Scientiae Mathematicae Japonicae Online, Vol. 8, (2003), 249–282

NON-ABSOLUTE MULTIPLE INTEGRAL DEFINED CONSTRUCTIVELY ON THE EUCLIDEAN SPACE AND ITERATED INTEGRAL

Shizu Nakanishi

Received January 10, 2003; revised February 10, 2003

Abstract. In this paper, we propose a non-absolute multiple integration in Euclidean spaces defined constructively, and show that the integral is obtained as the iterated integral of one-dimensional integral. The integral is defined as an extension of the special Denjoy integral to higher dimensions.

In [1], 1955, we have published a few constructive definitions which characterize the special Denjoy integral to investigate multidimensional generalizations of the special Denjoy integral (cf. [2]). In this paper, we propose a non-absolute multiple integration in multidimensional Euclidean spaces. The integral is defined as an extension of one definition chosen from among some other definitions for the spacial Denjoy integral shown in [1]([1, Theorems 3 and 4]) to higher dimensions. In this paper, we shall show that the multiple integral is obtained as the iterated integral of one dimensional integral which is equivalent to the special Denjoy integral.

Let E_n be the n-dimensional Euclidean space. Given a system of $2n$ real numbers $a_1, b_1; a_2, b_2; \dots; a_n, b_n$ with $a_i < b_i$ for $i = 1, 2, \dots, n$, the set $\{(x_1, x_2, \dots, x_n) : a_i \le x_i \le b_i$ for $i = 1, 2, \dots, n\}$ is called an interval in E_n. A finite system of intervals I_i $(i = 1, 2, \dots, i_0)$ in E_n is called an elementary system if $I_i \cap I_{i'} = \emptyset$ for $i \ne i'$, sometimes it is denoted by $S : \{I_i(i = 1, 2, \dots, i_0)\}$. μ_n denoted the Lebesgue measure on E_n. Sometimes, the Lebesgue measure of an interval I in E_n is denoted by $|I|$ and for an elementary system $S : \{I_i(i = 1, 2, \dots, i_0)\}$, S denotes the set $\cup_{i=1}^{i_0} I_i$, $|S|$ denotes the measure $\sum_{i=1}^{i_0} |I_i|$, and when $F(I)$ is a finitely additive interval function on an interval in E_n containing $S : \{I_i(i = 1, 2, \dots, i_0)\}$, $F(S)$ denotes $\sum_{i=1}^{i_0} F(I_i)$. N denotes the set $\{1, 2, \dots\}$. Measure means Lebesgue measure. The Lebesgue integral of a function $f(p)$ on a set E in E_n is denoted by $(L)\int_E f(p)dp$ or $(L)\int\int\cdots\int_E f(x_1, \dots, x_n)d(x_1, \dots, x_n)$.

We refer to S.Saks [4] for the terminology and the propositions concerning points of density for a set etc.

For a set $A \subset E_n$, $\overline{A}$ denotes the closure of A in E_n, and A° the interior of A in E_n. Sometimes, for an elementary $S : \{I_i(i = 1, 2, \dots, i_0)\}$, the interior of the set $\cup_{i=1}^{i_0} I_i$ is denoted by S°.

Definition 1. Let R_0 be an interval in the n_0-dimensional Euclidean space E_n and $f(p)$ a measureable function defined on R_0. The function $f(p)$ is said to be (D_0) integrable on R_0 if there exist a finitely additive interval function $F(I)$ defined on R_0, a nondecreasing sequence of measureable sets M_n $(n = 1, 2, \dots)$ such that $M_n \subset R_0$ and $\cup_{n=1}^{\infty} M_n = R_0$, and a nondecreasing sequence of closed sets F_n $(n = 1, 2, \dots)$ such that $F_n \subset M_n$ and $\mu_{n_0}(R_0 - \cup_{n=1}^{\infty} F_n) = 0$, satisfying the following conditions (1) and (2):

2000 Mathematics Subject Classification. 28A35,26A39.
Key words and phrases. non-absolute multiple integral, iterated integaral, special Denjoy integral.

