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INVERSE OF THE BERGE MAXIMUM THEOREM IN CONVEX METRIC SPACES

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Abstract. We consider an extension of Komiya's inverse of the Berge maximum theorem in convex metric spaces.

1. Introduction

The following theorem is called the Berge maximum theorem and is often used in the general equilibrium theory of mathematical economics([1, 2]).

Theorem 1.1 (Berge)**.** *Let X and Y be topological spaces. Let $F : X \multimap Y$ be a nonempty compact-valued continuous multi-valued mapping and $f : X \times Y \to \mathbb{R}$ a continuous function. Then the function $\hat{f} : X \to \mathbb{R}$ defined by $\hat{f}(x) = \max\{f(x,y) : y \in F(x)\}$ is continuous. Moreover, the multi-valued mapping $\Gamma : X \multimap Y$ defined by $\Gamma(x) = \{y \in F(x) : f(x,y) = \hat{f}(x)\}$ is compact-valued and upper semicontinuous.*

Komiya studied an inverse problem in the case where X and Y are Euclidean spaces and he obtained the following theorem([4, Theorem 2.1]):

Theorem 1.2. *Let X be a subset of $\mathbb{R}^l$ and let $K : X \multimap \mathbb{R}^m$ be a nonempty compact convex-valued upper semicontinuous multi-valued mapping. Then there exists a continuous function $v : X \times \mathbb{R}^m \to [0,1]$ such that for any $x \in X$,*

(i) $K(x) = \{y \in \mathbb{R}^m : v(x,y) = \max_{z \in \mathbb{R}^m} v(x,z)\}$;
(ii) $v(x,\cdot)$ *is quasi-concave.*

Recently, Komiya and Park studied this problem in the case where X is a topological space and Y is a metric topological vector space whose balls are convex([5]). In this paper, we investigate this inverse problem in a convex metric space with some conditions.

2. Preliminaries

Let (X,d) be a metric space. We denote by $B(x,r)$ the open ball whose center is $x \in X$ and radius is $r > 0$. For each $C \subset X$ and $r > 0$, subsets of X defined by $\{x \in X : d(x,C) < r\}$, $\{x \in X : d(x,C) \le r\}$ are denoted by C^r, $C^{\overline{r}}$, respectively. If X is a normed space, then $\overline{C^r} = C^{\overline{r}}$. But $\overline{C^r} \subset C^{\overline{r}}$ in general metric spaces.

Let X and (Y,d) be metric spaces. A mapping F from X into 2^Y is called a multi-valued mapping and is denoted by $F : X \multimap Y$. A multi-valued mapping is said to be nonempty compact-valued, if, for each $x \in X$, $F(x) \neq \emptyset$ and $F(x)$ is compact. The graph of $F : X \multimap Y$ is denoted by $\mathrm{Gr}(F)$, i.e., $\mathrm{Gr}(F) = \{(x,y) \in X \times Y : y \in F(x)\}$. For each

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multi-valued mapping $F : X \multimap Y$ and $t > 0$, we define multi-valued mappings $F^t : X \multimap Y$ and $F^{\overline{t}} : X \multimap Y$ by

$$F^t(x) = \{y \in Y : d(y, F(x)) < t\},$$
$$F^{\overline{t}}(x) = \{y \in Y : d(y, F(x)) \leq t\}$$

for each $x \in X$, respectively. A multi-valued mapping $F : X \multimap Y$ is said to be lower semicontinuous in X, if, for each $x \in X$ and open set G with $F(x) \cap G \neq \emptyset$, there is a neighborhood U_x of x such that $F(x') \cap G \neq \emptyset$ for each $x' \in U_x$. A multi-valued mapping $F : X \multimap Y$ is said to be upper semicontinuous in X, if, for each $x \in X$ and open set G with $F(x) \subset G$, there is a neighborhood U_x of x such that $F(x') \subset G$ for each $x' \in U_x$. A multi-valued mapping $F : X \multimap Y$ is called continuous, if F is both lower and upper semicontinuous in X.

