\lambda V^{\otimes r} \simeq \mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda, V^{\otimes r})$, which we will explore in section 2.3. It also suggests that $\chi_\lambda^{(r)}$ is a generalization of the Schur function s_λ.

2.2 χ_λ **as a multilinearization** Just as one can go from a bilinear form B to a quadratic form Q via $Q(x) := B(x, x)$ and then from a quadratic form to a bilinear one via $B(x, y) := \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$, we can go from s_λ back up to $\chi_\lambda^{(r)}$.

First we clarify what we mean by multilinearization. Suppose we are given a homogeneous r^{th} degree polynomial $Q(\vec{x}) = Q(x_1, \ldots, x_n)$ in the (possibly noncommuting) variables $\{x_i\}_1^n$. Let $V = \mathbb{C}^n$ with standard basis $\{\vec{e_i}\}_1^n$, and let V^* have dual basis $\{\phi_i\}_1^n$. Then $\vec{x} = (x_1, \ldots, x_n) = \sum_1^n x_i \vec{e_i}$ and $\phi_i(\vec{x}) = x_i$. We'll associate to Q an element of $(V^*)^{\otimes r} \simeq (V^{\otimes r})^*$. If $Q(\vec{x}) = \sum c_{i_1 i_2 \ldots i_r} x_{i_1} \cdots x_{i_r}$ let $\phi_Q = \sum c_{i_1 i_2 \ldots i_r} \phi_{i_1} \otimes \phi_{i_2} \otimes \cdots \otimes \phi_{i_r}$. Notice $\phi_Q(\vec{x} \otimes \cdots \otimes \vec{x}) = Q(\vec{x})$.

We define the *multilinearization* M_Q of Q to be the homogeneous r^{th} degree polynomial in the nr noncommuting variables $\{x_{ij}\}_{\substack{1 \le i \le n \\ 1 \le j \le r}}$ via

$$M_Q(\vec{X_1}, \vec{X_2}, \ldots \vec{X_r}) = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} \phi_Q(\vec{X}_{\sigma(1)} \otimes \vec{X}_{\sigma(2)} \otimes \cdots \otimes \vec{X}_{\sigma(r)})$$

where $\vec{X_i} = (x_{1i}, x_{2i}, \ldots, x_{ni})$.

One can also define M_Q more directly from Q via

$$\begin{aligned}
r! M_Q(\vec{X_1}, \ldots \vec{X_r}) \quad = \quad & Q(\vec{X_1} + \cdots + \vec{X_r}) - \sum_i Q(\vec{X_1} + \cdots + \widehat{\vec{X_i}} + \cdots \vec{X_r}) + \\
& \sum_{i<j} Q(\vec{X_1} \cdots + \widehat{\vec{X_i}} \cdots + \widehat{\vec{X_j}} \cdots + \vec{X_r}) - \ldots \\
& + (-1)^{n-1} \sum_k Q(\vec{X_k}).
\end{aligned}$$

It is tedious, but not too difficult, to verify these definitions are equivalent.

Because $\phi_Q \in (V^{\otimes r})^*$, it is easy to see M_Q is linear in each entry, i.e. r-multilinear. Also, it is clear that $Q(\vec{x}) = M_Q(\vec{x}, \vec{x}, \ldots, \vec{x})$. So we have a recipe to go from an r-multilinear form to an r-form and back.

We can multilinearize a given Q, by summing the multilinearizations of the monomials of Q. So, in the following examples, we consider just monomials.

Example

1. Let $p(x) = x^r$. Then $M_p(\vec{X}) = M_p(x_1, \ldots, x_r) = \frac{1}{r!}\sum_{\sigma \in \mathfrak{S}_r} x_{\sigma(1)} \cdots x_{\sigma(r)}$.
2. Let $q(x, y) = x^2 y$. Then $M_q(x_1, y_1, x_2, y_2, x_3, y_3) =$
$\frac{1}{6}(x_1 x_2 y_3 + x_2 x_1 y_3 + x_1 x_3 y_2 + x_3 x_1 y_2 + x_2 x_3 y_1 + x_3 x_2 y_1)$.
3. Let $p(z) = z^2$. Let $r(x, y, z) = q(x, y)p(z) = x^2 y z^2$.
Notice $M_r(x_1, y_1, z_1, \ldots, x_5, y_5, z_5) =$
$\frac{3!2!}{5!}\sum_{\sigma \in \mathfrak{S}_5} M_q(x_{\sigma(1)}, y_{\sigma(1)}, x_{\sigma(2)}, y_{\sigma(2)}, x_{\sigma(3)}, y_{\sigma(3)}) M_p(z_{\sigma(4)}, z_{\sigma(5)})$.

This process of multilinearization gives us a way to generalize symmetric functions to functions in noncommuting variables. Now let's focus on Schur functions.

Claim 2 *Let M_{s_λ} denote the multilinearization of s_λ. Then we have $s_\lambda(g) = \chi_{S_\lambda V}(g) = \chi_\lambda(g, g, \ldots, g)$ and $\chi_\lambda(g_1, g_2, \ldots, g_r) = M_{s_\lambda}(g_1, g_2, \ldots, g_r)$*

Proof: The first statement follows from claim 1. Let $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$ denote the power sum symmetric function. If Y is an $n \times n$ matrix with eigenvalues $\{y_1, y_2, \ldots, y_n\}$, and $\lambda \vdash r$, then let us denote $p_\lambda(Y) = p_\lambda(y_1, y_2, \ldots, y_n) = \mathrm{Tr}(Y^{\lambda_1})\mathrm{Tr}(Y^{\lambda_2}) \cdots \mathrm{Tr}(Y^{\lambda_l})$. Similar to the above examples, upon multilinearization we get

$$r! M_{p_\lambda}(X_1, \ldots, X_r) = \sum_{\sigma \in \mathfrak{S}_r} \mathrm{Tr}\left(X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(\lambda_1)}\right) \cdots \mathrm{Tr}\left(X_{\sigma(r-\lambda_l+1)} \cdots X_{\sigma(r)}\right).$$

