BOUNDEDNESS OF FRACTIONAL INTEGRAL OPERATORS ON GENERALIZED MORREY SPACES

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Abstract. On the generalized Morrey spaces, an inequality concerning fractional integrals is proved and a boundedness of fractional integral operators is shown. The target spaces of the result is smaller than that of any results as far as we know.

1 Introduction The boundedness of fractional integral operators on the Morrey spaces was studied by Adams ([Ad]), Chiarenza and Frasca ([CF]) etc. Chiarenza and Frasca showed that the Hardy-Littlewood maximal operator is bounded on the Morrey spaces. By use of this fact and establishing a pointwise estimate of fractional integrals with the Hardy-Littlewood maximal function, they showed the boundedness of fractional integral operators on the Morrey spaces ([CF, Theorem 2]). On the other hand, not using this pointwise estimate, Olsen showed the same bound as [CF, Theorem 2] ([Ol, Theorem 9]). Olsen proved an interesting inequality concerning fractional integrals on the Morrey spaces ([Ol, Theorem 2]) and showed his result by use of this inequality and a bootstrapping argument.

The purpose of this paper is to prove Olsen's inequality on the generalized Morrey spaces (Theorem 1) and to show a boundedness of fractional integral operators on the generalized Morrey spaces (Theorem 4). This is a generalization of [Ol, Theorem 9] to the generalized Morrey spaces.

The notion of the generalized Morrey spaces can be found in [P1] and the theory of the generalized Morrey spaces has been developed by many authors (see the references at the end of this paper). The reasons for considering the generalized Morrey spaces are emphasized in [P1]. The boundedness of fractional integral operators on the generalized Morrey spaces has been investigated by several authors. In [Na1] by use of the Hardy-Littlewood-Sobolev inequality Nakai obtained the boundedness with some (broader) target spaces. (See Remark 3.) The result corresponds to [CF, Corollary] which is weaker than [CF, Theorem 2]. We emphasize that our result (Theorem 4) is a generalization of [CF, Theorem 2] to the generalized Morrey spaces. (See [KNS] for a related result.)

Another purpose of this paper is to give a simplified proof. In [CP] Curz-Uribe and Pérez established a weighted inequality relating the Hardy-Littlewood maximal function to the sharp maximal function. The inequality resembles the good-λ inequality of Fefferman and Stein. By use of this inequality they showed two-weighted norm inequalities for the several operators. We believe that our proof of Theorem 1 will be simple compared with the one in [Ol] by following the arguments established in [CP].

We first recall definitions and notations. All cubes are assumed to have their sides parallel to the coordinate axes. For $x \in \mathbf{R}^n$ and $l > 0$ we will use the notation $Q(x,l)$ to denote a cube with center at x and sidelength l. Also, cQ will denote a cube with the same center as Q, but with sidelength cl.

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We will say that a function f is in the generalized Morrey space $L_q^\phi(\mathbf{R}^n)$, $1 \le q \le \infty$ and $\phi : (0,\ \infty) \mapsto (0,\ \infty)$, if for all cubes $Q(\cdot, l)$ we have

$$\left(\frac{1}{|Q|} \int_Q |f|^q \, dx \right)^{1/q} \le C\phi(l). \tag{1}$$

Here, $|Q|$ denotes the volume of the cube Q. The smallest constant C such that (1) holds is the Morrey norm of f with respect to q and ϕ, and it is denoted by $\|f\|_{q,\phi}$. Applying Hölder's inequality to (1), we see that $\|f\|_{q_1,\phi} \ge \|f\|_{q_2,\phi}$ for all $q_1 \ge q_2$. This tells us that $L_{q_1}^\phi(\mathbf{R}^n) \subset L_{q_2}^\phi(\mathbf{R}^n)$ for $q_1 \ge q_2$. If $\phi(l) \equiv l^{-n/r}$, $1 \le q \le r \le \infty$, then $L_q^\phi(\mathbf{R}^n)$ is the Morrey space defined in [Ol]. The function space $L_q^\phi(\mathbf{R}^n)$ is a Banach space.

A function $\theta : (0,\ \infty) \mapsto (0,\ \infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that $\theta(l) \le C\theta(l')$ ($\theta(l) \ge C\theta(l')$) for $l \le l'$. The following observation can be seen in [Na3].