(1) $f(p)$ is Lebesgue integrable on F_n for each $n \in N$;

(2) Given any $n \in N$ and a number $\varepsilon > 0$, there exists a number $\delta(n,\varepsilon) > 0$ for which the following holds: if I_i $(n = 1,\ 2,\ \dots,\ i_0)$ is an elementary system in R_0 such that

(2.1) $I_i \cap M_n \neq \emptyset$ for $i = 1,\ 2,\ \dots,\ i_0$;

(2.2) $\mu_{n_0}(\cup_{i=1}^{i_0} I_i - M_n) < \delta(n,\varepsilon)$,

then the following inequality holds:

$$\left| \sum_{i=1}^{i_0} F(I_i) - \sum_{i=1}^{i_0} (L) \int_{I_i \cap F_n} f(p)dp \right| < \varepsilon.$$

In this case, $F(R_0)$ is called the (D_0) *integral* of $f(p)$ on R_0, and it is denoted by $(D_0)\int_{R_0} f(p)dp$ or $(D_0)\int\cdots\int_{R_0} f(x_1,\ \dots,\ x_{n_0})d(x_1,\ \dots,\ x_{n_0})$. Further, the sequence $M_n (n = 1,\ 2,\ \dots)$ is called a *characteristic sequence* of the (D_0) integral and the sequence F_n $(n = 1,\ 2,\ \dots)$ is called a *fundamental sequence* of the (D_0) integral.

We remark that in Definition 1 we can suppose that $\delta(n,\varepsilon)$ has the following property:

$$\delta(n,\varepsilon) \geq \delta(m,\varepsilon) \text{ for } m > n \text{ and } \delta(n,\varepsilon) \geq \delta(n,\varepsilon') \text{ for } \varepsilon > \varepsilon'.$$

The following Propositions 1-3 follow immediately from the definition of (D_0) integral.

Proposition 1. If a function $f(p)$ is (D_0) integrable on an interval R_0 in E_{n_0}, then $f(p)$ is (D_0) integrable on any sub-interval R of R_0, and if $F(I)$ is the interval function indicated in the definition of (D_0) integral of $f(p)$, $F(R)$ is the (D_0) integral of $f(p)$ on R.

Proposition 2. When $f(p)$ and $f^*(p)$ are functions defined on an interval R_0 in E_{n_0} such that $f(p) = f^*(p)$ almost everywhere on R_0, $f(p)$ is (D_0) integrable on R_0 if and only if $f^*(p)$ is (D_0) integrable on R_0, and the (D_0) integrals of $f(p)$ and $f^*(p)$ on R_0 coincide.

Proposition 3. If $f(p)$ and $g(p)$ are (D_0) integrable functions on an interval R_0 in E_{n_0}, then the function $\alpha f(p)+\beta g(p)$, where α, β are real numbers, is (D_0) integrable on R_0 and $(D_0)\int_{R_0}(\alpha f(p)+\beta g(p))dp = \alpha(D_0)\int_{R_0} f(p)dp + \beta(D_0)\int_{R_0} g(p)dp$.

Proposition 4. When $f(p)$ is a function defined on an interval I_0 in the one-dimensional Euclidean space, the function $f(p)$ is (D_0) integrable on I_0 if and only if it is special Denjoy integrable on I_0, and both integrals on I_0 coincide.

Proof. This follows from Theorems 3 and 4 in [1, pp. 82-83].

Proposition 5. Let $f(p)$ be a (D_0) integrable function on an interval R_0 in E_{n_0}, and let $F(I) = (D_0)\int_I f(p)dp$ for an interval I in R_0. Then, if $I_j (j = 1,\ 2,\ \dots)$ is a decreasing sequence of intervals in R_0 such that $\lim_{j\to\infty} \mu_{n_0}(I_j) = 0$, then $\lim_{j\to\infty} F(I_j) = 0$.

This follows from that there exist a point $p \in \cap_{j=1}^{\infty} I_j$ and an $n \in N$ such that $p \in M_n$, as an immediate consequence of the definition of (D_0) integral.