Writing $\tau \in \mathfrak{S}_r$ in cycle notation $\tau = \left(\sigma(1)\sigma(2)\ldots\sigma(\lambda_1)\right)\left(\sigma(\lambda_1+1)\ldots\sigma(\lambda_1+\lambda_2)\right)\cdots\left(\sigma(r-\lambda_l+1)\ldots\sigma(r)\right)$, then the inner summand of the above is just $\mathrm{Tr}_\tau(X_1, \ldots, X_r)$.

As σ ranges over $\mathfrak{S}_r$, τ ranges over all permutations of shape λ in $\mathfrak{S}_r$ and this is a z_λ-to-1 correspondence. (In fact, all of $C_{\mathfrak{S}_r}(\tau)$ gets sent to τ.) Hence $r! M_{p_\lambda}(X_1, \ldots, X_r) = z_\lambda \sum_{\tau : sh(\tau) = \lambda} \mathrm{Tr}_\tau(X_1, \ldots, X_r)$.

So it is easy to see that the multilinearization of the Schur function $s_\lambda = \sum_{\mu \vdash r} \frac{1}{z_\mu}\lambda(\mu) p_\mu$ is just

$$\begin{aligned} M_{s_\lambda}(X_1, \ldots, X_r) &= \frac{1}{r!}\sum_{\mu \vdash r} \frac{1}{z_\mu}\lambda(\mu)\ z_\mu \sum_{\tau : sh(\tau) = \mu} \mathrm{Tr}_\tau(X_1, \ldots, X_r) \\ &= \frac{1}{r!}\sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma)\mathrm{Tr}_\sigma(X_1, \ldots, X_r). \end{aligned}$$

When $\rho : G \to GL_n(\mathbb{C})$ is a representation with character χ and we set $X_i = \rho(g_i)$, the above says the multilinearization of the Schur function evaluated at the X_i is exactly $\frac{1}{r!}\chi_\lambda^{(r)}(g_1, g_2, \ldots, g_r)$.

□

Corollary 1 *If $\lambda \vdash r$ has $\ell(\lambda) > \chi(1)$, then $\chi_\lambda^{(r)}(g_1, g_2, \ldots, g_r) = 0$.*

Proof: We know that if the length of λ is greater than $n = \dim V$ then L_λ does not occur in $V^{\otimes r}$. See [8] or [3]. Hence $\chi_{S_\lambda V} = s_\lambda = 0$ and so its multilinearization is also 0.

□

We now have the machinery to compute $(\chi + \psi)_\lambda$.

Motivated by Example 3 above, we make the following definition, adopting Johnson's notation [6].

Definition 4 *For two functions $f_i : G^{r_i} \to \mathbb{C}$, let $f_1 \circ f_2 : G^{r_1+r_2} \to \mathbb{C}$ be given by*

$$f_1 \circ f_2(g_1, \dots, g_{r_1+r_2}) = \frac{1}{r_1! r_2!} \sum_{\sigma \in \mathfrak{S}_{r_1+r_2}} f_1(g_{\sigma(1)}, \dots g_{\sigma(r_1)}) f_2(g_{\sigma(r_1+1)}, \dots g_{\sigma(r_1+r_2)}).$$

Remark: Notice that, up to a scalar, $f_1 \circ f_2$ multilinearizes the tensor of the functions $f_1 \cdot f_2(\underline{g}, \underline{h}) = f_1(\underline{g}) f_2(\underline{h})$.

Theorem 2 *Let $\chi, \psi \in \mathrm{Irr}(G), \lambda \vdash r$. Then*

$$(\chi + \psi)_\lambda = \sum C^\lambda_{\mu\nu} (\chi_\mu \circ \psi_\nu),$$

where $C^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients.

Proof: First recall that the Littlewood-Richardson coefficients appear when we induce (or restrict) representations of $\mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2}$ to $\mathfrak{S}_{r_1+r_2}$. In particular, $\mathrm{Ind}^{\mathfrak{S}_{r_1+r_2}}_{\mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2}} \mu \boxtimes \nu = \sum C^\lambda_{\mu\nu} \lambda$. Using Schur Weyl duality, as in [3], we conclude that over $GL(V) \times GL(W)$

$$S_\lambda(V \oplus W) = \bigoplus C^\lambda_{\mu\nu}\, S_\mu V \otimes S_\nu W.$$

In terms of characters this says $\chi_{S_\lambda(V \oplus W)} = \sum C^\lambda_{\mu\nu} \chi_{S_\mu V} \cdot \chi_{S_\nu W}$. Applying claims 1 and 2 and the remark above, we see immediately that $(\chi + \psi)_\lambda = \sum C^\lambda_{\mu\nu} (\chi_\mu \circ \psi_\nu)$.

□

Corollary 3 *As previously, let $\chi^{(r)}$ denote the higher character $\chi_{(1^r)}$. Then*

$$(\chi + \psi)^{(r)} = \sum_{a+b=r} \chi^{(a)} \circ \psi^{(b)}.$$

Proof: We know the $C^{(1^r)}_{\mu\nu} = 1$ exactly when $\mu = (1^a), \nu = (1^b)$ and 0 otherwise. Or, in other words, $\wedge^r(V \oplus W) = \bigoplus_{a+b=r} \wedge^a V \otimes \wedge^b W$.

