ON SOME VARIATIONS OF GLEASON'S GAME

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Abstract. We consider a two-person zero-sum game where the players alternate their moves until each of them has made a total of n moves. A move of either player consists of instructing a referee to move a chip either clockwise or counterclockwise to the next node around a three-node board. These three nodes are arranged in a circle and are labeled +1, +2 and -3. The main feature here is neither player is informed of any of his opponent's past or current moves. Whenever the chip visits a node there is an intermediate payoff equal to the label on that node. The payoff is taken to be the sum of these intermediate payoffs at termination. Each player can remember all his own past moves and therefore may use a history of such moves to decide his next move. This game is solved for all positive integral values of n.

1 Introduction In the early 1950's Andrew Gleason of Harvard proposed an interesting two-person zero-sum game. This stochastic game with an information lag for both players has a very simple description but turns out to be quite difficult to solve. Ferguson and Shapley[2] described Gleason's Game as follows. The two players move a chip around a three-node board (see Figure 1). The nodes are arranged in a circle, and are labeled +1, +2 and -3. Initially the chip rests on node +1 and player 1 starts. Thereafter, the players move alternately. There is a one move delay in informing the players of the position of the chip, so that, except for the first move, the players make their move only knowing the node from which the opponent has just moved. A move consists of instructing a referee to move the chip either clockwise or counterclockwise to the next node; the players are not allowed to leave the chip where it is. After each move is given to the referee, the referee announces the node that the chip has just left, and requires player 2 to pay player 1 an amount equal to the label of that node. The problem is for player 1 to maximize and for player 2 to minimize the limiting average payoff.

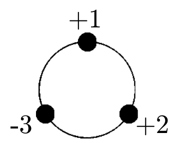

Figure 1: The three-node board.

Ferguson and Shapley explained why Gleason's Game seems easy but is actually hard: "When the referee announces the state just vacated, both players know the history of the game up to that point. Indeed, this information is common knowledge (both players know the other knows, both know the other knows he knows, etc.). At first sight, it might be thought that one needs not remember back past that point in choosing a strategy. This is

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not so because when the opponent made his last move, he had to choose it not knowing the actual state, and you should be able to take advantage of that. And the opponent chose his strategy trying to take advantage of your lack of knowledge of the previous state, so that should be taken into account, and so on." They solved Gleason's Game by first converting it to a stochastic game using the notion of generalized subgames. The functional equations associated with these generalized subgames are then solved with an iterative method that involves alternately solving a finite game with perfect information and a Markov decision problem with limiting average payoff. This leads to $-0.09336 < v < -0.09323$, where v is the value of Gleason's Game. It was also shown by them that no strategy that remembers only a bounded number of past moves can be optimal. Games in which a player needs to remember a history of all its past moves in order to play optimally are difficult to analyze, and only a very small number of them had ever been solved. The best-known example is undoubtedly the Big Match due to Blackwell and Ferguson[1].

In this paper we consider a game that differs from Gleason's Game in terms of the information made available to the players. More specifically, we assume neither player is informed of any moves, past or current, made by the other player. It is surprising there exist optimal strategies that use a mixture of at most four pure strategies irrespective of the duration of the game. Games in which each player is not informed of any of the opponent's moves are sometimes called games with no information. The published literature on games with no information is limited. Because of their nature, these games are almost exclusively certain search games where a mobile seeker searches a mobile hider in darkness over some specified region. A good example is the Princess and Monster Game; see, for example, Foreman[3], Gal[4], Garnaev[5], Wilson[8], and Worsham[9]. It is therefore of interest to find a multi-stage game with no information that is not a search game and that can be solved exactly.

2 The Game Γ_n We consider the two-person zero-sum game Γ_n where n is any positive integer. A referee with the board shown in Figure 1 is stationed in one room. The players, called player 1 and player 2, are isolated from one another and also from the referee in another two rooms. Using an intercom, the referee can talk to either player. The players know n, how the board looks like, and that the chip is initially at node $+1$. Here is how the game proceeds. The referee calls player 1 and asks him to make his first move. Player 1 is only allowed to decide between moving clockwise or counterclockwise. If player 1 chooses clockwise (counterclockwise), the referee moves the chip clockwise (counterclockwise) from its initial position at node $+1$ to the next node on the board and records its label as the intermediate payoff. That is, the intermediate payoff is taken to be the label of the node that the chip visits as the result of a player's move. For example, if the first move of player 1 is to move clockwise, the referee will move the chip to node $+2$, and he will record $+2$ as the intermediate payoff. The referee next calls player 2 and asks him to make his first move. Player 2 also has to decide between moving clockwise or counterclockwise. If player 2 chooses clockwise (counterclockwise), the referee moves the chip clockwise (counterclockwise) from its current position to the next node on the board. He then records the label of that node as another intermediate payoff. The referee then calls player 1 and asks him to make his second move. This process continues with the players moving alternately. While the game is still in progress the referee reveals nothing else to the players. The game terminates after each player has completed n moves, making a total of $2n$ moves between the two players. After the game terminates, the payoff is taken to be the sum of the $2n$ intermediate payoffs recorded by the referee. The objective in Γ_n is for player 1 to maximize and for player 2 to minimize this payoff.

There are certain differences between Γ_n and Gleason's Game. The former is a finite game for each n while the latter is an infinite game. We avoid matrix games with countably many pure strategies since such games exhibit several undesirable properties not found among finite games. Furthermore, since n can be made arbitrarily large, nothing of significance is lost by considering finite games in our case. In Gleason's Game, the payoff is the average of the intermediate payoffs, chosen out of necessity to avoid unbounded payoffs. Such a problem does not arise in Γ_n, so the payoff is chosen to be the sum of the intermediate payoffs for ease of exposition. The main difference between the two games occurs in how the information of the opponent's moves is revealed. In Gleason's Game, this information is given one move late while in Γ_n it is not given at all.