The concept of a convex metric space was introduced by Takahashi([6]). We say that a metric space (Y, d) has a convex structure, if there exists a mapping $W : Y \times Y \times [0, 1] \to Y$ such that for each $(x, y, \lambda) \in Y \times Y \times [0, 1]$ and $z \in Y$,

$$d(z, W(x, y, \lambda)) \leq \lambda d(z, x) + (1 - \lambda)d(z, y).$$

A metric space (Y, d) having a convex structure is called a convex metric space and denoted by (Y, d, W). A subset K of a convex metric space (Y, d, W) is said to be convex if $W(x, y, \lambda) \in K$ for each x, $y \in K$ and $\lambda \in [0, 1]$. Convex sets of a convex metric space have the following property([6, Proposition 1]):

Lemma 2.1. *Let $\{K_\alpha\}_{\alpha \in A}$ be a sequence of convex sets of a convex metric space (Y, d, W). Then $\bigcap_{\alpha \in A} K_\alpha$ is convex.*

A function $f : K \to \mathbb{R}$ is said to be quasi-concave, if, for each $s \in \mathbb{R}$, $\{y \in K : f(y) \geq s\}$ is convex in Y. A function $f : K \to \mathbb{R}$ is said to be quasi-convex, if, for each $s \in \mathbb{R}$, $\{y \in K : f(y) \leq s\}$ is convex in Y.

Let X be a metric space and Y a convex metric space. We say that a multi-valued mapping $F : X \multimap Y$ has Property (σ), if there is a sequence $\{A_n\}_{n \in \mathbb{N}}$ of multi-valued mappings satisfying the following conditions:

(i) for each $n \in \mathbb{N}$, $A_n : X \multimap Y$ is nonempty compact convex-valued continuous multi-valued mapping;
(ii) for each $x \in X$ and n, $n' \in \mathbb{N}$ with $n > n'$, $F(x) \subset A_n(x) \subset A_{n'}(x)$;
(iii) for each $x \in X$, $F(x) = \bigcap_{n=1}^{\infty} A_n(x)$.

If X is a subset of $\mathbb{R}^n$, a nonempty compact convex-valued upper semicontinuous multi-valued mapping $F : X \multimap \mathbb{R}^m$ has Property (σ)([4, Lemma 2.1]).

We say that a convex metric space (Y, d, W) has Property (K), if, for each x, y, x', $y' \in Y$ and $\lambda \in [0, 1]$,

$$d(W(x, y, \lambda), W(x', y', \lambda)) \leq \lambda d(x, x') + (1 - \lambda)d(y, y').$$

To prove our result, we need the following; see [3, Lemma 4.2, Lemma 4.3].

Lemma 2.2. *Let X be a topological space and D a dense subset of positive real numbers. Let $\{F_t\}_{t \in D}$ be a sequence of open sets of X and satisfy the following conditions:*

(i) *if $t < s$, then $\overline{F_t} \subset F_s$;*
(ii) $\bigcup_{t \in D} F_t = X$.

Then a real-valued function f defined by $f(x) = \inf\{t : x \in F_t\}$ is continuous. Moreover, for each non-negative real number s,

$$\{x \in X : f(x) \leq s\} = \bigcap_{\substack{t \in D \\ t > s}} F_t.$$

3. Main results

To prove our theorem, we show some lemmas. We extend [4, Lemma 2.2] to the following:

Lemma 3.1. *Let X and (Y,d) be metric spaces. Let $A : X \multimap Y$ be a nonempty compact-valued lower semicontinuous multi-valued mapping. Then, for each $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$A(x) \subset A^{\epsilon}(x') \quad \textit{for each} \quad x' \in B(x,\delta).$$

Proof. Let $x \in X$ and $\epsilon > 0$. Since A is lower semicontinuous, for each $y \in A(x)$, there exists $\delta(y) > 0$ such that

$$A(x') \cap B(y,\epsilon/2) \neq \emptyset \quad \text{for each} \quad x' \in B(x,\delta(y)).$$

Since $A(x) \subset \bigcup_{y\in A(x)} B(y,\epsilon/2)$ and $A(x)$ is compact, we can find a finite set $\{y_i\} \subset A(x)$ such that $A(x) \subset \bigcup_i B(y_i,\epsilon/2)$. Setting $\delta = \min_i \delta(y_i)$, we shall show that

$$A(x) \subset A^{\epsilon}(x') \quad \text{for each} \quad x' \in B(x,\delta).$$

