INFINITE PRODUCT PROBLEMS ON $\delta\theta$-REFINABLE SPACES

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ABSTRACT. Suppose that X=$\prod_{n<\omega}$X$_n$,if each space $\prod_{i\leq n}$X$_n$ is $\delta\theta$-refinable (i.e., submetalindelof), is X also $\delta\theta$-refinable? K.Chiba asked in [1]. This paper first show that an inverse limit theorem for $\delta\theta$-refinable spaces. Using this, we obtain the result: Let X=$\prod_{\alpha\in\Lambda}$X$_\alpha$ be $|\Lambda|$- paracompact, X is $\delta\theta$-refinable iff $\prod_{\alpha\in F}$X$_\alpha$ is $\delta\theta$-refinable for each F$\in[\Lambda]^{<\omega}$. Then, the above problem is answered positively. Next, we show that there are similar results on hereditarily $\delta\theta$-refinable spaces.

In the paper [1], K.Chiba asked:Suppose that X=$\prod_{n<\omega}$X$_n$,if each space $\prod_{i\leq n}$X$_n$ is $\delta\theta$-refinable (i.e., submetalindelof), is X also $\delta\theta$-refinable? This paper first prove respectively the following:

Theorem 1. Let X be the inverse limit of an inverse system $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ and let the projection π_α be an open and onto map for each $\alpha \in \Lambda$. If X is $|\Lambda|$-paracompact and each X$_\alpha$ is $\delta\theta$-refinable, then X is $\delta\theta$-refinable.

Theorem 2. Let X be the inverse limit of an inverse system $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ and let the projection π_α be an open and onto map for each $\alpha \in \Lambda$. If X is hereditarily $|\Lambda|$-paracompact and each X$_\alpha$ is hereditarily $\delta\theta$-refinable, then X is also hereditarily $\delta\theta$-refinable.

Using the aboves, we obtain the results:

Theorem 3. Let X=$\prod_{\alpha\in\Lambda}$X$_\alpha$ be $|\Lambda|$-paracompact (resp. hereditarily $|\Lambda|$-paracompact), X is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable) iff $\prod_{\alpha\in F}$X$_\alpha$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable) for each F$\in[\Lambda]^{<\omega}$.

Therefore, the following holds trivially:

Theorem 4. Let X=$\prod_{i\in\omega}$X$_i$ is countable paracompact (resp. hereditarily countable paracompact), then the following are equivalent:

(1) X is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable).

(2) $\prod_{i\in F}$X$_i$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable) for each $F \in [\Sigma]^{<\omega}$.

(3) $\prod_{i\leq n}$X$_i$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable) for each n$\in\omega$.

(3)$\Rightarrow$(1) in Theorem 4 is a positively answer of Problem 5 in [1].

We use that N$_Y$(x) denotes the neighourhood system of a point x of a subspace Y of a space X. Espectly, N(x) denotes N$_Y$(x) when Y=X; |A|, clA, IntA and A^c denote respectively the cardinality, the closure, the interior and the complementary set of a set A; $(\mathcal{U})_x$, $(\mathcal{U})|_A$ and $\bigwedge_{n\in F}\mathcal{H}_n$ denote respectively $\{U\in\mathcal{U} : x \in U\}$, $\{U\bigcap A : U\in\mathcal{U}\}$ and $\{\bigcap_{n\in F}H_n : H_n \in \mathcal{H}_n\}$; ω and $[\Sigma]^{<\omega}$ denote, respectively, the first infinite ordinal number and the collection of all non-empty finite subsets of a non-empty set Σ. And assume that all spaces are Hausdoff spaces throughout this paper.

Definition 1. Let κ be a cardinal number, A space is κ-paracompact iff its every open cover $\mathcal{U}$ of cardinal $|\mathcal{U}| \leq \kappa$ has a locally finite open refinement; A space is $|\Sigma|$-paracompact iff it is κ- paracompact, where κ=$|\Sigma|$.

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Definition 2[3]. A space X is said to be $\delta\theta$-refinable (submetalindelof) if its every open cover $\mathcal{U}$ has a sequence $\langle \mathcal{G}_n \rangle_{n\in\omega}$ of open refinements such that for every x∈X there is a n∈ ω with ord(x,$\mathcal{G}_n$) $\leq \omega$; A space X is said to be weakly $\delta\theta$-refinable if its every open cover $\mathcal{U}$ has an open refinement $\mathcal{G}=\bigcup_{n\in\omega}\mathcal{G}_n$ such that for every x∈X there is a n∈ ω such that 1≤ord(x,$\mathcal{G}_n$) $\leq \omega$.