Although a player has no access to his opponent's moves we assume he can remember all his own past moves. Hence he may use a history of such moves to decide his next move.

3 A Solution of Γ_n To obtain a solution, we first represent Γ_n in **extensive form** as a tree with its information sets. In standard terminology, this representation is a game of **perfect recall**. We next examine the set of pure strategies, and reduce their number by using the **reduced normal form** of an extensive game. This method of reducing the number of pure strategies is due to Kuhn[6, 7]. After performing such reduction to our case, we are still left with 2^n pure strategies for each player. This is the major obstacle to solve a game in extensive form since the size of a mixed strategy generally grows exponentially in the size of the game tree. Such a huge increase in the size often renders a problem computationally intractable.

Each pure strategy may be identified by a path of length n. Let m be a **move** of a player. We take it that a move has the value 1 or -1: $m = 1$ ($m = -1$) stands for a clockwise (counterclockwise) move. An ordered tuple $(m_1, m_2, \ldots, m_n)$ of n moves is called a **path** of length n. All the moves in a path belong to the same player. In the above path, m_i denotes his i^{th} move. A path for a player is a description how he moves. For example, the path $(-1, 1, 1)$ for player 2 means his first move is counterclockwise, his second and third moves are clockwise. For brevity, we hereafter refer a pure strategy as a path.

We next construct the game matrix of size $2^n \times 2^n$. We adopt the standard convention that player 1 chooses a row and player 2 chooses a column. Let $node = +1, +2,$ or -3, $s_n = (m_1, m_2, \ldots, m_n)$ be a path of player 1, $t_n = (m'_1, m'_2, \ldots, m'_n)$ be a path of player 2, and m and m' be moves. The length of a given path is indicated by its subscript. Let

$\langle s_n, t_n\rangle$	Payoff to player 1 in Γ_n if player 1 uses s_n and player 2 uses t_n
$\text{leaf}\,(s_n, t_n)$	Node where the chip is found at termination in Γ_n
$\langle node : m, m'\rangle$	Sum of the two intermediate payoffs if player 1 chooses m and then player 2 chooses m', assuming that the chip is at $node$ just before player 1 chooses m
$s_n \circ m$	$(m_1, m_2, \ldots, m_n, m)$
$\text{sum}\,(s_n, t_n)$	$\sum_{i=1}^{n}(m_i + m'_i)$

Suppose $s_3 = (1, 1, -1)$ and $t_3 = (-1, 1, 1)$. Then $\langle s_3, t_3\rangle = 2+1+2+(-3)+2+(-3) = 1$; $\text{leaf}\,(s_3, t_3) = -3$; $\langle +2 : -1, -1\rangle = 1 + (-3) = -2$; $\langle -3 : -1, 1\rangle = 2 + (-3) = -1$; $s_3 \circ 1 = (1, 1, -1, 1)$; $\text{sum}\,(s_3, t_3) = 2$. The following relations are easy to establish.

$$\langle s_n \circ m, t_n \circ m'\rangle = \langle s_n, t_n\rangle + \langle \text{leaf}\,(s_n, t_n) : m, m'\rangle,$$

(1) $$\text{sum}\,(s_n, t_n) \equiv 0 \pmod 3 \text{ if and only if leaf}\,(s_n, t_n) = +1,$$

(2) $$\text{sum}\,(s_n, t_n) \equiv 1 \pmod 3 \text{ if and only if leaf}\,(s_n, t_n) = +2,$$

(3) $$\text{sum}\,(s_n, t_n) \equiv 2 \pmod 3 \text{ if and only if leaf}\,(s_n, t_n) = -3.$$

Proposition 1 The value of Γ_1 is -1, and the value of Γ_n is $-\frac{2}{9}$ for $n \geq 2$.
Define the paths of length $n \geq 1$ by

$$s_n^1 = (1, 1, 1, 1, -1, 1, \ldots, (-1)^n), \qquad s_n^2 = (1, 1, -1, 1, -1, 1, \ldots, (-1)^n),$$
$$s_n^3 = (1, -1, 1, 1, -1, 1, \ldots, (-1)^n), \qquad s_n^4 = (1, -1, -1, 1, -1, 1, \ldots, (-1)^n),$$
$$t_n^1 = (1, 1, 1, -1, 1, -1, \ldots, (-1)^{n+1}), \qquad t_n^2 = (1, 1, -1, 1, -1, 1, \ldots, (-1)^n),$$
$$t_n^3 = (1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}), \qquad t_n^4 = (1, -1, -1, 1, -1, 1, \ldots, (-1)^n).$$

Let $x_n^* = \frac{1}{3}\, s_n^1 + \frac{1}{9}\, s_n^2 + \frac{2}{9}\, s_n^3 + \frac{1}{3}\, s_n^4$ and $y_n^* = \frac{2}{9}\, t_n^1 + \frac{1}{3}\, t_n^2 + \frac{1}{3}\, t_n^3 + \frac{1}{9}\, t_n^4$.
Then x_n^* is an optimal strategy of player 1 and y_n^* is an optimal strategy of player 2 in Γ_n for $n \geq 1$.