If $\inf_{l' \le l} \phi(l') = 0$ for some $l > 0$, then $L_q^\phi(\mathbf{R}^n) = \{0\}$. Let $\inf_{l' \le l} \phi(l') > 0$ for every $l > 0$ and $\phi_1(l) = \inf_{l' \le l} \phi(l')$. Then ϕ_1 is decreasing and $L_q^\phi(\mathbf{R}^n) = L_q^{\phi_1}(\mathbf{R}^n)$ with equivalent norms. If $\inf_{l' \ge l} \phi(l') l'^{n/q} = 0$ for some $l > 0$, then $L_q^\phi(\mathbf{R}^n) = \{0\}$. Let $\inf_{l' \ge l} \phi(l') l'^{n/q} > 0$ for every $l > 0$ and $\phi_2(l) = l^{-n/q} \inf_{l' \ge l} \phi(l') l'^{n/q}$. Then $\phi_2(l) l^{n/q}$ is increasing and $L_q^\phi(\mathbf{R}^n) = L_q^{\phi_2}(\mathbf{R}^n)$ with equivalent norms. Thus, to consider our problems we may assume that ϕ is almost decreasing and $\phi(l) l^{n/q}$ is almost increasing. For such ϕ one knows that

$$\frac{1}{A} \le \frac{\phi(l')}{\phi(l)} \le A \text{ for } \frac{1}{2} \le \frac{l'}{l} \le 2, \tag{2}$$

where $A > 0$ is independent of $l, l' > 0$.

Given p, $1 < p < \infty$, the fractional integral operator T_p is defined by

$$T_p f(x) = \sum_{k \in \mathbf{Z}} |Q(x, 2^k)|^{-1/p} \int_{Q(x,2^k)} |f| \, dy,$$

where $Q(x, 2^k)$ denotes the cube with center at x and sidelength 2^k. We notice that

$$T_p f(x) \sim \int_{\mathbf{R}^n} \frac{|f(y)|}{|x-y|^{n/p}} \, dy.$$

THEOREM **1** *Given p, $1 < p < \infty$. Let $1 < q < p/(p-1) \le v \le \infty$, $1/s = 1/v + 1/q + 1/p - 1$ (i.e. $q \le s$) and $s < u \le v$. Suppose that ϕ and η are almost decreasing, and, $\phi(l) l^{n/q}$ and $\eta(l) l^{n/v}$ are almost increasing. Let $\psi(l) = \phi(l) l^{n(1-1/p)}$ and suppose further that*

$$\int_L^\infty \psi(l) l^{-1} \, dl \le A\psi(L), \quad L > 0 \tag{3}$$

for some constant $A > 0$. Then we have

$$\|g \cdot T_p f\|_{s,\psi\eta} \le C \|f\|_{q,\phi} \|g\|_{u,\eta}.$$

Here, the constant C is independent of f and g.

REMARK **2** *It is easy to see that $\phi(l) \equiv l^{-n/r} \log(2+l)$, $q \le r < p/(p-1)$, satisfies the condition (3). Note that $L_q^\phi(\mathbf{R}^n)$ does not belong to any (classical) Morrey spaces.*

REMARK **3** *In Theorem 1 if we put $u = v = \infty$ and $g \equiv \eta \equiv 1$, then the relation between q and s corresponds to [CF, Corollary] and our theorem states a generalization of it. In [Na1] the author treated more general Morrey spaces than those of ours and obtained another generalization of [CF, Corollary]. We emphasize that in this paper one of our purposes is to obtain a generalization of [CF, Theorem 2] which is stronger than [CF, Corollary]. (See Theorem 4.)*

The following Theorem 4 can be proved by using a bootstrapping argument, which is due to Olsen, based on Theorem 1. We shall show this in the last section.

THEOREM **4** *Given p, $1 < p < \infty$. Let $1 < q \le r < p/(p-1)$ and $1/t = 1/r + 1/p - 1$. Suppose that ϕ is almost decreasing, and, $\phi(l)l^{n/r}$ is almost increasing. Let $\psi(l) = \phi(l)l^{n(1-1/p)}$ and suppose further that*

$$\int_L^\infty \psi(l)l^{-1}\,dl \le A\psi(L), \quad L > 0$$

for some constant $A > 0$. Then for $s/t = q/r$ we have

$$\|T_p f\|_{s,\psi} \le C\|f\|_{q,\phi}.$$

Here, the constant C is independent of f.