Let $x' \in B(x,\delta)$ and $y' \in A(x)$. By $A(x) \subset \bigcup_i B(y_i,\epsilon/2)$, there exists y_i such that $y' \in B(y_i,\epsilon/2)$. Since $x' \in B(x,\delta) \subset B(x,\delta(y_i))$ and A is lower semicontinuous, $A(x') \cap B(y_i,\epsilon/2) \neq \emptyset$. Therefore, for $z \in A(x') \cap B(y_i,\epsilon/2)$,

$$d(y',z) \leq d(y',y_i) + d(y_i,z) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and then $d(y',A(x')) = \inf_{w\in A(x')} d(y',w) \leq d(y',z) < \epsilon$. Hence, $y' \in A^{\epsilon}(x')$. Consequently, we obtain $A(x) \subset A^{\epsilon}(x')$. □

Using Lemma 3.1, we can immediately extend [4, Lemma 2.3] to the case of metric spaces as follows.

Lemma 3.2. *Let X and (Y,d) be metric spaces. Let $A : X \multimap Y$ be a nonempty compact-valued lower semicontinuous multi-valued mapping. Then the graph of $A^{\epsilon} : X \multimap Y$ is open in $X \times Y$ for each $\epsilon > 0$.*

Proof. Let $(x,y) \in \mathrm{Gr}(A^{\epsilon})$, i.e., $x \in X$ and $y \in A^{\epsilon}(x)$. Let $\epsilon' = \frac{1}{3}(\epsilon - d(A(x),y))$. By Lemma 3.1, for $\epsilon' > 0$, there exists $\delta > 0$ such that

$$A(x) \subset A^{\epsilon'}(x') \quad \text{for each} \quad x' \in B(x,\delta).$$

We shall show that $B(x,\delta) \times B(y,\epsilon') \subset \mathrm{Gr}(A^{\epsilon})$. If $(x',y') \in B(x,\delta) \times B(y,\epsilon')$, then we can easily prove the following.

$$y' \in B(y,\epsilon') \subset A^{\epsilon-\epsilon'}(x) \subset A^{\epsilon}(x').$$

Therefore, $(x',y') \in \mathrm{Gr}(A^{\epsilon})$. Hence, $\mathrm{Gr}(A^{\epsilon})$ is open in $X \times Y$. □

Furthermore, some propositions in convex metric spaces are described.

Lemma 3.3. *Let C be a nonempty subset of a convex metric space (Y,d,W). Then $C^{\overline{r}} = \overline{C^r}$ for each $r > 0$.*

Proof. First we show $C^{\overline{r}} \subset \overline{C^r}$. Let $x \in C^{\overline{r}}$. If $d(x,C) < r$, it is clear that $x \in \overline{C^r}$. Supposing $d(x,C) = r$, we can choose a sequence $\{a_n\}$ such that

$$a_n \in C \quad \text{and} \quad d(x,a_n) < r + \frac{r}{n} = \frac{r(n+1)}{n}$$

for each $n \in \mathbb{N}$. We define a sequence $\{x_n\}$ by

$$x_n = W\left(a_n, x, \frac{1}{n+1}\right)$$

for each $n \in \mathbb{N}$. Since

$$d(C, x_n) \leq d(a_n, x_n) \leq \frac{1}{n+1} d(a_n, a_n) + \frac{n}{n+1} d(a_n, x) = \frac{n}{n+1} d(a_n, x) < r,$$

$x_n \in C^r$ for each $n \in \mathbb{N}$. Furthermore,

$$d(x, x_n) \leq \frac{1}{n+1} d(x, a_n) + \frac{n}{n+1} d(x, x) = \frac{1}{n+1} d(x, a_n) < \frac{r}{n}$$

yields $x_n \to x$. Hence, $x \in \overline{C^r}$

On the other hand, $C^{\overline{r}} \supset \overline{C^r}$ is trivial. Consequently, we conclude $C^{\overline{r}} = \overline{C^r}$. □

Lemma 3.4. *Let X and (Y, d) be metric spaces. Let $A : X \multimap Y$ be a nonempty compact-valued upper semicontinuous multi-valued mapping. Then the graph of $A^{\overline{t}} : X \multimap Y$ is closed in $X \times Y$ for each $t > 0$.*