Lemma 1[2]. Let λ be a cardinal number. Suppose X is λ-paracompact, Λ is a directed set with $|\Lambda| = \lambda$ and $\mathcal{H} = \{H_\alpha : \alpha \in \Lambda\}$ is an open cover of X such that $H_\alpha \subset H_\beta$ for each α, $\beta \in \Lambda$ satisfying $\alpha \leq \beta$. Then there is an open cover $\mathcal{K} = \{K_\alpha : \alpha \in \Lambda\}$ of X such that $clK_\alpha \subset H_\alpha$ for each $\alpha \in \Lambda$ and $K_\alpha \subset K_\beta$ for each α, $\beta \in \Lambda$ satisfying $\alpha \leq \beta$.

Lemma 2. A space X is hereditarily $\delta\theta$-refinable (resp. hereditarily weakly $\delta\theta$-refinable) iff each open subspace of X is $\delta\theta$-refinable (resp.weakly $\delta\theta$-refinable).

This lemma is a direct result of Definition 2. Now we prove main theorems of this paper.

Proof of Theorem 1. Let $\mathcal{U} = \{U_\xi\colon \xi \in \Xi\}$ be an arbitrary open cover of X. For each $\alpha \in \Lambda$ and each $\xi \in \Xi$, let us put

$$V_{\alpha\xi}=\bigcup\{V\colon V \text{ is in } X_\alpha \text{ and } \pi_\alpha^{-1}(V)\subset U_\xi\}$$

and put $V_\alpha = \bigcup\{V_{\alpha\xi} : \xi \in \Xi\}$, then

(1) $\bigcup\{\pi_\alpha^{-1}(V_\alpha):\alpha \in \Lambda\}=X$, and $\pi_\alpha^{-1}(V_\alpha) \subset \pi_\beta^{-1}(V_\beta)$ if $\alpha \leq \beta$.

Since X is $|\Lambda|$-paracompact, there is an open cover $\{W_\alpha : \alpha \in \Lambda\}$ of X such that

(2) $clW_\alpha \subset \pi_\alpha^{-1}(V_\alpha)$ for each $\alpha \in \Lambda$, and $W_\alpha \subset W_\beta$ if $\alpha \leq \beta$.

For each $\alpha \in \Lambda$, let us put $T_\alpha=X_\alpha - \pi_\alpha(X-clW_\alpha)$, then T_α is closed in X_α because π_α is an open map. Again let C_α=Int T_α for each $\alpha \in \Lambda$, then

(3) $\{C_\alpha:\alpha \in \Lambda\}$ is an open cover of X.

In fact, for each x∈X there is $\alpha \in \Lambda$ such that x∈W_α. There are some $\beta \in \Lambda$ and some open set V in X_β such that x∈ $\pi_\beta^{-1}(V)\subset W_\alpha$ since $W_\alpha$is open in X. We choose a $\gamma \in \Lambda$ satisfying $\gamma \geq \alpha$ and $\gamma \geq \beta$, then x∈C_γ because $\pi_\beta^{-1}(V)\subset \pi_\gamma^{-1}(T_\gamma)$. To show this, let $y=(y_\delta)_{\delta\in\Lambda} \in \pi_\beta^{-1}(V)-\pi_\alpha^{-1}(T_\gamma)$, then $y_\beta \in V$ and $y_\gamma \in \pi_\gamma(X-clW_\gamma)$. I.e., there is an element $z=(z_\delta)_{\delta\in\Lambda} \in X-clW_\gamma$ such that $z_\gamma = \pi_\gamma(z)=y_\gamma$, $y_\beta = \pi_\beta^\gamma(z_\gamma)\in V$, $z\in \pi_\beta^{-1}(V)=\pi_\gamma^{-1}(\pi_\beta^\gamma)^{-1}(V)\subset W_\alpha$, then z∈$W_\gamma$. This is a contradiction.