REMARK **5** *If we put $\phi(l) \equiv l^{-n/r}$ and $\eta(l) \equiv l^{-n/v}$, then Theorems 1 and 4 are the same as Olsen's theorems Theorems 2 and 9, respectively. (See [Ol].) But Olsen's results need the condition that f have the support contained in a bounded domain. We note that the functions f with compact support are not dense in the Morrey spaces.*

In the following C's will denote constants independent of f and g. It will be different in each occasion.

2 Proof of Theorem 1

2.1 Reduction of the proof of Theorem 1 We denote the collection of all dyadic cubes by Δ and, for given $z \in \mathbf{R}^n$, denote the collection of all cubes Q such that $Q - z \in \Delta$ by Δ_z. Given p, $1 < p < \infty$, and $z \in \mathbf{R}^n$ define the translated dyadic fractional integral operator $T^d_{p,z}$ by

$$T^d_{p,z} f(x) = \sum_{x \in Q \in \Delta_z} |Q|^{-1/p} \int_Q |f|\,dy.$$

If $z = 0$ we write T^d_p for $T^d_{p,0}$. Given p, $1 \le p < \infty$, define the dyadic fractional maximal operator M^d_p by

$$M^d_p f(x) = \sup_{x \in Q \in \Delta} |Q|^{-1/p} \int_Q |f|\,dy.$$

If $p = 1$ this is the dyadic Hardy-Littlewood maximal operator and we write $M^d f$ for $M^d_1 f$.

The following argument is due to the first part of the proof of [Ol, Theorem 2]. We need to prove that

$$\left(\int_{Q(\cdot,l)} |gT_p f|^s\,dx\right)^{1/s} \le C\|f\|_{q,\phi}\|g\|_{u,\eta}\psi(l)\eta(l)|Q(\cdot,l)|^{1/s} \tag{4}$$

holds for all cubes $Q(\cdot, l)$. It will suffice to prove (4) with $\mathcal{Q} = Q(0, 2^K)$, $K \in \mathbf{Z}$, by (2). Decompose f according to $\mathcal{Q}$, that is, $f = f_1 + f_2$ and $f_1(x) = \chi_{3\mathcal{Q}}(x) f(x)$. We then have by Minkowski's inequality that

$$\left(\int_{\mathcal{Q}} |gT_p f|^s \, dx\right)^{1/s} \le \left(\int_{\mathcal{Q}} |gT_p f_1|^s \, dx\right)^{1/s} + \left(\int_{\mathcal{Q}} |gT_p f_2|^s \, dx\right)^{1/s} \equiv \mathrm{I} + \mathrm{II}. \tag{5}$$

The estimate of II. It follows from condition (3) that for $x \in \mathcal{Q}$

$$\begin{aligned} T_p f_2(x) &\le \sum_{k=K}^{\infty} |Q(x, 2^k)|^{-1/p} \int_{Q(x,2^k)} |f| \, dy \\ &\le \|f\|_{q,\phi} \sum_{k=K}^{\infty} |Q(x, 2^k)|^{1-1/p} \phi(2^k) \\ &\sim \|f\|_{q,\phi} \int_{2^K}^{\infty} \psi(t) t^{-1} \, dt \le C \|f\|_{q,\phi} \psi(2^K). \end{aligned} \tag{6}$$

Estimate (6) and $s < u$ imply

$$\mathrm{II} \le C \|f\|_{q,\phi} \psi(2^K) \left(\int_{\mathcal{Q}} |g|^s \, dx\right)^{1/s} \le C \|f\|_{q,\phi} \|g\|_{u,\eta} \psi(2^K) \eta(2^K) |\mathcal{Q}|^{1/s}. \tag{7}$$

The estimate of I.