We clarify certain points in Proposition 1. We only discuss the case for player 1; similar discussion applies to player 2. (1) The four paths s_n^1 to s_n^4 of player 1 are defined with respect to n starting from the left end of the tuples. For example, $s_2^3 = (1, -1)$. (2) Beginning with the fourth move, the moves in each path always alternate between 1 and -1. (3) When $n = 1$ or $n = 2$, s_n^1 to s_n^4 are not distinct. In these cases, x_n^* are interpreted as follows. For $n = 2$, $x_2^* = \frac{1}{3}\, s_2^1 + \frac{1}{9}\, s_2^2 + \frac{2}{9}\, s_2^3 + \frac{1}{3}\, s_2^4 = \frac{1}{3}\,(1,1) + \frac{1}{9}\,(1,1) + \frac{2}{9}\,(1,-1) + \frac{1}{3}\,(1,-1) = \frac{4}{9}\,(1,1) + \frac{5}{9}\,(1,-1)$. For $n = 1$, it is easy to see $x_1^* = 1(1)$, that is, player 1 always moves clockwise.

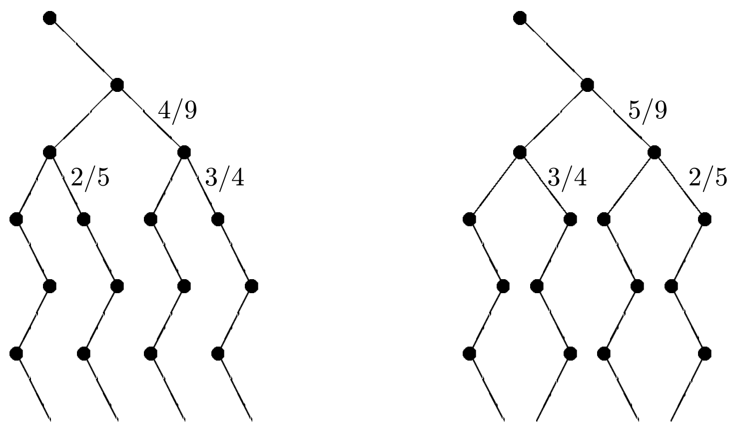

Figure 2: Optimal behavior strategies for player 1(left) and player 2(right) in Γ_n.

To better visualize the optimal strategies in Proposition 1, we represent them as behavior strategies in Figure 2. In this figure, an outgoing edge towards the lower right (left) is a clockwise (counterclockwise) move. If a node has only one outgoing edge, that edge is clearly chosen with certainty. If there are two outgoing edges, the probabilities of choosing them sum to one; we only show the probability choosing the clockwise move. One interesting observation is that the optimal strategies embed among themselves. By this is meant that if a player removes the last move from his optimal strategy in Γ_n, he will obtain an optimal strategy in Γ_{n-1}. This implies the players do not even need to know the value of n to play optimally. We now derive some results required to prove Proposition 1.

Lemma 1 Let $n \geq 3$ and let s_n^1 to s_n^4 be defined as in Proposition 1. For any path t_n of player 2, one of the following three cases holds.

$$
\begin{aligned}
&\text{leaf}\,(s_n^1, t_n) = +1,\ \text{leaf}\,(s_n^2, t_n) = +2,\ \text{leaf}\,(s_n^3, t_n) = +2,\ \text{leaf}\,(s_n^4, t_n) = -3.\\
&\text{leaf}\,(s_n^1, t_n) = +2,\ \text{leaf}\,(s_n^2, t_n) = -3,\ \text{leaf}\,(s_n^3, t_n) = -3,\ \text{leaf}\,(s_n^4, t_n) = +1.\\
&\text{leaf}\,(s_n^1, t_n) = -3,\ \text{leaf}\,(s_n^2, t_n) = +1,\ \text{leaf}\,(s_n^3, t_n) = +1,\ \text{leaf}\,(s_n^4, t_n) = +2.
\end{aligned} \tag{4}
$$

Proof Since the chip has to terminate at a node, leaf $(s_n^1, t_n) = +1$, $+2$ or -3. Suppose leaf $(s_n^1, t_n) = +1$. We need to show leaf $(s_n^2, t_n) = +2$, leaf $(s_n^3, t_n) = +2$, and leaf $(s_n^4, t_n) = -3$. This is the first case in (4). Let sum $(s_n^1, t_n) = r$. Then sum $(s_n^2, t_n) = r - 2$, sum $(s_n^3, t_n) = r - 2$, and sum $(s_n^4, t_n) = r - 4$. To see that sum $(s_n^2, t_n) = r - 2$, we just need to examine those moves that differ in s_n^1 and s_n^2. Only their third moves are different, so the result follows.
¿From (1), leaf $(s_n^1, t_n) = +1$ implies sum $(s_n^1, t_n) \equiv 0 \pmod 3$, that is, $r \equiv 0 \pmod 3$.

$$
\begin{aligned}
&\text{sum}\,(s_n^2, t_n) = r - 2 \equiv r + 1 \equiv 1 \pmod 3 \text{ implies from (2) that leaf}\,(s_n^2, t_n) = +2.\\
&\text{sum}\,(s_n^3, t_n) = r - 2 \equiv r + 1 \equiv 1 \pmod 3 \text{ implies from (2) that leaf}\,(s_n^3, t_n) = +2.\\
&\text{sum}\,(s_n^4, t_n) = r - 4 \equiv r + 2 \equiv 2 \pmod 3 \text{ implies from (3) that leaf}\,(s_n^4, t_n) = -3.
\end{aligned}
$$

The second and third cases of (4) can be treated similarly.◇

Lemma 2 is the companion to Lemma 1. It is slightly more involved because we have to separate the cases for n odd or even.