By $|\Lambda|$-paracompactness of X, there is a locally finite open cover $\{O_\alpha : \alpha \in \Lambda\}$ of X such that $O_\alpha \subset C_\alpha$ for each $\alpha \in \Lambda$. Since $T_\alpha \subset V_\alpha = \bigcup\{V_{\alpha\xi} : \xi \in \Xi\}$ and T_α is closed in X_α then there is a sequence $\langle\mathcal{G}_n(\alpha)\rangle_{n\in\omega}$ of open sets of X_α, satisfying

(4) Each $\mathcal{G}_n(\alpha)$ is a part refinement of $\{V_{\alpha\xi} : \xi \in \Xi\}$ and $T_\alpha \subset \bigcup\mathcal{G}_n(\alpha)$ for each $n \in \omega$.

(5) For each x∈T_α there is a n∈ ω such that ord(x,$\mathcal{G}_n(\alpha)$) $\leq \omega$, and $G_1 \bigcap G_2 \in \mathcal{G}_n(\alpha)$ if $G_1, G_2 \in \mathcal{G}_n(\alpha)$.

For each $n \in \omega$, let $\mathcal{H}_n = \{\pi_\alpha^{-1}(G)\bigcap O_\alpha : G\in \mathcal{G}_n(\alpha) \text{ and } \alpha \in \Lambda\}$, then

(6) $\mathcal{H}_n$ is an open refinement of $\mathcal{U}$ for each $n \in \omega$.

In fact, for each $x \in$X, there is $\alpha \in \Lambda$ such that $x \in O_\alpha \subset C_\alpha \subset \pi_\alpha^{-1}(T_\alpha)$ and there is G∈ $\mathcal{G}_n(\alpha)$ such that $x \in \pi_\alpha^{-1}(G)\bigcap O_\alpha$, i.e., $\mathcal{H}_n$ is a cover of X. Again since for each $\alpha \in \Lambda$ and each G∈ $\mathcal{G}_n(\alpha)$ there is some $\xi(G) \in \Xi$ such that $G\subset V_{\alpha\xi(G)}$, then $\pi_\alpha^{-1}(G)\bigcap O_\alpha \subset \pi_\alpha^{-1}(G)\subset \pi_\alpha^{-1}(V_{\alpha\xi(G)})\subset U_{\xi(G)}$. So, (6) is true.

For each $F \in [\omega]^{<\omega}$, let us put $\mathcal{H}_F=\bigwedge_{n\in F}\mathcal{H}_n$, then

(7) Each $\mathcal{H}_F$ is an open refinement of $\mathcal{U}$.

Finally, we prove:

(8) For each $x \in$X, there is a $F \in [\omega]^{<\omega}$ such that ord(x,$\mathcal{H}_F$)$\leq \omega$.

Let $x \in$X, since $\{O_\alpha : \alpha \in \Lambda\}$ is a locally open cover of X, $\triangle=\{\alpha \in \Lambda : x \in O_\alpha\}$ is a nonempty finite set. And for each $\alpha \in \triangle$, since $x \in O_\alpha \subset \pi_\alpha^{-1}(T_\alpha)$, there is some $n_\alpha \in \omega$ such that ord($\pi_\alpha(x)$,$\mathcal{G}_{n_\alpha}(\alpha)$) $\leq \omega$. Put $F = \{n_\alpha : \alpha \in \triangle\}$ and let $\mathcal{G}_{n_\alpha}^{-1}(\alpha)=\{\pi_\alpha^{-1}(G):G\in \mathcal{G}_{n_\alpha}(\alpha)\}$, then

$$(\mathcal{H}_F)_x \subset \{G\bigcap[\bigcap_{\alpha\in\triangle'}O_\alpha]\colon G\in \bigwedge_{\alpha\in\triangle'}(\mathcal{G}_{n_\alpha}^{-1}(\alpha))_x \text{ and } \triangle' \in [\triangle]^{<\omega}\}$$

Therefore, ord(x,$\mathcal{H}_F$)$\leq \omega$. □

Proof of Theorem 2. Let $\mathcal{U}$ = {U_ξ: $\xi \in \Xi$} be an open cover of open subspace Y of X. For each $\alpha \in \Lambda$ and each $\xi \in \Xi$, we put $V_{\alpha\xi}=\bigcup\{V$: V is in X_α and $\pi_\alpha^{-1}(V)\subset U_\xi\}$ and $V_\alpha = \bigcup\{V_{\alpha\xi} : \xi \in \Xi\}$, then

(1) $\{\pi_\alpha^{-1}(V_\alpha):\alpha \in \Lambda\}$ is an open cover of Y and $\pi_\alpha^{-1}(V_\alpha) \subset \pi_\beta^{-1}(V_\beta)$ if $\alpha \leq \beta$.