Claim 6 *([Ol, p.2017], also [SW, p.826]) We claim that there exists a constant C such that for $x \in \mathcal{Q}$*

$$T_p f_1(x) \le C \int_{3\mathcal{Q}} T_{p,z}^d f_1(x) \, \frac{dz}{|3\mathcal{Q}|}.$$

Proof. Fix $x \in \mathcal{Q}$. We rewrite

$$\begin{aligned} &\int_{3\mathcal{Q}} T_{p,z}^d f_1(x) \, \frac{dz}{|3\mathcal{Q}|} \\ &\quad = \int_{3\mathcal{Q}} \sum_{x \in Q \in \Delta_z} |Q|^{-1/p} \int_Q |f_1(y)| \, dy \, \frac{dz}{|3\mathcal{Q}|} \\ &\quad = \sum_{k \in \mathbf{Z}} (2^k)^{-n/p} \int_{3\mathcal{Q}} \int_{3\mathcal{Q}} |f_1(y)| \chi_{Q:\, x \in Q \in \Delta_z,\ |Q| = 2^{nk}}(y) \, dy \, \frac{dz}{|3\mathcal{Q}|}. \end{aligned} \tag{8}$$

By geometric consideration (see [SW]) we see for every $y \in Q(x, 2^k)$ that

$$|\{z \in 3\mathcal{Q} : \text{ exists } Q \in \Delta_z \text{ such that } x, y \in Q,\ |Q| = 2^{nk}\}| \ge 3^{-n} |3\mathcal{Q}|.$$

Thus, we have by Fubini's theorem

$$(2^k)^{-n/p} \int_{3\mathcal{Q}} \int_{3\mathcal{Q}} |f_1(y)| \chi_{Q:\, x \in Q \in \Delta_z,\, |Q| = 2^{nk}}(y) \, dy \, \frac{dz}{|3\mathcal{Q}|} \ge C |Q(x, 2^k)|^{-1/p} \int_{Q(x,2^k)} |f_1(y)| \, dy. \tag{9}$$

The claim follows from (8) and (9). ∎

This claim, Hölder's inequality and Fubini's theorem yield

$$\mathrm{I} \le C \left(\int_{3\mathcal{Q}} \int_{\mathcal{Q}} |g(x) T_{p,z}^d f_1(x)|^s \, dx \, \frac{dz}{|3\mathcal{Q}|}\right)^{1/s}. \tag{10}$$

Thus, we can reduce the problem to proving the following lemma.

LEMMA **7** *For all $z \in 3\mathcal{Q}$ we have*

$$\left(\int_{\mathcal{Q}} |gT^d_{p,z}f_1|^s \, dx\right)^{1/s} \le C\|g\|_{u,\eta}\eta(2^K)(2^K)^{n/v}\left(\int_{3\mathcal{Q}} |f_1|^q \, dx\right)^{1/q}.$$

Assume that Lemma 7 is true. Then it follows from (10) that

$$\mathrm{I} \le C\|g\|_{u,\eta}\eta(2^K)(2^K)^{n/v}\left(\int_{3\mathcal{Q}} |f|^q \, dx\right)^{1/q} \le C\|f\|_{q,\phi}\|g\|_{u,\eta}\psi(2^K)\eta(2^K)|\mathcal{Q}|^{1/q+1/p+1/v-1}. \tag{11}$$

From (5), (7) and (11) we have (4), since $1/s = 1/q + 1/p + 1/v - 1$.

2.2 Preliminaries Except for the last part, the proof of the lemma follows the argument in [CP, pp.707-710]. For reader's convenience the full proof is given here.

LEMMA **8** *[Lemma 4.1 of [CP]] Given a non-negative function f and p, $1 < p < \infty$, there exists a constant C_p depending only on p and the dimension n such that for any dyadic cube Q_0*

$$\sum_{Q:\, Q\in\Delta,\ Q\subset Q_0} |Q|^{1-1/p}\int_Q f\,dy \le C_p|Q_0|^{1-1/p}\int_{Q_0} f\,dy.$$

This lemma can be proved by the definition and $1 - 1/p > 0$.

LEMMA **9** *[Lemma 4.4 of [CP]] Given p, $1 < p < \infty$, there exists a constant D_p such that for any non-negative function f, dyadic cube Q_0 and $x_0 \in Q_0$*

$$\frac{1}{|Q_0|}\int_{Q_0} |T^d_p f - (T^d_p f)_{Q_0}|\,dx \le D_p M^d_p f(x_0).$$

Here, F_Q denotes the average of F over Q.