□

2.3 χ_λ as a trace In claim 1, we saw that $\frac{\lambda(1)}{r!} \chi^{(r)}_\lambda(g, \dots, g)$ is the trace of the diagonal action of g on $e_\lambda V^{\otimes r}$ and therefore equals the trace of the action of $(g, g, \dots, g) e_\lambda$ on $V^{\otimes r}$. One can then ask what is $\widetilde{\chi}((g_1, g_2, \dots, g_r) e_\lambda)$? Although it is not a representation of G_r, $\mathrm{Hom}(\mathrm{Ind}^{G_r}_{\mathfrak{S}_r} E_\lambda, V^{\otimes r})$ is a representation of $\mathrm{Hom}(\mathrm{Ind}^{G_r}_{\mathfrak{S}_r} E_\lambda,\ \mathrm{Ind}^{G_r}_{\mathfrak{S}_r} E_\lambda)$, and we are in effect computing the trace of this action.

Claim 3 *For $\sigma \in \mathfrak{S}_r$ and $\underline{g} = (g_1, g_2, \dots, g_r) \in G^r$, $\widetilde{\chi}(\underline{g}\sigma) = \chi_{\sigma^{-1}}(\underline{g})$.*

Proof: This is best seen on an example. Let $\sigma = (124)(3)$ and let $\underline{g} = (g_1, g_2, g_3, g_4)$. Let $\{v_i\}_1^n$ be a basis for V and write $[a_{ij}^{(k)}]$ for the matrix of g_k with respect to this basis. Then

$$\begin{aligned}
\underline{g}\sigma(v_{i_1}\otimes v_{i_2}\otimes v_{i_3}\otimes v_{i_4}) &= \underline{g}(v_{i_4}\otimes v_{i_1}\otimes v_{i_3}\otimes v_{i_2}) \\
&= \sum_{j=1}^n a_{ji_4}^{(1)} v_j \otimes \sum_{j=1}^n a_{ji_1}^{(2)} v_j \otimes \sum_{j=1}^n a_{ji_3}^{(3)} v_j \otimes \sum_{j=1}^n a_{ji_2}^{(4)} v_j \\
&= a_{i_1 i_4}^{(1)} a_{i_2 i_1}^{(2)} a_{i_3 i_3}^{(3)} a_{i_4 i_2}^{(4)} v_{i_1}\otimes v_{i_2}\otimes v_{i_3}\otimes v_{i_4} + \text{independent terms.}
\end{aligned}$$

So

$$\begin{aligned}
\widetilde{\chi}(\underline{g}\sigma) &= \sum_{i_k=1}^n a_{i_1 i_4}^{(1)} a_{i_2 i_1}^{(2)} a_{i_3 i_3}^{(3)} a_{i_4 i_2}^{(4)} \\
&= \left(\sum_{i_1=1}^n \sum_{i_4, i_2} a_{i_1 i_4}^{(1)} a_{i_4 i_2}^{(4)} a_{i_2 i_1}^{(2)}\right)\left(\sum_{i_3=1}^n a_{i_3 i_3}^{(3)}\right) \\
&= \mathrm{Tr}(a^{(1)} a^{(4)} a^{(2)})\mathrm{Tr}(a^{(3)}) = \chi(g_1 g_4 g_2)\chi(g_3) \\
&= \chi_{\sigma^{-1}}(\underline{g}).
\end{aligned}$$

The proof for general σ is similar, but for the sake of notation, we leave it to the reader.

□

Theorem 4 $\widetilde{\chi}(\underline{g}e_\lambda) = \frac{\lambda(1)}{r!}\chi_\lambda^{(r)}(\underline{g})$.

Proof: From claim 3, it follows

$$\begin{aligned}
\widetilde{\chi}(\underline{g}e_\lambda) &= \frac{\lambda(1)}{r!}\sum_{\sigma\in\mathfrak{S}_r} \lambda(\sigma^{-1})\widetilde{\chi}(\underline{g}\sigma) \\
&= \frac{\lambda(1)}{r!}\sum_{\sigma\in\mathfrak{S}_r} \lambda(\sigma^{-1})\chi_{\sigma^{-1}}(\underline{g}) = \frac{\lambda(1)}{r!}\sum_{\sigma\in\mathfrak{S}_r} \lambda(\sigma)\chi_\sigma(\underline{g}) \\
&= \frac{\lambda(1)}{r!}\chi_\lambda^{(r)}(\underline{g}).
\end{aligned}$$

□

In other words, $\chi_\lambda^{(r)}(\underline{g}) = \frac{r!}{\lambda(1)}\widetilde{\chi}(e_\lambda \underline{g} e_\lambda)$ (since $e_\lambda^2 = e_\lambda$), and $\widetilde{\chi}(e_\lambda \underline{g} e_\lambda)$ simply computes the trace of an element of $e_\lambda \mathbb{C}[G_r]e_\lambda \simeq \mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda)$ on the module $e_\lambda V^{\otimes r} \simeq \mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda, V^{\otimes r})$. In section 3 we will generalize this situation by replacing $V^{\otimes r}$ with any representation of G_r.

2.4 χ_λ as an invariant For the moment, let's consider $G = GL(V)$. Then $S_\lambda V$ is irreducible. We can see $\chi_\lambda^{(r)}$ as a multilinear invariant map (with respect to the $\Delta(G)$-action) of $\mathrm{End}(V)^{\times r}$. The above sections 2.2 and 2.3 all translate into the language of invariants used in Procesi's work [10], which we'll outline below.