Since X is hereditarily $|\Lambda|$-paracompact, the open cover $\{\pi_\alpha^{-1}(V_\alpha):\alpha \in \Lambda\}$ of the subspace Y of X has an open refinement $\{W_\alpha : \alpha \in \Lambda\}$ such that

(2) cl$W_\alpha \subset \pi_\alpha^{-1}(V_\alpha)$ for each $\alpha \in \Lambda$, and $W_\alpha \subset W_\beta$ if $\alpha \leq \beta$.

For each $\alpha \in \Lambda$, put $E_\alpha=\bigcup\{E$:E is open in X_α and $\pi_\alpha^{-1}(E)\subset W_\alpha\}$, then

(3) $\pi_\alpha^{-1}(E_\alpha)\subset W_\alpha$ for each $\alpha \in \Lambda$ and $\pi_\alpha^{-1}(E_\alpha)\subset \pi_\beta^{-1}(E_\beta)$ if $\alpha \leq \beta$.

Now, we assert that:

(4) $\{\pi_\alpha^{-1}(E_\alpha):\alpha \in \Lambda\}$ is an open cover of Y.

In fact, for each x$\in$Y there is a $\alpha \in \Lambda$ such that x$\in W_\alpha$. There are some $\beta \in \Lambda$ and some open set V in X_β such that x$\in \pi_\beta^{-1}(V)\subset W_\alpha$ by [4,Theorem 2.5.5]. Let us put $\gamma \in \Lambda$ such that both $\gamma \geq \alpha$ and $\gamma \geq \beta$, then x$\in \pi_\beta^{-1}(V)=(\pi_\beta^\gamma\pi_\gamma)^{-1}(V)=\pi_\gamma^{-1}(\pi_\beta^\gamma)^{-1}(V)\subset W_\alpha \subset W_\gamma$, then x$\in \pi_\gamma^{-1}(E_\gamma)$.

Put F_α=cl(E_α)$\bigcap$[cl(V_α)-V_α] for each $\alpha \in \Lambda$, we assert that

(5) $\pi_\alpha^{-1}(F_\alpha)\bigcap Y=\phi$.

In fact, if there is some $x = (x_\alpha)_{\alpha\in\Lambda} \in \pi_\alpha^{-1}(F_\alpha)\bigcap Y$, then $x_\alpha \in F_\alpha \subset$cl($V_\alpha$)-$V_\alpha$, $x \notin \pi_\alpha^{-1}(V_\alpha)$ since $x_\alpha \notin V_\alpha$. Next, we have $x \in \mathrm{cl}_Y\pi_\alpha^{-1}(E_\alpha)$. To prove this, let us put H$\in N_Y(x)$, then there are some $\beta \in \Lambda$ and some open set V in X such that $x \in \pi_\beta^{-1}(V)\subset$H. Pick $\gamma \geq \alpha$, $\gamma \geq \beta$ and let $V'=(\pi_\beta^\gamma)^{-1}(V)$, then

$$x \in \pi_\gamma^{-1}(V') =\pi_\gamma^{-1}(\pi_\beta^\gamma)(V)=(\pi_\beta^\gamma\pi_\gamma)^{-1}(V) =\pi_\beta^{-1}(V)\subset H.$$

Since $x_\alpha \in F_\alpha \subset$cl($E_\alpha$) and $x_\gamma \in V'$, then $\pi_\alpha^\gamma(V')\bigcap E_\alpha \neq \phi$. Let us put b$\in \pi_\alpha^\gamma(V')\bigcap E_\alpha$, then there is c$\in V'$ such that π_α^γ(c)=b. There is y=$(y_\delta)_{\delta\in\Lambda}$) $\in$X such that $y_\gamma = \pi_\gamma$(y)=c. I.e., $y_\alpha = \pi_\alpha^\gamma(y_\gamma) = \pi_\alpha^\gamma$(c)=b,

$$y\in \pi_\gamma^{-1}(V')\bigcap \pi_\alpha^{-1}[\pi_\alpha^\gamma(V')\bigcap E_\alpha] \subset \pi_\gamma^{-1}\ (V')\bigcap \pi_\alpha^{-1}(E_\alpha)\subset H\bigcap \pi_\alpha^{-1}\ (E_\alpha),$$

i.e., H$\bigcap \pi_\alpha^{-1}(E_\alpha)\neq \phi$, thus x$\in \mathrm{cl}_Y\pi_\alpha^{-1}(E_\alpha)\subset \pi_\alpha^{-1}(V_\alpha)$. This contradicts to x$\notin \pi_\alpha^{-1}(V_\alpha)$.