Proof. Let $x \in X$ and $\{x_n\}$ be a sequence of X that converges x. Let $\{y_n\}$ be a sequence of Y such that $y \to y_n$ and $y_n \in A^{\overline{t}}(x_n)$ for each $n \in \mathbb{N}$. Then we shall show $y \in A^{\overline{t}}(x)$, i.e., $d(A(x), y) \leq t$. Since $y_n \in A^{\overline{t}}(x_n)$ and $A(x_n)$ is compact for each $n \in \mathbb{N}$, there is a sequence $\{z_n\}$ such that

$$d(z_n, y_n) = d(A(x_n), y_n) \leq t \quad \text{and} \quad z_n \in A(x_n) \quad \text{for each} \quad n \in \mathbb{N}.$$

Because A is upper semicontinuous and $x_n \to x$, it is easy to show that $d(A(x), z_n) \to 0$. Similarly, by the compactness of $A(x)$, there exists a sequence $\{w_n\}$ such that

$$d(w_n, z_n) = d(A(x), z_n) \quad \text{and} \quad w_n \in A(x) \quad \text{for each} \quad n \in \mathbb{N},$$

and $\{w_n\}$ has a convergent subsequence $\{w_{n'}\}$. Let w be the limit of $\{w_{n'}\}$. Let $\{z_{n'}\}$ and $\{y_{n'}\}$ be the corresponding subsequences of $\{z_n\}$ and $\{y_n\}$, respectively. It is clear that

$$d(w, z_{n'}) \leq d(w, w_{n'}) + d(w_{n'}, z_{n'}) = d(w, w_{n'}) + d(A(x), z_{n'})$$

and hence $z_{n'} \to w$. Furthermore, we get

$$\begin{aligned} d(A(x), y) \leq d(w, y) &\leq d(w, z_{n'}) + d(z_{n'}, y_{n'}) + d(y_{n'}, y) \\ &= d(w, z_{n'}) + d(A(x_{n'}), y_{n'}) + d(y_{n'}, y) \\ &\leq d(w, z_{n'}) + t + d(y_{n'}, y). \end{aligned}$$

It is follows from $z_{n'} \to w$ and $y_{n'} \to y$ that $d(A(x), y) \leq t$. This completes the proof. □

A convex metric space with Property (K) has the following property:

Lemma 3.5. *Let (Y, d, W) be a convex metric space with Property (K) and let C be a nonempty convex subset of Y. Then C^r is convex for each $r > 0$.*

Proof. Let x, $y \in C^r$ and $\lambda \in [0, 1]$. There exist x_0, $y_0 \in C$ such that $d(x_0, x) < r$ and $d(y_0, y) < r$. Since Y has Property (K),

$$\begin{aligned} d(W(x, y, \lambda), W(x_0, y_0, \lambda)) &\leq \lambda d(x, x_0) + (1 - \lambda) d(y, y_0) \\ &< \lambda r + (1 - \lambda) r = r. \end{aligned}$$

By assumption, $W(x_0, y_0, \lambda) \in C$. Therefore,

$$d(W(x, y, \lambda), C) \leq d(W(x, y, \lambda), W(x_0, y_0, \lambda)) < r.$$

Hence, $W(x, y, \lambda) \in C^r$. Consequently, C^r is convex. □

Using lemmas above, we can obtain the following theorem, which is our main result of this paper. This theorem is an extension of Theorem 1.2 to a convex metric space. We shall use the techniques developed in [4]; however, we modify them so as to apply to the convex metric space case.

Theorem 3.6. *Let X be a metric space and (Y, d, W) a convex metric space with Property (K). Let $\Gamma : X \multimap Y$ be a nonempty multi-valued mapping that has Property (σ). Then there exists a continuous function $f : X \times Y \to [0, 1]$ such that for any $x \in X$,*

(i) $\Gamma(x) = \{y \in Y : f(x, y) = \max_{z\in Y} f(x, z)\}$;
(ii) $f(x, \cdot)$ *is quasi-concave.*