Proof. By the definition of T^d_p for $x \in Q_0$

$$T^d_p f(x) = \sum_{x\in Q\in\Delta,\ Q\subset Q_0} |Q|^{-1/p}\int_Q f\,dy + \sum_{x\in Q\in\Delta,\ Q_0\subset Q} |Q|^{-1/p}\int_Q f\,dy.$$

Hence,

$$\frac{1}{|Q_0|}\int_{Q_0} T^d_p f\,dx = \frac{1}{|Q_0|}\sum_{Q\in\Delta,\ Q\subset Q_0} |Q|^{1-1/p}\int_Q f\,dy + \sum_{Q\in\Delta,\ Q_0\subset Q} |Q|^{-1/p}\int_Q f\,dy.$$

Therefore, by Lemma 8

$$\begin{aligned}&\frac{1}{|Q_0|}\int_{Q_0} |T^d_p f - (T^d_p f)_{Q_0}|\,dx \\ &\quad\le \frac{2}{|Q_0|}\sum_{Q\in\Delta,\ Q\subset Q_0} |Q|^{1-1/p}\int_Q f\,dy \le 2C_P|Q_0|^{-1/p}\int_{Q_0} f\,dy \le 2C_p M^d_p f(x_0). \quad \blacksquare\end{aligned}$$

LEMMA **10** *[Calderón-Zygmund decomposition] Given p, $1 \le p < \infty$, and a non-negative function $f \in L^q(\mathbf{R}^n)$ for some q, $1 \le q < p/(p-1)$. Then for each $\lambda > 0$ there exists a disjoint collection of dyadic cubes $\{C^\lambda_i\}$ such that for each i*

$$\lambda < |C^\lambda_i|^{-1/p}\int_{C^\lambda_i} f\,dy \le 2^{n/p}\lambda$$

and

$$\{x \in \mathbf{R}^n : M_p^d f(x) > \lambda\} = \bigcup_i C_i^\lambda.$$

Moreover, the cubes are maximal: if $Q \in \Delta$ such that $Q \subset \{M_p^d f(x) > \lambda\}$, then $Q \subset C_i^\lambda$ for some i.

DEFINITION **11** *[Definition 3.1 of [CP]] Given $a > 1$ and a weight w define the set function A_w^a on measurable sets $E \subset \mathbf{R}^n$ by*

$$A_w^a(E) = |E|^{1/a'} \left(\int_E w^a \, dx \right)^{1/a} = |E| \left(\frac{1}{|E|} \int_E w^a \, dx \right)^{1/a}.$$

Here, $1/a' = 1 - 1/a$.

LEMMA **12** *[Lemma 3.2 of [CP]] For any $a > 1$ and weight w the set function A_w^a has the following properties:*
(a) *If $E \subset F$, then*

$$A_w^a(E) \le (|E|/|F|)^{1/a'} A_w^a(F);$$

(b) *$w(E) \le A_w^a(E)$;* (c) *If $\{E_j\}$ is a sequence of disjoint sets and $\bigcup_j E_j = E$, then*

$$\sum_j A_w^a(E_j) \le A_w^a(E).$$

Here, $w(E)$ denotes $w(x)dx$ measure of E.

This lemma can be proved by the definition and using Hölder's inequality.

2.3 Proof of Lemma 7 First, we note that it will suffice to prove the result for T_p^d. Since in the proof that follows it will be clear that the general case follows by a simple modification of the argument. Second, we may assume that f_1 is bounded. Since the general case follows by using a simple limiting argument.

Fix $1 < p < \infty$. Set $w = |g|^s \chi_Q$, $H = T_p^d f_1$, $a = u/s > 1$. Let $\{C_i^\lambda\}$ be the Calderón-Zygmund decomposition of H with M^d and let $\{Q_j^\lambda\}$ be the Calderón-Zygmund decomposition of $|f_1|$ with M_p^d.