Make the identifications $(\mathrm{End}(V)^{\otimes r})^* \simeq ((V^* \otimes V)^{\otimes r})^* \simeq (V^{\otimes r})^* \otimes V^{\otimes r} \simeq \mathrm{End}(V^{\otimes r})$ and so can view a G-invariant map $\mathrm{End}(V)^{\otimes r} \to \mathbb{C}$ as an element of $\mathrm{End}(V^{\otimes r})$ which commutes with the action of $G = GL(V)$.

Again, $\mathfrak{S}_r$ acts naturally on $V^{\otimes r}$ (as does GL_r) and is centralized by the diagonal action of $G = GL(V) \simeq GL_n$. We know all operators commuting with the G-action can be realized as elements of $\mathbb{C}[\mathfrak{S}_r]$, and those commuting with both the G-action and the $\mathfrak{S}_r$-action are thus linear combinations of the e_λ.

In his theorem 1.2 of [10], Procesi shows that under the above identifications $\sigma \in \mathfrak{S}_r$ corresponds to an element he calls $\lambda_\sigma \in \operatorname{End}(V^{\otimes r})$ via

$$\lambda_\sigma : w_1 \otimes w_2 \otimes \cdots \otimes w_r \mapsto w_{\sigma^{-1}(1)} \otimes w_{\sigma^{-1}(2)} \otimes \cdots \otimes w_{\sigma^{-1}(r)}$$

which in turn corresponds to an invariant $\mu_\sigma \in (\operatorname{End}(V)^{\otimes r})^*$, and that this is exactly our $\chi_{\sigma^{-1}}$. (This gives an alternate proof of claim 3.) Thus any multilinear invariant on r $n \times n$ matrices is a linear combination of the χ_σ.

Using the fact that the center of $\mathbb{C}[\mathfrak{S}_r]$ is spanned by the e_λ, we also see the invariants that commute with both the G-action and the $\mathfrak{S}_r$-action are linear combinations in the $\chi_\lambda^{(r)}$. Corollary 1 is essentially equivalent to his theorem 4.3 [10] that all "trace identities" are linear combinations of the $\chi_\lambda^{(r)}$ for which $\ell(\lambda) > n$.

3 w-functions We saw in theorem 4 that $\chi_\lambda^{(r)}(\underline{g}) = \frac{r!}{\lambda(1)} \widetilde{\chi}(\underline{g} e_\lambda)$ where $\widetilde{\chi}$ was the character of $V^{\otimes r}$ as a representation of the wreath product G_r. Let us generalize the situation to any character of G_r and examine the corresponding functions.

Definition 5 *Define the w-orbits of G^r to be the orbits of G^r under the conjugation action of $\mathfrak{S}_r \times \Delta(G)$.*

Observe these are *not* the same as the conjugacy classes in the group G_r.

Recall $\sigma^{-1}(g_1, g_2, \ldots, g_r)\sigma = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \ldots, g_{\sigma^{-1}(r)})$.

Definition 6 *Let $f : G^r \to \mathbb{C}$ be any function. Then we'll call f a w-function (for lack of a better name) if*

$$f(\sigma^{-1}\underline{g}\sigma) = f(\underline{g}) = f(\underline{a}^{-1}\underline{g}\underline{a}),$$

for all $\sigma \in \mathfrak{S}_r, \underline{a} = (a, a, \ldots, a) \in \Delta(G) \subset G^r, \underline{g} = (g_1, g_2, \ldots, g_r) \in G^r$. In other words, f is constant on all w-orbits.

Definition 7 *Let χ be any character of G_r. Define $f_\lambda^\chi : G^r \to \mathbb{C}$ by*

$$f_\lambda^\chi : \underline{g} \mapsto \chi(\underline{g} e_\lambda).$$

Claim 4 *f_λ^χ is a w-function.*

Proof: First, recall χ is a class function on G_r, and e_λ and $\Delta(G)$ commute with all of $\mathbb{C}[\mathfrak{S}_r]$. Thus for $\sigma \in \mathfrak{S}_r$, $\underline{a} \in \Delta(G)$,

$$\begin{aligned} f_\lambda^\chi(\sigma^{-1}\underline{g}\sigma) &= \chi(\sigma^{-1}\underline{g}\sigma e_\lambda) = \chi(\sigma^{-1}(\underline{g}e_\lambda)\sigma) = \chi(\underline{g}e_\lambda) \\ &= f_\lambda^\chi(\underline{g}) \\ f_\lambda^\chi(\underline{a}^{-1}\underline{g}\underline{a}) &= \chi(\underline{a}^{-1}\underline{g}\underline{a}e_\lambda) = \chi(\underline{a}^{-1}(\underline{g}e_\lambda)\underline{a}) = \chi(\underline{g}e_\lambda) \\ &= f_\lambda^\chi(\underline{g}) \end{aligned}$$

□

Notice, in general, the character of any $e_\lambda\mathbb{C}[G_r]e_\lambda$-module will yield a w-function in this manner. (Warning: they are not usually class functions of G^r.) In the following sections, we will give some conditions under which the f_λ^χ are orthogonal to each other, find the span of *all* the f_λ^χ, and give a condition under which they span all the w-functions.