Lemma 2 Let $n \geq 3$ and t_n^1 to t_n^4 be defined as in Proposition 1. Let s_n be any path of player 1. If n is odd, one of the following three cases holds.

$$
\begin{aligned}
&\text{leaf}\,(s_n, t_n^1) = +1,\ \text{leaf}\,(s_n, t_n^2) = +2,\ \text{leaf}\,(s_n, t_n^3) = +2,\ \text{leaf}\,(s_n, t_n^4) = -3.\\
&\text{leaf}\,(s_n, t_n^1) = +2,\ \text{leaf}\,(s_n, t_n^2) = -3,\ \text{leaf}\,(s_n, t_n^3) = -3,\ \text{leaf}\,(s_n, t_n^4) = +1.\\
&\text{leaf}\,(s_n, t_n^1) = -3,\ \text{leaf}\,(s_n, t_n^2) = +1,\ \text{leaf}\,(s_n, t_n^3) = +1,\ \text{leaf}\,(s_n, t_n^4) = +2.
\end{aligned} \tag{5}
$$

If n is even, one of the following three cases holds.

$$
\begin{aligned}
&\text{leaf}\,(s_n, t_n^1) = +1,\ \text{leaf}\,(s_n, t_n^2) = +1,\ \text{leaf}\,(s_n, t_n^3) = +2,\ \text{leaf}\,(s_n, t_n^4) = +2.\\
&\text{leaf}\,(s_n, t_n^1) = +2,\ \text{leaf}\,(s_n, t_n^2) = +2,\ \text{leaf}\,(s_n, t_n^3) = -3,\ \text{leaf}\,(s_n, t_n^4) = -3.\\
&\text{leaf}\,(s_n, t_n^1) = -3,\ \text{leaf}\,(s_n, t_n^2) = -3,\ \text{leaf}\,(s_n, t_n^3) = +1,\ \text{leaf}\,(s_n, t_n^4) = +1.
\end{aligned} \tag{6}
$$

Proof Proceed as in the proof of Lemma 1. If sum $(s_n, t_n^1) = r$, we need to verify that:
For n odd, sum $(s_n, t_n^2) = r - 2$, sum $(s_n, t_n^3) = r - 2$, and sum $(s_n, t_n^4) = r - 4$.
For n even, sum $(s_n, t_n^2) = r$, sum $(s_n, t_n^3) = r - 2$, and sum $(s_n, t_n^4) = r - 2$.◇

Let

$$
\begin{aligned}
U &= \{(+1, +2, +2, -3),\ (+2, -3, -3, +1),\ (-3, +1, +1, +2)\},\\
V &= \{(+1, +1, +2, +2),\ (+2, +2, -3, -3),\ (-3, -3, +1, +1)\}.
\end{aligned}
$$

Each ordered tuple in U is a case in (4) (or (5)), and each tuple in V is a case in (6). The proof of Lemma 3 below is by straightforward exhaustive evaluation and is omitted.

Lemma 3 Let m and m' be any moves.
(i) For each (k_1, k_2, k_3, k_4) in U,

$$
\frac{1}{3}\langle k_1 : m, m' \rangle + \frac{1}{9}\langle k_2 : m, m' \rangle + \frac{2}{9}\langle k_3 : m, m' \rangle + \frac{1}{3}\langle k_4 : m, m' \rangle = 0. \tag{7}
$$

(ii) For each (k_1, k_2, k_3, k_4) in U,

$$\frac{2}{9}\langle k_1 : m, -1\rangle + \frac{1}{3}\langle k_2 : m, 1\rangle + \frac{1}{3}\langle k_3 : m, -1\rangle + \frac{1}{9}\langle k_4 : m, 1\rangle = 0. \tag{8}$$

(iii) For each (k_1, k_2, k_3, k_4) in V,

$$\frac{2}{9}\langle k_1 : m, 1\rangle + \frac{1}{3}\langle k_2 : m, -1\rangle + \frac{1}{3}\langle k_3 : m, 1\rangle + \frac{1}{9}\langle k_4 : m, -1\rangle = 0. \tag{9}$$

Let $E(x_n, t_n)$ denote the expected payoff (to player 1) if player 1 uses the mixed strategy x_n and player 2 uses the path t_n. Let $E(s_n, y_n)$ denote the expected payoff if player 1 uses the path s_n and player 2 uses the mixed strategy y_n.

Lemma 4 Let $n \geq 3$. Let x^*_n and y^*_n be defined as in Proposition 1. Then

$$E(x^*_{n+1}, t_n \circ m') = E(x^*_n, t_n) \text{ for all paths } t_n \text{ and for all moves } m', \tag{10}$$
$$E(s_n \circ m, y^*_{n+1}) = E(s_n, y^*_n) \text{ for all paths } s_n \text{ and for all moves } m. \tag{11}$$

Proof We only prove (11). Let $n \geq 3$, s_n be any path, and m be any move. Let $\tilde{m}$ denote the $(n+1)^{\text{th}}$ move in the paths t^i_{n+1} $(i = 1, 2, 3, 4)$ that are defined in Proposition 1. When $i = 1, 3$, this $\tilde{m}$ move is clockwise (counterclockwise) for n even (odd). When $i = 2, 4$, this move is clockwise (counterclockwise) for n odd(even). That is, for $i = 1, 3$, $\tilde{m} = (-1)^n$, and for $i = 2, 4$, $\tilde{m} = (-1)^{n+1}$. The above observation may also be seen from the right diagram in Figure 2 where these paths are numbered from right to left. Hence for $i = 1, 3$, $t^i_{n+1} = t^i_n \circ (-1)^n$, and for $i = 2, 4$, $t^i_{n+1} = t^i_n \circ (-1)^{n+1}$. We have