Fix $N = 2^n + 1$. For each $\lambda > 0$ let $\Omega_\lambda = \bigcup_i C_i^\lambda$. By maximality for each k we have $C_k^{N\lambda} \subset C_i^\lambda$ for some i. By Lemma 12, (b) and (c),

$$\begin{aligned} w(\Omega_{N\lambda}) & \\ &= \sum_k w(C_k^{N\lambda}) \le \sum_k A_w^a(C_k^{N\lambda}) \\ &= \sum_i \sum_{C_k^{N\lambda} \subset C_i^\lambda} A_w^a(C_k^{N\lambda}) \\ &\le \sum_i A_w^a(\Omega_{N\lambda} \cap C_i^\lambda). \end{aligned}$$

Fix $\epsilon < N^{-sa'}$. Divide the indices i into two sets: $i \in F$ if

$$\frac{1}{|C_i^\lambda|} \int_{C_i^\lambda} |H - H_{C_i^\lambda}| \, dx \le \epsilon\lambda$$

and $i \in G$ if the opposite inequality holds. Now we have two relations:

$$w(\Omega_{N\lambda}) \leq \sum_k A_w^a(C_k^{N\lambda}), \tag{12}$$

$$\sum_k A_w^a(C_k^{N\lambda}) \leq \sum_{i\in F} A_w^a(\Omega_{N\lambda}\cap C_i^\lambda) + \sum_{i\in G} A_w^a(\Omega_{N\lambda}\cap C_i^\lambda). \tag{13}$$

If $i \in G$, then by Lemma 9

$$\sum_{i\in G} A_w^a(\Omega_{N\lambda}\cap C_i^\lambda) \leq \sum_{i\in G} A_w^a(C_i^\lambda) \leq \sum_j A_w^a(Q_j^{(\epsilon/D_p)\lambda}). \tag{14}$$

If $i \in F$, then we claim that

$$A_w^a(\Omega_{N\lambda}\cap C_i^\lambda) \leq \epsilon^{1/a'} A_w^a(C_i^\lambda).$$

By Lemma 12, (a), it suffices to show that $|\Omega_{N\lambda}\cap C_i^\lambda| \leq \epsilon |C_i^\lambda|$. By the maximality if $x \in \Omega_{N\lambda}\cap C_i^\lambda$, then $M^d H(x) = M^d(H\chi_{C_i^\lambda})(x)$. Hence,

$$\begin{aligned}\Omega_{N\lambda}\cap C_i^\lambda &= \{x \in C_i^\lambda : M^d(H\chi_{C_i^\lambda})(x) > N\lambda\}\\ &= \{x \in C_i^\lambda : M^d(H\chi_{C_i^\lambda})(x) - H_{C_i^\lambda} > N\lambda - H_{C_i^\lambda}\}\\ &\subset \{x \in C_i^\lambda : M^d(|H - H_{C_i^\lambda}|\chi_{C_i^\lambda})(x) > \lambda\}.\end{aligned}$$

Since M^d is weak-type $(1,\ 1)$ with constant 1, and since $i \in F$,

$$|\Omega_{N\lambda}\cap C_i^\lambda| \leq \frac{1}{\lambda}\int_{C_i^\lambda} |H - H_{C_i^\lambda}|\,dx \leq \epsilon|C_i^\lambda|.$$

Therefore,

$$\sum_{i\in F} A_w^a(\Omega_{N\lambda}\cap C_i^\lambda) \leq \epsilon^{1/a'}\sum_{i\in F} A_w^a(C_i^\lambda) \leq \epsilon^{1/a'}\sum_i A_w^a(C_i^\lambda). \tag{15}$$

By the Lebesgue differentiation theorem

$$\int_{\mathcal{Q}} wH^s\,dx \leq s\int_0^\infty \lambda^{s-1} w(\{x \in \mathcal{Q} : M^d H(x) > \lambda\})\,d\lambda. \tag{16}$$

It follows from (13)–(15) that

$$\begin{aligned}&\int_0^\infty \lambda^{s-1}\left(\sum_k A_w^a(C_k^{N\lambda})\right)d\lambda\\ &\quad\leq \int_0^\infty \lambda^{s-1}\left(\sum_j A_w^a(Q_j^{(\epsilon/D_p)\lambda})\right)d\lambda + \epsilon^{1/a'}\int_0^\infty \lambda^{s-1}\left(\sum_i A_w^a(C_i^\lambda)\right)d\lambda.\end{aligned}$$