3.1 orthogonality

Theorem 5 *Let $\chi, \phi \in \mathrm{Irr}(G_r)$, $\lambda, \mu \in \mathrm{Irr}(\mathfrak{S}_r)$. If $\langle \lambda \cdot \chi, \mu \cdot \phi \rangle = 0$ then*

$$\sum_{\underline{g} \in G^r} f_\lambda^\chi(\underline{g}) \overline{f_\mu^\phi(\underline{g})} = 0.$$

Proof: Notice, because $\mathfrak{S}_r$ is a quotient of G_r we can lift λ, μ up to characters of G_r which will have G^r in their kernels. In other words, $\lambda(\underline{g}\sigma) = \lambda(\sigma), \mu(\underline{g}\sigma) = \mu(\sigma), \ \forall \underline{g} \in G^r$. Thus the inner tensor products $\lambda \cdot \chi, \mu \cdot \phi$ make sense. The orthogonality relations for characters of G_r give us that for any $\tau \in \mathfrak{S}_r \subset G_r$, $\sum_{w \in G_r} \lambda \cdot \chi(w) \overline{\mu \cdot \phi(\tau w)} = 0$. Scaling, summing over τ and reordering summations yields

$$\begin{aligned}
0 &= \tfrac{\lambda(1)}{r!} \tfrac{\mu(1)}{r!} \sum_{\tau \in \mathfrak{S}_r} \sum_{\sigma \underline{g} \in G_r} \lambda \cdot \chi(\sigma \underline{g}) \overline{\mu \cdot \phi(\tau \sigma \underline{g})} \\
&= \tfrac{\lambda(1)}{r!} \tfrac{\mu(1)}{r!} \sum_{\underline{g} \in G^r} \sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma) \chi(\sigma \underline{g}) \sum_{\tau \in \mathfrak{S}_r} \overline{\mu(\tau\sigma) \phi(\tau \sigma \underline{g})} \\
&= \sum_{\underline{g} \in G^r} \chi\Big(\sum_{\sigma \in \mathfrak{S}_r} \tfrac{\lambda(1)}{r!} \lambda(\sigma) \sigma \underline{g} \Big) \overline{\phi\Big(\sum_{\tau \in \mathfrak{S}_r} \tfrac{\mu(1)}{r!} \mu(\tau\sigma) \tau \sigma \underline{g} \Big)} = \sum_{\underline{g} \in G^r} \chi(e_\lambda \underline{g}) \overline{\phi(e_\mu \underline{g})} \\
&= \sum_{\underline{g} \in G^r} \chi(\underline{g} e_\lambda) \overline{\phi(\underline{g} e_\mu)} = \sum_{\underline{g} \in G^r} f_\lambda^\chi(\underline{g}) \overline{f_\mu^\phi(\underline{g})}
\end{aligned}$$

□

Theorem 6 *If $\lambda \cdot \chi$ is a sum of distinct irreducible characters of G_r, then*

$$\frac{1}{|G^r|} \sum_{\underline{g} \in G^r} f_\lambda^\chi(\underline{g}) \overline{f_\lambda^\chi(\underline{g})} = \frac{\lambda(1)}{\chi(1)} \langle \lambda, \mathrm{Res}_{\mathfrak{S}_r}^{G_r} \chi \rangle = \frac{1}{\chi(1)} \chi(e_\lambda)$$

Proof: Now, because irreducible characters occur in $\lambda \cdot \chi$ with multiplicity 1, the relations for characters of G_r gives us $\frac{1}{|G_r|} \sum_{w \in G_r}, \lambda \cdot \chi(w) \overline{\lambda \cdot \chi(\tau w)} = \frac{1}{\lambda \cdot \chi(1)} \overline{\lambda \cdot \chi(\tau)}$. Then, using the work above, we get

$$\begin{aligned}
\frac{1}{|G^r|} \sum_{\underline{g} \in G^r} f_\lambda^\chi(\underline{g}) \overline{f_\lambda^\chi(\underline{g})} &= \frac{\lambda(1)}{\chi(1)} \frac{1}{r!} \sum_{\tau \in \mathfrak{S}_r} \lambda(\tau) \overline{\chi(\tau)} = \frac{1}{\chi(1)} \chi(e_\lambda) \\
&= \frac{\lambda(1)}{\chi(1)} \langle \lambda, \mathrm{Res}_{\mathfrak{S}_r}^{G_r} \chi \rangle.
\end{aligned}$$

□

Corollary 7 *In particular, it follows that the higher characters $\chi_\lambda^{(r)}$ are orthogonal. Further, $\frac{1}{r!} \chi_\lambda^{(r)}$ has norm $\frac{1}{n^r} s_\lambda(1, \ldots, 1)$ if it is non-zero and if $\lambda \cdot \widetilde{\chi}$ consists of distinct characters.*

Proof As before, if $\chi_i \in \mathrm{Irr}(G)$ is the character of a representation V_i, then let $\widetilde{\chi_i} \in \mathrm{Irr}(G_r)$ be the irreducible character of the associated action of G_r on $V_i^{\otimes r}$. Notice that

$$\langle \mathrm{Res}_{G^r}^{G_r} \lambda \cdot \widetilde{\chi_i}, \mathrm{Res}_{G^r}^{G_r} \mu \cdot \widetilde{\chi_j} \rangle_{G^r} = \lambda(1) \mu(1) \langle \chi_i^{\otimes r}, \chi_j^{\otimes r} \rangle_{G^r} = 0$$

if $i \neq j$. Thus we must also have $\langle \lambda \cdot \widetilde{\chi}_i, \mu \cdot \widetilde{\chi_j} \rangle_{G_r} = 0$, and the result follows from theorem 5. Observe that we know the action of $\sigma \in \mathfrak{S}_r$ on $V^{\otimes r}$, so that $\widetilde{\chi}(\sigma) = n^{\ell(\mathrm{sh}(\sigma))}$ where $n = \chi(1)$. Hence, if $\lambda \cdot \widetilde{\chi}$ is a sum of distinct irreducible characters, then theorems 4 and 6 give that the norm of $\frac{1}{r!}\chi_\lambda^{(r)}$ is $\frac{1}{\widetilde{\chi}(1)} \frac{1}{r!} \sum_{\tau \in \mathfrak{S}_r} \lambda(\tau) n^{\ell(\mathrm{sh}(\tau))} = \frac{1}{n^r} s_\lambda(1, 1, \ldots, 1)$.