$$\begin{aligned}
E(s_n \circ m, y^*_{n+1}) &= E(s_n \circ m, \frac{2}{9} t^1_{n+1} + \frac{1}{3} t^2_{n+1} + \frac{1}{3} t^3_{n+1} + \frac{1}{9} t^4_{n+1}) \\
&= \frac{2}{9}\langle s_n \circ m, t^1_{n+1}\rangle + \frac{1}{3}\langle s_n \circ m, t^2_{n+1}\rangle + \frac{1}{3}\langle s_n \circ m, t^3_{n+1}\rangle + \frac{1}{9}\langle s_n \circ m, t^4_{n+1}\rangle \\
&= \frac{2}{9}\langle s_n \circ m, t^1_n \circ (-1)^n\rangle + \frac{1}{3}\langle s_n \circ m, t^2_n \circ (-1)^{n+1}\rangle \\
&\quad + \frac{1}{3}\langle s_n \circ m, t^3_n \circ (-1)^n\rangle + \frac{1}{9}\langle s_n \circ m, t^4_n \circ (-1)^{n+1}\rangle \\
&= \frac{2}{9}\langle s_n, t^1_n\rangle + \frac{1}{3}\langle s_n, t^2_n\rangle + \frac{1}{3}\langle s_n, t^3_n\rangle + \frac{1}{9}\langle s_n, t^4_n\rangle \\
&\quad + \frac{2}{9}\langle \text{leaf}(s_n, t^1_n) : m, (-1)^n\rangle + \frac{1}{3}\langle \text{leaf}(s_n, t^2_n) : m, (-1)^{n+1}\rangle \\
&\quad + \frac{1}{3}\langle \text{leaf}(s_n, t^3_n) : m, (-1)^n\rangle + \frac{1}{9}\langle \text{leaf}(s_n, t^4_n) : m, (-1)^{n+1}\rangle \\
&= E(s_n, y^*_n) + \theta \text{ where} \\
\theta &= \frac{2}{9}\langle \text{leaf}(s_n, t^1_n) : m, (-1)^n\rangle + \frac{1}{3}\langle \text{leaf}(s_n, t^2_n) : m, (-1)^{n+1}\rangle \\
&\quad + \frac{1}{3}\langle \text{leaf}(s_n, t^3_n) : m, (-1)^n\rangle + \frac{1}{9}\langle \text{leaf}(s_n, t^4_n) : m, (-1)^{n+1}\rangle.
\end{aligned}$$

We are done if we can show $\theta = 0$.
Let n be odd. From (5),

$$\theta = \frac{2}{9}\langle k_1 : m, -1\rangle + \frac{1}{3}\langle k_2 : m, 1\rangle + \frac{1}{3}\langle k_3 : m, -1\rangle + \frac{1}{9}\langle k_4 : m, 1\rangle$$

for some (k_1, k_2, k_3, k_4) in U. Using (8), $\theta = 0$.
Let n be even. From (6),

$$\theta = \frac{2}{9}\langle k_1 : m, 1\rangle + \frac{1}{3}\langle k_2 : m, -1\rangle + \frac{1}{3}\langle k_3 : m, 1\rangle + \frac{1}{9}\langle k_4 : m, -1\rangle$$

for some (k_1, k_2, k_3, k_4) in V. Using (9), $\theta = 0.\diamondsuit$

When constructing the payoff matrix of Γ_n, it is useful to enumerate the 2^n paths of a player in some consistent way from path 1 to path 2^n. For $1 \le i \le 2^n$, we define path i as follows. First convert $i-1$ to binary as a string of 0 and 1. If necessary, left pad by adding extra 0 in front until we have a string of length n. Replace all 1 by -1, and then replace all 0 by 1. As an illustration suppose $n = 5$ and $i = 14$. In binary, $13 = 1101$ and we need to add one extra 0 in front to get 01101. Replace all 1 by -1 to obtain $0, -1, -1, 0, -1$. Replace all 0 by 1 to obtain $1, -1-1, 1, -1$. Hence path 14 is $(1, -1, -1, 1, -1)$.

Proof of Proposition 1 When $n = 1$, the payoff matrix is

-1	3
-2	-1

This matrix has a saddle-point in pure strategies; the **-1** in bold is its saddle-point value.

Now let $n \ge 2$. The conclusion of Proposition 1 follows if we can prove

$$E(x_n^*, t_n) \ \ge \ -\frac{2}{9} \quad \text{for all paths } t_n, \tag{12}$$

$$E(s_n, y_n^*) \ \le \ -\frac{2}{9} \quad \text{for all paths } s_n. \tag{13}$$

We only prove (12) since the proof of (13) is similar. When $n = 2, 3$, the payoff matrices are respectively

2	-3	2	6
-2	2	1	2
-3	1	-3	-2
-4	-3	2	-3

0	1	0	-5	5	0	5	9
5	0	-4	0	1	5	4	5
1	-4	1	5	0	4	0	1
-3	1	0	1	-1	0	5	0
0	-5	0	4	-4	0	-4	-3
-4	0	-1	0	-5	-4	1	-4
-5	-1	-5	-4	0	1	0	-5
-6	-5	0	-5	5	0	-4	0

We verify directly, using the above matrices, that (12) is true for $n = 2, 3$.
Assume now (12) is true for $n = k \ge 3$.
Using (10) with n replaced by k, for all paths t_k and all moves m',

$$E(x_{k+1}^*, t_k \circ m') = E(x_k^*, t_k).$$

Hence for all moves m',

$$E(x_{k+1}^*, t_k \circ m') = E(x_k^*, t_k) \ge -\frac{2}{9} \quad \text{for all paths } t_k.$$

If t_k runs through all the 2^k paths of length k, and m' runs through the two moves $m' = 1$ and $m' = -1$, then $t_k \circ m'$ will run through all the 2^{k+1} paths of length $k+1$. Writing t_{k+1} for $t_k \circ m'$, we obtain

$$E(x^*_{k+1}, t_{k+1}) \geq -\frac{2}{9} \quad \text{for all paths } t_{k+1}.$$

By induction (12) is true for $n \geq 3$. Including the case for $n = 2$ that has been verified by direct calculation, (12) is true for $n \geq 2$.$\diamondsuit$

There are many optimal strategies. We restrict the discussion to those optimal strategies that address the following problem: What are some alternate optimal strategies if all we can change are the fourth or later moves in the paths in Proposition 1? The weights in Proposition 1 cannot be changed.