Since f_1 and w have compact support and f_1 is bounded, the integrals of the both sides of above inequality are finite. Thus, we can re-arrenge the terms and hence,

$$\int_0^\infty \lambda^{s-1}\left(\sum_i A_w^a(C_i^\lambda)\right)d\lambda \leq C\int_0^\infty \lambda^{s-1}\left(\sum_j A_w^a(Q_j^\lambda)\right)d\lambda \tag{17}$$

by $\epsilon < N^{-sa'}$. Inequalities (16), (12) and (17) imply

$$\int_{\mathcal{Q}} wH^s\,dx \le Cs\int_0^\infty \lambda^{s-1}\left(\sum_j A_w^a(Q_j^\lambda)\right)\,d\lambda. \tag{18}$$

Claim **13** *For every dyadic cube $Q = Q(\cdot, 2^k)$ such that $Q \cap \mathcal{Q} \neq \emptyset$ we claim that*

$$A_w^a(Q) \le C\left(\|g\|_{u,\eta}\eta(2^K)(2^K)^{n/v}\right)^s |Q|^{1-s/v}.$$

Proof. Since $\eta(l)l^{n/v}$ is almost increasing, if $k < K$ then

$$\begin{aligned} A_w^a(Q) &= |Q|\left(\frac{1}{|Q|}\int_Q |g|^u\,dy\right)^{s/u} \le |Q|\left(\|g\|_{u,\eta}\eta(2^k)\right)^s \\ &= \left(\|g\|_{u,\eta}\eta(2^k)(2^k)^{n/v}\right)^s |Q|^{1-s/v} \le C\left(\|g\|_{u,\eta}\eta(2^K)(2^K)^{n/v}\right)^s |Q|^{1-s/v}. \end{aligned}$$

If $k \ge K$ then by $u \le v$

$$\begin{aligned} A_w^a(Q) &= |Q|^{1-s/u}\left(\int_{Q\cap\mathcal{Q}} |g|^u\,dy\right)^{s/u} \\ &\le |Q|^{1-s/u}|\mathcal{Q}|^{s/u}\left(\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} |g|^u\,dy\right)^{s/u} \\ &\le \left(\|g\|_{u,\eta}\eta(2^K)(2^K)^{n/v}\right)^s |Q|^{1-s/u}|\mathcal{Q}|^{s/u-s/v} \le \left(\|g\|_{u,\eta}\eta(2^K)(2^K)^{n/v}\right)^s |Q|^{1-s/v}. \quad \blacksquare \end{aligned}$$

By this claim and (18) now we need to prove that

$$s\int_0^\infty \lambda^{s-1}\left(\sum_j |Q_j^\lambda|^{1-s/v}\right)\,d\lambda \le C\left(\int |f_1|^q\,dx\right)^{s/q}. \tag{19}$$

Since $1 - 1/p > 0$, f_1 has compact support and is bounded, we note that the volume of $Q \in \{Q_j^\lambda\}$ have a positive lower bound. For every $Q \in \{Q_j^\lambda\}$ define the *mother* cube $m(Q)$ by a minimal cube Q' such that $Q \subset Q' \in \{Q_j^\lambda\}$ and $|Q| < |Q'|$. Then we can rewrite the left-hand side of (19) as

$$\sum_{Q\in\{Q_j^\lambda\}} |Q|^{1-s/v}\left(\left(|Q|^{-1/p}\int_Q |f_1|\,dy\right)^s - \left(|m(Q)|^{-1/p}\int_{m(Q)} |f_1|\,dy\right)^s\right). \tag{20}$$

Since $|Q| < |m(Q)|$, (20) is bounded by

$$\sum_{Q\in\{Q_j^\lambda\}} |Q|\left(\left(|Q|^{-1/p-1/v}\int_Q |f_1|\,dy\right)^s - \left(|m(Q)|^{-1/p-1/v}\int_{m(Q)} |f_1|\,dy\right)^s\right). \tag{21}$$