□

Notice, when $\chi \in \mathrm{Irr}(G_r)$ has $\mathrm{Res}_{G^r}^{G_r} \chi$ irreducible, i.e. the restriction is of the form $\psi_1^{\mu_1} \otimes \cdots \otimes \psi_\ell^{\mu_\ell}$ for distinct $\psi_i \in \mathrm{Irr}(G), \mu \vdash r$, then the f_λ^χ are symmetrized products of the $\psi_i{}_\lambda^{(\mu_i)}$ (as with the multilinearizations in the examples of section 2.2), and so also include the functions denoted by $\chi_i \circ \chi_j$ in Johnson's work [6]. The above proof extends to show that this larger class of f_λ^χ are orthogonal.

3.2 counting w-orbits and w-functions

Claim 5 *The dimension of the space of all w-functions on a finite group G is*

$$\frac{1}{|G|} \sum_{g \in G} \sum_{\mu \vdash r} \frac{1}{z_\mu} |C_G(g^{\mu_1})| \cdots |C_G(g^{\mu_\ell})|.$$

Proof: Counting the dimension of the space of w-functions is equivalent to counting the number of orbits of the conjugation action of $\mathfrak{S}_r \times \Delta(G)$ on G^r. We count them as follows. G acts on G by conjugation with character $\psi(g) = |C_G(g)|$. G acts on G^r under the diagonal action with character $\psi^{\otimes r}$. Asking how many $\mathfrak{S}_r$ orbits there are is the same as asking how many times the trivial character occurs in $Sym^r(\mathbb{C}[G]) = S_{(r)}(\mathbb{C}[G]) = S_1(\mathbb{C}[G])$. We know the character of G on $S_1(\mathbb{C}[G])$ is just $\psi_1^{(r)}|_{\Delta(G)}$, recalling $\lambda = (r)$ denotes the trivial representation of $\mathfrak{S}_r$, which by abuse of notation we also sometimes refer to as 1. Hence the number of w-orbits is

$$\begin{aligned} \langle 1, \psi_1^{(r)} \rangle_{\Delta(G)} &= \frac{1}{|G|} \sum_{g \in G} \sum_{\mu \vdash r} \frac{1}{z_\mu} \psi(g^{\mu_1}) \psi(g^{\mu_2}) \cdots \psi(g^{\mu_\ell}) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{\mu \vdash r} \frac{1}{z_\mu} |C_G(g^{\mu_1})| \cdots |C_G(g^{\mu_\ell})|. \end{aligned}$$

□

Theorem 8 *Let $B = \{f_1^\chi : \chi(e_1) \neq 0\}$. Then B is an orthogonal basis for span $\{f_\lambda^\chi : \lambda \in \mathrm{Irr}(\mathfrak{S}_r), \chi \in \mathrm{Irr}(G_r)\}$. In particular, the higher characters are all linear combinations of the f_1^χ.*

Proof: By abuse of notation, we write 1 for the trivial representation $\lambda = (r) \in \mathrm{Irr}(\mathfrak{S}_r)$. Theorem 5 gives us that the f_1^χ are orthogonal. Take any $f_\lambda^\chi : \underline{g} \mapsto \chi(\underline{g}\sigma)$. Recall (as in the proof of theorem 5) we can lift λ up to a character of G_r, also denoted λ, which will have G^r in its kernel.

Write $\lambda \cdot \chi = \sum_{\chi_i \in \mathrm{Irr}(G_r)} m_i \chi_i$. Then

$$\begin{aligned} \chi(\underline{g} e_\lambda) &= \chi(\underline{g} \sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma)\sigma) = \sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma) \chi(\underline{g}\sigma) = \sum_{\sigma \in \mathfrak{S}_r} \lambda(\underline{g}\sigma) \chi(\underline{g}\sigma) \\ &= \sum_{\sigma \in \mathfrak{S}_r} \lambda \cdot \chi(\underline{g}\sigma) = \lambda \cdot \chi(\underline{g} \sum_{\sigma \in \mathfrak{S}_r} \sigma) = \lambda \cdot \chi(\underline{g} e_1) \\ &= \sum_{\chi_i \in \mathrm{Irr}(G_r)} m_i \chi_i(\underline{g} e_1). \end{aligned}$$

Hence we see $f_\lambda^\chi = \sum_{\chi_i \in \mathrm{Irr}(G_r)} m_i f_1^\chi$. By the remark below, $f_1^\chi = 0$ if $f_1^\chi \notin B$, thus $f_\lambda^\chi \in span(B)$.

□

Remark: Notice, the set $\{\chi \in \mathrm{Irr}(G_r) : \chi(e_\lambda) \neq 0\}$ are exactly the support of $\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} L_\lambda$, since

$$\chi(e_\lambda) = \frac{\lambda(1)}{r!} \sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma)\chi(\sigma) = \lambda(1)\langle \mathrm{Res}_{\mathfrak{S}_r}^{G_r} \chi, \lambda \rangle_{\mathfrak{S}_r} = \lambda(1)\langle \chi, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r} \lambda \rangle.$$

Let U be the representation of G_r with character χ. Then U and $\mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} 1, U)$ are isomorphic as $\mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} 1, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r} 1)$-modules and f_1^χ is the trace of this action. When $\chi(e_1) = 0$ then $\mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} 1, U) = 0$, and so its trace f_1^χ is also 0.