Using (7) it can be shown the following sets of paths also work for player 1:

$$s^1_n = (1, 1, 1, m_4, \dots, m_n), \qquad s^2_n = (1, 1, -1, m_4, \dots, m_n),$$
$$s^3_n = (1, -1, 1, m_4, \dots, m_n), \quad s^4_n = (1, -1, -1, m_4, \dots, m_n)$$

where for $4 \leq k \leq n$, m_k may take the value +1 or -1 independent of k. Thus there are at least 2^{n-3} sets of such paths.

Alternate optimal strategies with at most 4 paths seem to occur less frequent for player 2. Besides the one given in Proposition 1, the only other set of paths that we are able to find is

$$t^1_n = (1, 1, 1, -1, 1, -1, \dots, (-1)^{n+1}), \quad t^2_n = (1, 1, -1, -1, 1, -1, \dots, (-1)^{n+1}),$$
$$t^3_n = (1, -1, 1, 1, -1, 1, \dots, (-1)^n), \qquad t^4_n = (1, -1, -1, 1, -1, 1, \dots, (-1)^n).$$

For a (mixed) strategy x_n and a path s_n, let $x_n(s_n)$ denote the probability that x_n assigns to s_n. The support of x_n is defined as the set of paths s_n with $x_n(s_n) > 0$. The size of x_n is then defined to be the cardinality of its support. Proposition 1 says each player in Γ_n has an optimal strategy of size 4 or less for all positive n. We show this remains true even when we change the initial position of the chip at the start of the game.

4 Games with Other Initial Positions In Γ_n, the chip rests initially at node +1 when the game starts. Let Γ^2_n and Γ^{-3}_n denote the game where the chip rests initially at node +2 and at node -3 respectively. All other aspects of Γ_n are assumed to remain unchanged, like player 1 still making the first move.

First consider Γ^2_n. We solve Γ^2_n in the same way as we have solved Γ_n. The basic method is to solve Γ^2_n numerically for small values of n, make a guess, and then verify the guess.

Proposition 2 The value of Γ^2_1 is $-\frac{7}{6}$, and the value of Γ^2_n is $-\frac{7}{9}$ for $n \geq 2$.
In Γ^2_1, an optimal strategy of player 1 is to choose the clockwise move with probability $\frac{5}{6}$, and an optimal strategy of player 2 is to choose the clockwise move with probability $\frac{1}{6}$.
Define the paths of length $n \geq 2$ by

$$\hat{s}^1_n = (1, -1, 1, 1, -1, 1, \dots, (-1)^n), \qquad \hat{s}^2_n = (1, -1, -1, 1, -1, 1, \dots, (-1)^n),$$
$$\hat{s}^3_n = (-1, -1, 1, 1, -1, 1, \dots, (-1)^n), \qquad \hat{s}^4_n = (-1, -1, -1, 1, -1, 1, \dots, (-1)^n),$$
$$\hat{t}^1_n = (-1, 1, 1, -1, 1, -1, \dots, (-1)^{n+1}), \quad \hat{t}^2_n = (-1, 1, -1, -1, 1, -1, \dots, (-1)^{n+1}),$$
$$\hat{t}^3_n = (-1, -1, 1, 1, -1, 1, \dots, (-1)^n), \qquad \hat{t}^4_n = (-1, -1, -1, 1, -1, 1, \dots, (-1)^n).$$

Let $\hat{x}^*_n = \frac{1}{3}\,\hat{s}^1_n + \frac{1}{9}\,\hat{s}^2_n + \frac{2}{9}\,\hat{s}^3_n + \frac{1}{3}\,\hat{s}^4_n$ and $\hat{y}^*_n = \frac{1}{9}\,\hat{t}^1_n + \frac{1}{3}\,\hat{t}^2_n + \frac{1}{3}\,\hat{t}^3_n + \frac{2}{9}\,\hat{t}^4_n$.
Then $\hat{x}^*_n$ is an optimal strategy of player 1 and $\hat{y}^*_n$ is an optimal strategy of player 2 in Γ^2_n for $n \geq 2$.

We have given the optimal strategies separately for $n = 1$ since we cannot extend the definition of $\hat{x}^*_n$ and $\hat{y}^*_n$ above to include this case. Let us see what happen if we do otherwise. For example, $\hat{x}^*_1 = \frac{4}{9}(1) + \frac{5}{9}(-1)$, that is, to choose the clockwise move with probability $\frac{4}{9}$. It is easily seen that this strategy is not optimal. We can prove Proposition 2 with the same method we have used to prove Proposition 1. Since the chip now starts initially from node $+2$, we have to make some minor adjustments to reflect this fact. For example, (1) has to be replaced by

$$\text{sum}(s_n, t_n) \equiv 0 \pmod 3 \text{ if and only if } \text{leaf}(s_n, t_n) = +2.$$

We omit the proof of Proposition 2.