(6) (X_α-F_α)$\bigcap$cl(E_α)$\subset V_\alpha = \bigcup\{V_{\alpha\xi} : \xi \in \Xi\}$ for each $\alpha \in \Lambda$.

In fact, for each t$\in$(X_α-F_α)$\bigcap$cl(E_α), we have t$\notin F_\alpha$ and t$\in$cl(E_α). Since t$\notin$cl(V_α)-V_α and $E_\alpha \subset V_\alpha$, then t$\in V_\alpha$.

By $\delta\theta$-refinableness of X_α-F_α, there is a sequence $\langle\mathcal{G}_n(\alpha)\rangle_{n\in\omega}$ of open covers of (X_α-F_α)$\bigcap$cl(E_α) such that

(7) Each $\mathcal{G}_n(\alpha)$ is part refinement of $\{V_{\alpha\xi} : \xi \in \Xi\}$ and $G_1 \bigcap G_2 \in \mathcal{G}_n(\alpha)$ if G_1, $G_2 \in \mathcal{G}_n(\alpha)$

(8) For each x$\in$(X_α-F_α)$\bigcap$cl(E_α) there is a n$\in \omega$ such that ord(x,$\mathcal{G}_n(\alpha)$)$\leq \omega$.

Next, since X is hereditarily $|\Lambda|$-paracompact, the open cover $\{\pi_\alpha^{-1}\ (E_\alpha):\alpha \in \Lambda\}$ of the subspace Y has a locally finite open refinement $\{O_\alpha : \alpha \in \Lambda\}$ such that $O_\alpha \subset \pi_\alpha^{-1}(E_\alpha)$ for each $\alpha \in \Lambda$

Define $\mathcal{H}_n$ = $\{O_\alpha \bigcap \pi_\alpha^{-1}(G)$: G$\in \mathcal{G}_n(\alpha)$, $\alpha \in \Lambda\}$ and let $\mathcal{H}_F=\bigwedge_{n\in F} \mathcal{H}_n$ for each $F \in [\omega]^{<\omega}$, then

(9) Each $\mathcal{H}_F$ is an open refinement of $\mathcal{U}$.

In fact, for each $x \in$Y and each n$\in \omega$, there is some $\alpha \in \Lambda$ such that x$\in O_\alpha \subset \pi_\alpha^{-1}(E_\alpha)$, then $x_\alpha \in$(X_α-F_α)$\bigcap$cl(E_α). There is G$\in \mathcal{G}_n(\alpha)$ such that $x_\alpha \in$G, x$\in O_\alpha \bigcap \pi_\alpha^{-1}(G)$, i.e., $\mathcal{H}_n$ is an open cover of Y. Since for each G$\in \mathcal{G}_n(\alpha)$ there is $\xi \in \Xi$ such that G$\subset V_{\alpha\xi}$, then $O_\alpha \bigcap \pi_\alpha^{-1}(G)\subset \pi_\alpha^{-1}(G)\subset \pi_\alpha^{-1}(V_{\alpha\xi})\subset U_\xi$, hence $\mathcal{H}_F$ is an open refinement of $\mathcal{U}$ for each $F \in [\omega]^{<\omega}$.

Finally, we assert that

(10) For each $x \in$X, there is some $F \in [\omega]^{<\omega}$ such that ord(x,$\mathcal{H}_F$)$\leq \omega$.