Remark: We could also replace the trivial character $\lambda = (r)$ by the sign character $\lambda = (1^r)$ in theorem 8 and theorem 9 (to follow).

3.3 abelian G By comparing the number of w-orbits to the size of the support of $\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} 1$, we see that in general not all w-functions occur as f_λ^χ. For example, when $r = 2, G = \mathfrak{S}_3$ there are 8 w-orbits but $|B| = 7$. When $r = 3, G = \mathfrak{S}_3$ there are 17 w-orbits but $|B| = 13$. See the appendix. However, when G is abelian, we do indeed capture all the w-functions. To prove this, we require a simple case of the following lemma.

Lemma 1 *Let $\lambda \in \mathrm{Irr}(\mathfrak{S}_r), \underline{h} \in G^r$, and $\tau \in \mathfrak{S}_r$ be of shape $sh(\tau) = \mu$. Define $\underline{h}_\tau = (h_{m_1} h_{\tau(m_1)} \cdots h_{\tau^{\mu_1}(m_1)}, \ldots, h_{m_l} h_{\tau(m_l)} \cdots h_{\tau^{\mu_\ell}(m_l)})$. Then*

$$\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} \lambda(\tau \underline{h}) = \begin{cases} \lambda(\tau)|G|^{\ell(\mu)} & \textit{if } \underline{h}_\tau = (1^{\ell(\mu)}) \\ 0 & \textit{otherwise} \end{cases}$$

Proof:

$$\begin{aligned}
\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} \lambda(\tau \underline{h}) &= \frac{1}{r!} \sum_{\substack{\sigma \underline{g} \in G_r: \\ (\sigma \underline{g})\tau \underline{h} (\sigma \underline{g})^{-1} \in \mathfrak{S}_r}} \lambda((\sigma \underline{g})\tau \underline{h} (\sigma \underline{g})^{-1}) \\
&= \frac{1}{r!} \sum_{\substack{\sigma \underline{g} \in G_r: \\ \sigma\tau\sigma^{-1}[\underline{g}^\tau \underline{h} \underline{g}^{-1}]^{\sigma^{-1}} \in \mathfrak{S}_r}} \lambda(\sigma\tau\sigma^{-1}[\underline{g}^\tau \underline{h} \underline{g}^{-1}]^{\sigma^{-1}}) \\
&= \sum_{\substack{\underline{g} \in G^r: \\ \underline{g}^\tau \underline{h} \underline{g}^{-1} = \underline{1}}} \lambda(\tau)
\end{aligned}$$

So, we'll be done if we can show that $\#\{\underline{g} \in G^r : \underline{g}^\tau \underline{h} \underline{g}^{-1} = \underline{1}\} = |G|^{\ell(\mu)}$ when $\underline{h}_\tau = (1, 1, \ldots, 1) = (1^{\ell(\mu)})$, and 0 otherwise. For instance, we have $|G|$-many choices for g_{m_1}. The equation $\underline{g}^\tau \underline{h} \underline{g}^{-1} = \underline{1}$ then specifies for us the values of $g_{\tau^q(m_1)}$, and also imposes the requirement that the product $h_{m_1} h_{\tau(m_1)} \cdots h_{\tau^{\mu_1}(m_1)} = 1$. The same argument holds for $m_2, \ldots, m_{\ell(\mu)}$. Our requirement is met for all the m_j precisely when $\underline{h}_\tau = (1^{\ell(\mu)})$, and in this case, we have $|G|^{\ell(\mu)}$ choices for $\underline{g}$.

□

Theorem 9 *If G is abelian, then $B = \{f_1^\chi : \chi(e_1) \neq 0\}$ is an orthogonal basis for the space of all w-function on G^r.*

Proof: We simply do a dimension count. We saw in claim 5 that the number of w-orbits for an abelian group is $\sum_{\lambda\vdash r}\frac{1}{z_\lambda}|G|^{\ell(\lambda)}$. We also computed in lemma 1 that $\mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1(\tau) = |G|^{\ell(sh(\tau))}$. So

$$\begin{aligned}
\langle \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle &= \langle \mathrm{Res}_{\mathfrak{S}_r}^{G_r}\mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1, 1\rangle_{\mathfrak{S}_r} = \frac{1}{r!}\sum_{\tau\in\mathfrak{S}_r}\mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1(\tau)\ \overline{1(\tau)}\\
&= \frac{1}{r!}\sum_{\tau\in\mathfrak{S}_r}|G|^{\ell(sh(\tau))}\\
&= \sum_{\lambda\vdash r}\frac{1}{z_\lambda}|G|^{\ell(\lambda)}.
\end{aligned}$$

Next we'll show that if $\chi(e_1)\neq 0$ then $\langle\chi, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle = 1$. In other words, it occurs with multiplicity one. Take any $\chi\in\mathrm{Irr}(G_r)$. Since G is abelian, χ can be realized as $\mathrm{Ind}_{H\ltimes G^r}^{G_r}\rho\cdot\phi$ where $\phi\in\mathrm{Irr}(G^r)$, $\rho\in\mathrm{Irr}(H)$, and $H\subset\mathfrak{S}_r$ is the stabilizer of ϕ with action given by $\sigma\cdot\phi(\underline{g}) = \phi(\sigma^{-1}\underline{g}\sigma)$. Both ϕ and ρ extend to characters of $H\ltimes G^r$, because H is the stabilizer of ϕ and because H is a quotient of $H\ltimes G^r$. Using Mackey's criterion, we see $\mathrm{Res}_{\mathfrak{S}_r}^{G_r}\mathrm{Ind}_{H\ltimes G^r}^{G_r}\rho\cdot\phi = \mathrm{Ind}_H^{\mathfrak{S}_r}\rho$. Hence

$$\begin{aligned}
\langle\chi, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle &= \langle \mathrm{Ind}_{H\ltimes G^r}^{G_r}\rho\cdot\phi, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle = \langle \mathrm{Res}_{\mathfrak{S}_r}^{G_r}\mathrm{Ind}_{H\ltimes G^r}^{G_r}\rho\cdot\phi, 1\rangle\\
&= \langle \mathrm{Ind}_H^{\mathfrak{S}_r}\rho, 1\rangle_{\mathfrak{S}_r} = \langle\rho, \mathrm{Res}_H^{\mathfrak{S}_r}1\rangle_H = \langle\rho,1\rangle_H\\
&= 1 \text{ or } 0
\end{aligned}$$