Now consider Γ^{-3}_n. Here we may guess a solution of Γ^{-3}_n based on the known solution of Γ_n. Consider the game Γ_{n+1}. Using the optimal strategies in Proposition 1 both players choose clockwise for their first move. Thus the referee moves the chip to node -3 and he is about to call player 1 to choose his second move. From that moment the game to be played is precisely Γ^{-3}_n. Hence the guess is that for $n \geq 1$, value $(\Gamma_{n+1}) = 2 + (-3) + \text{value}(\Gamma^{-3}_n)$, or value $(\Gamma^{-3}_n) = \frac{7}{9}$. Furthermore, if we delete the first move from each player's optimal strategy in Γ_{n+1} we should obtain his optimal strategy in Γ^{-3}_n. It turns out the above guess is correct. For completeness we summarize the results in Proposition 3.

Proposition 3 The value of Γ^{-3}_n is $\frac{7}{9}$ for $n \geq 1$. Define the paths of length $n \geq 1$ by

$$\begin{aligned}
\tilde{s}^1_n &= (1, 1, 1, -1, 1, -1, \dots, (-1)^{n+1}), & \tilde{s}^2_n &= (1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}), \\
\tilde{s}^3_n &= (-1, 1, 1, -1, 1, -1, \dots, (-1)^{n+1}), & \tilde{s}^4_n &= (-1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}), \\
\tilde{t}^1_n &= (1, 1, -1, 1, -1, 1, \dots, (-1)^{n}), & \tilde{t}^2_n &= (1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}), \\
\tilde{t}^3_n &= (-1, 1, -1, 1, -1, 1, \dots, (-1)^{n}), & \tilde{t}^4_n &= (-1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}).
\end{aligned}$$

Let $\tilde{x}^*_n = \frac{1}{3}\tilde{s}^1_n + \frac{1}{9}\tilde{s}^2_n + \frac{2}{9}\tilde{s}^3_n + \frac{1}{3}\tilde{s}^4_n$ and $\tilde{y}^*_n = \frac{2}{9}\tilde{t}^1_n + \frac{1}{3}\tilde{t}^2_n + \frac{1}{3}\tilde{t}^3_n + \frac{1}{9}\tilde{t}^4_n$.
Then $\tilde{x}^*_n$ is an optimal strategy of player 1 and $\tilde{y}^*_n$ is an optimal strategy of player 2 in Γ^{-3}_n for $n \geq 1$.

5 The Game $\Omega_n(a, b, c)$ We may generalize further by adding a chance move at the very beginning before player 1 chooses his first move. Here is how the chance move may be implemented. The referee uses a probability distribution to select the node where the chip will rest initially. Let the probabilities of selecting nodes 1, 2 and -3 be a, b and c respectively where $a + b + c = 1$. Both players are informed the values of a, b and c but are not informed the outcome of this chance move. All other aspects of the game remain unchanged. Let $\Omega_n(a, b, c)$ denote this game. We have already solved three special cases of $\Omega_n(a, b, c)$: $\Omega_n(1, 0, 0) = \Gamma_n$, $\Omega_n(0, 1, 0) = \Gamma^2_n$, and $\Omega_n(0, 0, 1) = \Gamma^{-3}_n$.

It is helpful to recall what we have done earlier. Associated with each of Γ_n, Γ^2_n or Γ^{-3}_n is an $2^n \times 2^n$ payoff matrix where each entry is obtained by summing $2n$ intermediate payoffs. Let these matrices be called $\mathbf{A}(\Gamma_n)$, $\mathbf{A}(\Gamma^2_n)$ and $\mathbf{A}(\Gamma^{-3}_n)$ respectively. $\mathbf{A}(\Omega_n(a, b, c))$, the payoff matrix of $\Omega_n(a, b, c)$, is clearly given by

$$\mathbf{A}(\Omega_n(a, b, c)) = a\,\mathbf{A}(\Gamma_n) + b\,\mathbf{A}(\Gamma^2_n) + c\,\mathbf{A}(\Gamma^{-3}_n).$$

For any specific values of a, b and c, it is possible but laborious to use the above relation to solve $\Omega_n(a, b, c)$ by applying the same method as we have done for Γ_n. Instead of doing that, we restrict our attention now to explore when $\Omega_n(a, b, c)$ may be solved easily. Since we already know the solutions of Γ_n, Γ^2_n and Γ^{-3}_n, it is natural to investigate how to reduce $\Omega_n(a, b, c)$ to a game that is similar in some sense to one of these three games. Towards this goal we first prove the following lemma.

Lemma 5 For $n \geq 1$,

$$\mathbf{A}(\Gamma_n) + \mathbf{A}(\Gamma_n^2) + \mathbf{A}(\Gamma_n^{-3}) = \mathbf{0}. \tag{14}$$

Proof The $\mathbf{0}$ on the right side of (14) is an $2^n \times 2^n$ matrix whose entries are all zeroes. Let s_n and t_n be any paths of player 1 and player 2 respectively. We wish to show that, corresponding to these two paths,

$$\text{payoff in } \Gamma_n + \text{payoff in } \Gamma_n^2 + \text{payoff in } \Gamma_n^{-3} = 0. \tag{15}$$

The proof becomes obvious if we play the games Γ_n, Γ_n^2 and Γ_n^{-3} simultaneously. Prior to the start, suppose the referee places a white chip at node +1, a black chip at node +2, and a red chip at node -3. Note that at this moment there is exactly one chip at each node. According to the instructions contained in s_n and t_n, the referee moves the three chips simultaneously $2n$ times. But each time the chips move (we call this a stage), they do so with the same direction, either clockwise or counterclockwise. This is because the moves are determined from one particular element in the tuples s_n or t_n. It is not hard to see that, at the end of each stage, there is still exactly one chip at each node. This is all we need. The sum of the three intermediate payoffs at the end of each stage is therefore equal to the sum of the labels on the three nodes which is zero. The right side of (15) is $2n \times 0 = 0$.$\diamondsuit$

Lemma 4 implies $\Omega_n(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a trivial game since

$$\mathbf{A}(\Omega_n(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})) = \frac{1}{3}\,\mathbf{A}(\Gamma_n) + \frac{1}{3}\,\mathbf{A}(\Gamma_n^2) + \frac{1}{3}\,\mathbf{A}(\Gamma_n^{-3}) = \mathbf{0}.$$

Its value is zero, and any path of length n is an optimal strategy for either player.