Let x ∈Y, △={$\alpha \in \Lambda$:x ∈O_α} is an nonempty finite set. For each $\alpha \in \triangle$, $x \in O_\alpha \subset \pi_\alpha^{-1}(E_\alpha)$, we have $x_\alpha \in (X_\alpha\text{-}F_\alpha)\bigcap E_\alpha$ by (5), there is some $n_\alpha \in \omega$ such that ord(x,$\mathcal{G}_{n_\alpha}(\alpha)$) $\leq \omega$. Put $F = \{n_\alpha : \alpha \in \triangle\}$, then

$$(\mathcal{H}_F)_x \subset \{G\bigcap[\bigcap_{\alpha\in\triangle'}O_\alpha]\text{: } G\in \bigwedge_{\alpha\in\triangle'}(\mathcal{G}_{n_\alpha}^{-1}(\alpha))_x \text{ and } \triangle' \in [\triangle]^{<\omega}\},$$

i.e., ord(x,$\mathcal{H}_F$)$\leq \omega$.

So, X is a hereditarily $\delta\theta$-refine space. □

Now, we discuss Tychonoff products of infinite factors about both $\delta\theta$-refinable spaces and hereditarily $\delta\theta$-refinable spaces.

Proof of Theorem 3. (⇐) When $|\Lambda| < \omega$, it is obvious that X=$\prod_{\alpha\in\Lambda}X_\alpha$ is $\delta\theta$-refinable since F=Λ ∈$[\Lambda]^{<\omega}$. Without the loss of generality, we suppose $|\Lambda| \geq \omega$. Define the relation ≤: F≤E if and only if F⊂E for each F,E ∈$[\Lambda]^{<\omega}$. Then $[\Lambda]^{<\omega}$ is a directed set on the relation ≤. Put X_F=$\prod_{\alpha\in F}X_\alpha$ for each $F \in [\Lambda]^{<\omega}$ and define the projection:

π_F^E:$X_E \to X_F$ when $F \leq E$, where π_F^E(x)=$(x_\alpha)_{\alpha\in F} \in X_F$ for each x=$(x_\alpha)_{\alpha\in E} \in X_E$.

It is easy to prove that π_F^E is an open and onto map, {X_E, π_F^E, $[\Lambda]^{<\omega}$} is an inverse system of spaces X_E with bounding maps π_F^E: $X_E \to X_F$ when $E \geq F$.

Let X$'$ is the inverse limit of the inverse system {X_E, π_F^E, $[\Lambda]^{<\omega}$}, by [4, 2.5.3 Example], X$'$ is homeomorphic to X=$\prod_{\alpha\in\Lambda}X_\sigma$.

In other respects, since each X_F=$\prod_{\alpha\in F}X_\alpha$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable), the inverse system {X_E, π_F^E, $[\Lambda]^{<\omega}$} satisfies the condition of Theorem 1. X$'$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable). Therefore, so is X=$\prod_{\alpha\in\Lambda}X_\alpha$ also.

(⇐) Assume that the product X=$\prod_{\alpha\in\Lambda}X_\alpha$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable). For every F∈$[\Lambda]^{<\omega}$, let us put a point $x_\alpha \in X_\alpha$ when $\alpha \in$ Λ-F, then the closed subspace Y_F=$\prod_{\alpha\in F}X_\alpha \times \prod_{\alpha\in\Lambda-F}\{x_\alpha\}$ of X is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable). Thus, X_F=$\prod_{\alpha\in F}X_\alpha$ is also $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable). □

Proof of Theorem 4. The equivalence of both (1) and (2) is direct by Theorem 3. (2)⇒(3) hold obviously. Now, we prove (3)⇒(2). In fact, for each $F \in [\Lambda]^{<\omega}$, let m=maxF since F≠ ϕ. We pick a fixed $x_\alpha \in X_\alpha$ when $\alpha \in \{0,1,...,m\}-F$, then $\prod_{\alpha\in F}X_\alpha\times\prod_{\alpha\in\{0,1,...,m\}-F}\{x_\alpha\}$ is a closed set of $\prod_{i\leq m}X_i$. So, $\prod_{i\in F}X_i$ is $\delta\theta$-refinable (resp. hereditarily $\delta\theta$-refinable). □

Finally, we point out that there are similar results about both weakly $\delta\theta$ -refinable spaces and hereditarily weakly $\delta\theta$-refinable spaces

Corollary 1. Let X be the inverse limit of an inverse system $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ and let the projection π_α is an open and onto map for each $\alpha \in \Lambda$. If X is $|\Lambda|$-paracompact (resp. hereditarily $|\Lambda|$- paracompact) and each X_α is weakly $\delta\theta$-refinable (resp. hereditarily weakly $\delta\theta$-refinable), then X is weakly $\delta\theta$-refinable (resp. hereditarily weackly $\delta\theta$-refinable).