We conclude $|B| = \#\{\chi\in\mathrm{Irr}(G_r) : \chi(e_1)\neq 0\} = \#$ of χ in the support of $\mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1$ $= \langle \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle = \#$ of w-orbits.

□

Remark: Since characters of wreath products of the form G_r are well understood [8] and the combinatorics of the character theory of $\mathfrak{S}_n$ is well-understood, one could combine the two to get combinatorial formulas for the higher characters of $\mathfrak{S}_n$. We have computed some small examples, and as yet saw no structure in the data. (These computations were done in 1996-97, so perhaps more recent literature of which the author is unaware addresses this structure.)

4 Appendix In this appendix, we list the non-zero f_1^χ for $\chi\in\mathrm{Irr}(G)$ where $G=\mathfrak{S}_3$, $r=2, r=3$. The computations were done using the package GAP [11].

The first column lists w-orbit representatives, where $\tau=(23)$, $\sigma=(123)$. The second column is the size of each w-orbit. The very last row of the table gives the norm of f_1^χ. Notice the first row is just $\chi(e_1) = \langle\chi, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle$, and so one can read off the degree $\chi(1)$ by dividing the first row by the last.

The higher characters $\psi_\lambda^{(r)}$ for $\psi\in\mathrm{Irr}(\mathfrak{S}_3), \lambda\vdash r$ are contained in the table as well. Write $\mathrm{Irr}(\mathfrak{S}_3) = \{1, sgn, \mu\}$.

Then for $r=2$ (Table I.), the first column is $\frac{1}{2!}1_{(2)} = 1\otimes 1$, the second column is $\frac{1}{2!}sgn_{(2)} = sgn\otimes sgn$, the fourth column is $\frac{1}{2!}\mu_{(2)}$, and the fifth column is $\frac{1}{2!}sgn_{(11)}$. Notice the sum of the fourth and fifth columns is $\mu\otimes\mu\in\mathrm{Irr}(\mathfrak{S}_3^2)$.

For $r=3$ (Table II.), the first column is $\frac{1}{3!}1_{(3)} = 1\otimes 1\otimes 1$, the second column is $\frac{1}{3!}sgn_{(3)} = sgn\otimes sgn\otimes sgn$, the seventh column is $\frac{1}{3!}\mu_{(3)}$, and the thirteenth column is $\frac{1}{2}\frac{1}{3!}sgn_{(21)}$. Notice the second column plus twice the thirteenth column is $\mu\otimes\mu\otimes\mu\in\mathrm{Irr}(\mathfrak{S}_3^3)$.

I.

The f_1^χ for $G = \mathfrak{S}_3, r = 2$

$(1,1)$	1	1	1	1	3	1	2	2
$(1,\sigma\tau)$	6	1	-1	0	0	0	1	-1
$(1,\sigma)$	4	1	1	1	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
(τ,τ)	3	1	1	-1	1	-1	0	0
$(\tau,\sigma\tau)$	6	1	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
(τ,σ)	12	1	-1	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
(σ,σ)	2	1	1	1	0	1	-1	-1
(σ,σ^{-1})	2	1	1	1	$\frac{3}{2}$	$-\frac{1}{2}$	-1	-1
norm		1	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$

II.

The f_1^χ for $G = \mathfrak{S}_3, r = 3$

$(1,1,1)$	1	1	1	1	1	2	2	4	2	1	3	1	3	2
$(1,1,\tau)$	9	1	-1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$	$-\frac{4}{3}$	0	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	-1	0
$(1,1,\sigma)$	6	1	1	1	1	1	1	-2	1	0	0	0	0	-1
$(1,\tau,\tau)$	9	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$
$(1,\tau,\sigma\tau)$	18	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
$(1,\tau,\sigma)$	36	1	-1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	$-\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$	0
$(1,\sigma,\sigma)$	6	1	1	1	1	0	0	0	0	0	-1	0	-1	1
$(1,\sigma,\sigma^{-1})$	6	1	1	1	1	0	0	2	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
(τ,τ,τ)	3	1	-1	-1	1	0	0	0	0	-1	1	1	-1	0
$(\tau,\tau,\sigma\tau)$	18	1	-1	-1	1	0	0	0	0	0	0	0	0	0
(τ,τ,σ)	18	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$(\tau,\sigma\tau,\sigma)$	36	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
$(\tau,\sigma\tau,\tau\sigma)$	6	1	-1	-1	1	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0
(τ,σ,σ)	18	1	-1	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	0	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	0
$(\tau,\sigma,\sigma^{-1})$	18	1	-1	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	0	0	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	0
(σ,σ,σ)	2	1	1	1	1	-1	-1	1	-1	1	0	1	0	-1
$(\sigma,\sigma,\sigma^{-1})$	6	1	1	1	1	-1	-1	-1	-1	0	1	0	1	0
norm		1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{8}$

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