Let T_n denote one of Γ_n, Γ_n^2, Γ_n^{-3} or their variants. For any positive real number α, we define $\Omega_n(a,b,c) \sim \alpha T_n$ if $\mathbf{A}(\Omega_n(a,b,c)) = \alpha \mathbf{A}(T_n)$. If $\Omega_n(a,b,c) \sim \alpha T_n$, a basic result in game theory states the games $\Omega_n(a,b,c)$ and T_n have the same optimal strategies and the value of $\Omega_n(a,b,c)$ is α times the value of T_n.

Proposition 4 Let $\frac{1}{3} < a \leq 1$. Then

$$\begin{aligned} \Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2}) &\sim \frac{3a-1}{2}\Gamma_n, \\ \Omega_n(\frac{1-a}{2}, a, \frac{1-a}{2}) &\sim \frac{3a-1}{2}\Gamma_n^2, \\ \Omega_n(\frac{1-a}{2}, \frac{1-a}{2}, a) &\sim \frac{3a-1}{2}\Gamma_n^{-3}. \end{aligned} \tag{16}$$

Proof To prove (16),

$$\begin{aligned} \mathbf{A}(\Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2})) &= a\,\mathbf{A}(\Gamma_n) + \frac{1-a}{2}\,\mathbf{A}(\Gamma_n^2) + \frac{1-a}{2}\,\mathbf{A}(\Gamma_n^{-3}) \\ &= \frac{3a-1}{2}\,\mathbf{A}(\Gamma_n) + \frac{1-a}{2}\left(\mathbf{A}(\Gamma_n) + \mathbf{A}(\Gamma_n^2) + \mathbf{A}(\Gamma_n^{-3})\right) \\ &= \frac{3a-1}{2}\,\mathbf{A}(\Gamma_n) \end{aligned}$$

so that

$$\Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2}) \sim \frac{3a-1}{2}\Gamma_n.\diamondsuit$$

Suppose we reverse the sign of all the node labels, that is, node +1 becomes node -1, node +2 becomes node -2, and node -3 becomes node +3. Let $\hat{\Gamma}_n^{-1}$, $\hat{\Gamma}_n^{-2}$ and $\hat{\Gamma}_n^{3}$ denote the games when the chip initially rests at node -1, node -2 and node +3 respectively.

Proposition 5 Let $0 \le a < \frac{1}{3}$. Then

$$\begin{aligned} \Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2}) &\sim \frac{1-3a}{2} \hat{\Gamma}_n^{-1}, \\ \Omega_n(\frac{1-a}{2}, a, \frac{1-a}{2}) &\sim \frac{1-3a}{2} \hat{\Gamma}_n^{-2}, \\ \Omega_n(\frac{1-a}{2}, \frac{1-a}{2}, a) &\sim \frac{1-3a}{2} \hat{\Gamma}_n^{3}. \end{aligned} \tag{17}$$

Proof To prove (17),

$$\begin{aligned} \mathbf{A}(\Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2})) &= a\,\mathbf{A}(\Gamma_n) + \frac{1-a}{2}\mathbf{A}(\Gamma_n^2) + \frac{1-a}{2}\mathbf{A}(\Gamma_n^{-3}) \\ &= \frac{1-3a}{2}\{-\mathbf{A}(\Gamma_n)\} + \frac{1-a}{2}\left(\mathbf{A}(\Gamma_n) + \mathbf{A}(\Gamma_n^2) + \mathbf{A}(\Gamma_n^{-3})\right) \\ &= \frac{1-3a}{2}\mathbf{A}(\hat{\Gamma}_n^{-1}) \end{aligned}$$

so that

$$\Omega_n(a, \frac{1-a}{2}, \frac{1-a}{2}) \sim \frac{1-3a}{2} \hat{\Gamma}_n^{-1}. \diamondsuit$$

A solution of $\hat{\Gamma}_n^{-1}$ cannot be deduced from a solution of Γ_n. We need to repeat the whole process in Section 3 to obtain a solution.

6 Conclusion Motivated by Gleason's Game we formulate and solve a class of finite games. The surprising thing about these games is that they have optimal strategies with a small support. It is uncertain whether this is due to the fact that the labels on the nodes sum to zero. In the original formulation, Gleason probably chose the labels with a zero sum to make the game appeared fair to both the players. As pointed out by Ferguson and Shapley[2], Gleason's Game is not a fair game since it favors player 2. Talking about fairness, $\Omega_n(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a fair game, both Γ_n and Γ_n^2 favor player 2 while Γ_n^{-3} favors player 1.

We make one final remark. The left diagram in Figure 2 shows an optimal strategy of player 1 in Γ_n. From the fourth move onwards, player 1 always moves clockwise for his even moves and counterclockwise otherwise. This implies player 2 knows exactly how player 1 is going to move. But player 2 cannot gain any advantage from this knowledge because he does not know the current position of the chip. Player 1, by randomizing appropriately when he chooses his second and third moves, ensures the chip would move in a pattern that prevents player 2 from exploiting his future deterministic moves.

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