Proof. We only prove the situation of weakly $\delta\theta$-refinable spaces, the Proof of hereditarily weakly $\delta\theta$-refinable spaces is similar to Theorem 2.

Let $\mathcal{U}$ = {U_ξ: $\xi \in \Xi$} be an arbitrary open cover of X. For each $\alpha \in \Lambda$ and each $\xi \in \Xi$, the following are the same as the symbols in the proof of the above theorem: $V_{\alpha\xi}$, V_α, W_α, T_α, C_α and O_α. And there are the results which are same as (1)-(3) in Theorem 1.

Since $T_\alpha \subset V_\alpha$=$\bigcup$\{$V_{\alpha\xi}$:$\xi \in \Xi$\}, there is an open cover $\bigcup_{n\in\omega}\mathcal{G}_n(\alpha)$ of T_α such that

(4$'$) For each G∈ $\bigcup_{n\in\omega}\mathcal{G}_n(\alpha)$, there is some $\xi \in \Xi$ such that G⊂$V_{\alpha\xi}$, and $G_1\bigcap G_2 \in \mathcal{G}_n(\alpha)$ for each G_1, $G_2 \in \mathcal{G}_n(\alpha)$

(5$'$) For each x∈T_α there is a $n_\alpha \in \omega$ such that 1≤ord(x,$\mathcal{G}_n(\alpha)$) $\leq \omega$.

For each $n \in \omega$ and each $F \in [\omega]^{<\omega}$, let us put $\mathcal{H}_n$ = {π_α^{-1}(G)$\bigcap O_\alpha$:G∈ $\mathcal{G}_n(\alpha)$ and $\alpha \in \Lambda$} and $\mathcal{H}_F$=$\bigwedge_{n\in F}\mathcal{H}_n$, then

(6$'$) Each $\mathcal{H}_F$ is an open part refinement of $\mathcal{U}$.

Finally, we prove:

(7$'$) For each x ∈X, there is some $F \in [\omega]^{<\omega}$ such that ord(x,$\mathcal{H}_F$)$\leq \omega$.

Let $x \in$X, since $\{O_\alpha : \alpha \in \Lambda\}$ is a locally open cover of X, $\triangle=\{\alpha \in \Lambda: x \in O_\alpha\}$ is an nonempty finite set. And for each $\alpha \in \triangle$, since $x \in O_\alpha \subset \pi_\alpha^{-1}(T_\alpha)$, then $x_\alpha \in T_\alpha$. There is some $n_\alpha \in \omega$ such that $1\leq \mathrm{ord}(x,\mathcal{G}_{n_\alpha}(\alpha)) \leq \omega$. Put $F = \{n_\alpha : \alpha \in \triangle\}$, then

$$\phi \neq (\mathcal{H}_F)_x \subset \{G\bigcap[\bigcap_{\alpha\in\triangle'} O_\alpha]:\ G\in \bigwedge_{\alpha\in\triangle'}(\mathcal{G}_{n_\alpha}^{-1}(\alpha))_x \text{ and } \triangle' \in [\triangle]^{<\omega}\}.$$

So, $1\leq \mathrm{ord}(x,\mathcal{H}_F)\leq \omega$. □

Corollary 2. Let $X=\prod_{\alpha\in\Lambda}X_\alpha$ be $|\Lambda|$-paracompact, X is $\delta\theta$-refinable (resp. weakly $\delta\theta$-refinable) iff $\prod_{\alpha\in F}X_\alpha$ is $\delta\theta$-refinable (resp. weakly $\delta\theta$-refinable) for each $F\in[\Sigma]^{<\omega}$.

Corollary 3. Let $X=\prod_{i\in\omega}X_i$ is countable paracompact, then the following are equivalent:

(1) X is weakly $\delta\theta$-refinable (resp. hereditarily weakly $\delta\theta$-refinable).

(2) $\prod_{i\in F}X_i$ is weakly $\delta\theta$-refinable (resp. hereditarily weakly $\delta\theta$-refinable) for each $F \in [\Sigma]^{<\omega}$.

(3) $\prod_{i\leq n}X_i$ is weakly $\delta\theta$-refinable (resp. hereditarily weakly $\delta\theta$-refinable) for each $n\in \omega$.

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