Set $1/P = 1/p + 1/v \le 1$. For a.e. $x \in \mathbf{R}^n$ define the cube Q_x by a minimal cube Q such that $x \in Q \in \{Q_j^\lambda\}$, and define an operator S by

$$Sf_1(x) = |Q_x|^{-1/P}\int_{Q_x} |f_1|\,dy.$$

Then we claim that

$$(22)\ \sum_{Q\in\{Q_j^\lambda\}} |Q| \left(\left(|Q|^{-1/P} \int_Q |f_1|\, dy \right)^s - \left(|m(Q)|^{-1/P} \int_{m(Q)} |f_1|\, dy \right)^s \right) = \int (Sf_1)^s\, dx.$$

By Fubini's theorem

$$\begin{aligned} &\sum_{Q\in\{Q_j^\lambda\}} |Q| \left(\left(|Q|^{-1/P} \int_Q |f_1|\, dy \right)^s - \left(|m(Q)|^{-1/P} \int_{m(Q)} |f_1|\, dy \right)^s \right) \\ &\quad = \int \sum_{Q\in\{Q_j^\lambda\}} \chi_Q(x) \left(\left(|Q|^{-1/P} \int_Q |f_1|\, dy \right)^s - \left(|m(Q)|^{-1/P} \int_{m(Q)} |f_1|\, dy \right)^s \right) dx \\ &\quad = \int \left(|Q_x|^{-1/P} \int_{Q_x} |f_1|\, dy \right)^s dx = \int (Sf_1)^s\, dx. \end{aligned}$$

Clearly, $Sf_1(x) \le M_P^d f_1(x)$. Since M_P^d with $1/s = 1/q + 1/P - 1$ is bounded from $L^q(\mathbf{R}^n)$ to $L^s(\mathbf{R}^n)$, we obtain

$$(23) \qquad \int (M_P^d f_1)^s\, dx \le C \left(\int |f_1|^q\, dx \right)^{s/q}.$$

Now (19) follows from (20)–(23).

3 Proof of Theorem 4 We note that

$$(24) \qquad f \in L_q^\phi(\mathbf{R}^n) \iff f^\alpha \in L_{q/\alpha}^{\phi^\alpha}(\mathbf{R}^n) \text{ and } \|f^\alpha\|_{q/\alpha,\ \phi^\alpha} = \|f\|_{q,\phi}^\alpha$$

for $0 < \alpha/q \le 1$ and non-negative function f. Recall $1/t = 1/r + 1/p - 1$. Set $\kappa = [t/r] + 1 < \infty$, where $[t/r]$ denotes the largest integer not greater than t/r. For the integers $k \in [1,\ \kappa]$ define the sequences $\{\alpha_k\}$ and $\{s_k\}$ by

$$\alpha_k = \begin{cases} k, & k \in [1,\ \kappa - 1], \\ t/r, & k = \kappa \end{cases} \quad \text{and } 1/s_k = 1/t + 1/\alpha_k(1/q - 1/r).$$

Notice that $s_\kappa/t = q/r$.

The proof is by induction on k. Letting $u = v = \infty$ and $g \equiv \eta \equiv 1$ in Theorem 1, we have

$$(25) \qquad \|T_p f\|_{s_1,\psi} \le C\|f\|_{q,\phi}.$$

We assume that for $k \in [2,\ \kappa]$

$$(26) \qquad \|T_p f\|_{s_{k-1},\psi} \le C\|f\|_{q,\phi}.$$

Then, noting $0 < (\alpha_k - 1)/s_{k-1} \le 1/q < 1$, we have by (24) and (26)

$$(27)\ \ (T_p f)^{\alpha_k - 1} \in L_{s_{k-1}/(\alpha_k-1)}^{\psi^{\alpha_k-1}}(\mathbf{R}^n) \text{ and } \|(T_p f)^{\alpha_k-1}\|_{s_{k-1}/(\alpha_k-1),\ \psi^{\alpha_k-1}} \le C\|f\|_{q,\phi}^{\alpha_k-1}.$$

Our choices of α_k and s_k enable us to apply Theorem 1 to $g \equiv (T_p f)^{\alpha_k-1}$ and $s = s_k/\alpha_k$ (i.e. $v = t/(\alpha_k - 1)$), and hence we have by (27) and (24)

$$(28) \qquad \|T_p f\|_{s_k,\psi} \le C\|f\|_{q,\phi}.$$

Inequalities (25), (26) and (28) and relation $s_\kappa/t = q/r$ imply our desired inequality.

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