Proof. We define $D = \{n/2^{n'} : n, n' \in \mathbb{N}\}$. Then D is a dense subset of the positive real numbers. Now, for $t \in D \cap (0, 1)$, we consider its binary expansion, i.e.,

$$t = \sum_{i=1}^{n} \frac{t_i}{2^i} \quad (t_i = 0 \text{ or } t_i = 1),$$

and we define a function $\ell : D \cap (0, 1) \to \mathbb{N}$ by

$$\ell(t) = \min\{i : t_i = 1\} \quad \text{for each} \quad t \in D \cap (0, 1).$$

Since Γ has Property (σ), there is a sequence of continuous multi-valued mappings $\{A_n\}$ such that A_n has the conditions of Property (σ) for each $n \in \mathbb{N}$. Using $\{A_n\}$, for each $t \in D$, we define a multi-valued mapping $G_t : X \multimap Y$ by

$$G_t(x) = \begin{cases} A^t_{\ell(t)}(x) = \{y \in Y : d(A_{\ell(t)}(x), y) < t\}, & \text{if } 0 < t < 1, \\ Y, & \text{if } t \geq 1. \end{cases}$$

Then, for each $x \in X$ and $s, t \in D$ with $s < t$, it is easy to show that

$$\overline{G_s(x)} \subset G_t(x).$$

Moreover, for each $t \in D$, we define a multi-valued mapping $\overline{G_t} : X \multimap Y$ by

$$\overline{G_t}(x) = \overline{G_t(x)} \quad \text{for each} \quad x \in X.$$

It is follows from Lemma 3.3 and Lemma 3.4 that its graph $\mathrm{Gr}(\overline{G_t})$ is closed in $X \times Y$ for each $t \in D$. That is, for each $t \in D$, $\mathrm{Gr}(\overline{G_t}) = \overline{\mathrm{Gr}(\overline{G_t})}$. Thus, for each $s, t \in D$, if $s < t$, then we can prove

$$\overline{\mathrm{Gr}(G_s)} \subset \mathrm{Gr}(G_t).$$

In fact, we get

$$\overline{\mathrm{Gr}(G_s)} \subset \overline{\mathrm{Gr}(\overline{G_s})} = \mathrm{Gr}(\overline{G_s}) \subset \mathrm{Gr}(G_t).$$

On the other hand, by Lemma 3.2, $\mathrm{Gr}(G_t)$ is open in $X \times Y$ for each $t \in D$. Furthermore, $\bigcup_{t\in D} \mathrm{Gr}(G_t) = X \times Y$ by the definition of G_t.

With the help of Lemma 2.2, the function $g : X \times Y \to [0, 1]$ defined by

$$g(x, y) = \inf\{t : (x, y) \in \mathrm{Gr}(G_t)\}$$

is continuous and for each $s \geq 0$,

$$\bigcap_{\substack{t\in D \\ t>s}} \mathrm{Gr}(G_t) = \{(x, y) \in X \times Y : g(x, y) \leq s\}.$$

Therefore, for each $x \in X$ and $s \geq 0$,

$$\bigcap_{\substack{t\in D \\ t>s}} G_t(x) = \{y \in Y : g(x, y) \leq s\}.$$

It is follows from Lemma 2.1 and Lemma 3.5 that the left side of the equation above is convex and hence g is quasi-convex in its second variable. Moreover, for each $x \in X$, we have

$$\begin{aligned} \Gamma(x) &= \bigcap_{n=1}^{\infty} A_n(x) = \bigcap_{t \in D} G_t(x) \\ &= \{y \in Y : g(x,y) = 0\} = \{y \in Y : g(x,y) = \min_{z \in Y} g(x,z)\}. \end{aligned}$$

Hence, $f = -g + 1$ is the required function. □

References

[1] C. Berge, *Topological Spaces*, Dover Publications, Mineola, New York, 1997.

[2] G. Debreu, *Theory of value*, Yale University Press, New Haven, London, 1959.

[3] J. L. Kelley, *General Topology*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1955.

[4] H. Komiya, "Inverse of the Berge maximum theorem," *Econ. Th.* **9** (1997), 371–375.

[5] S. Park and H. Komiya, "Another Inverse of the Berge maximum theorem," *J. Nonlinear and Convex Anal.* **2** (2001), 105–109.

[6] W. Takahashi, "A Convexity in metric space and nonexpansive mappings, I," *Kōdai Math. Sem. Rep.* **22** (1970), 142–149.

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