Throughout this paper, R_0 denotes an interval in the n_0-dimensional Euclidean space E_{n_0}. When $f(p)$ is a (D_0) integrable function on R_0, $F(I)$ denotes the interval function indicated in the definition of (D_0) integral of $f(p)$ in R_0, M_n $(n = 1,\ 2,\ \dots)$ and F_n $(n = 1,\ 2,\ \dots)$ denote the characteristic and fundamental sequences of (D_0)-integral for $f(p)$ in R_0, respectively, and $\delta(n,\varepsilon)$ denotes the positive number indicated in the definition of (D_0)-integral of $f(p)$ in R_0, corresponding to $n \in N$ and $\varepsilon > 0$.

Lemma 1. Let R_0 be an interval in E_{n_0} and $f(p)$ a (D_0) integrable function on R_0. Then, given any $n \in N$ and a number $\varepsilon > 0$, if I_i $(i = 1,\ 2,\ \dots,\ i_0)$ is an elementary system in R_0 such that

(2.1*) $I_i \cap \overline{M_n} \neq \emptyset$ for $i = 1, 2, \ldots, i_0$;

(2.2*) $\mu_{n_0}(\cup_{i=1}^{i_0} I_i) < \delta(n, \varepsilon)$,

then the following inequality holds:

$$\left| \sum_{i=1}^{i_0} F(I_i) - \sum_{i=1}^{i_0} (L) \int_{I_i \cap F_n} f(p)dp \right| < \lambda_{n_0}\varepsilon,$$

where λ_{n_0} is a positive number depending only on the dimension of the space E_{n_0}. In particular $\lambda_2 = 4$.

The proof follows easily from the definition of (D_0) integral.

For sets A and B such that $A \subset E_{n_1}$ and $B \subset E_{n_2}$, $A \times B$ denotes the product set of A and B. When the space E_n is the product space $E_n = E_{n_1} \times E_{n_2}$ of E_{n_1} and E_{n_2} and A is a sub-set of E_n, $\underset{E_{n_1}}{\text{proj}_x}(A)$ denotes the projection of the set A on E_{n_1} and $\underset{E_{n_2}}{\text{proj}_y}(A)$ the projection of the set A on E_{n_2}. In particular, when $n = 2$ and $n_1 = n_2 = 1$, $\underset{E_{n_1}}{\text{proj}_x}(A)$ is denoted by $\text{proj}_x(A)$, and $\underset{E_{n_2}}{\text{proj}_y}(A)$ is denoted by $\text{proj}_y(A)$. When $A \subset E_n$, we denote: for a point $p \in E_{n_1}$, the set $\{(p, q) : (p, q) \in A,\ q \in E_{n_2}\}$ by A^p; for a point $q \in E_{n_2}$, the set $\{(p, q) : (p, q) \in A,\ p \in E_{n_1}\}$ by A^q.

An elementary system $S:\ I_i\ (i = 1, 2, \ldots, i_0)$ in E_n is called a (*)-elementary system if

$$\underset{E_{n-1}}{\text{proj}}\,_y(I_1) = \underset{E_{n-1}}{\text{proj}}\,_y(I_2) = \ldots = \underset{E_{n-1}}{\text{proj}}\,_y(I_{i_0}).$$

An elementary system S is called a (**)-elementary system if it is composed of finite (*)-elementary systems $S_l\ (l = 1, 2, \ldots, l_0)$ such that

$$\underset{E_{n-1}}{\text{proj}}\,_y(S_l) \cap \underset{E_{n-1}}{\text{proj}}\,_y(S_{l'}) = \emptyset \quad \text{for } l \neq l'.$$

Let $f(p)$ be a (D_0) integrable function on an interval R_0 in E_{n_0}. For $n \in N$ and $\varepsilon > 0$, let $\eta(n, \varepsilon)$ be a positive number such that

$$\text{if } \mu_{n_0}(E) < \eta(n, \varepsilon), \text{ then } (L) \int_{E \cap F_n} |f(p)|dp < \varepsilon. \qquad (1°)$$

Without loss of generality, we can suppose that

$$\eta(n, \varepsilon) \geq \eta(m, \varepsilon) \text{ for } m > n \text{ and } \eta(n, \varepsilon) \geq \eta(n, \varepsilon') \text{ for } \varepsilon > \varepsilon'.$$

Throughout this paper, let $\varepsilon_n\ (n = 1, 2, \ldots)$ be a sequence of positive numbers such that

$$\varepsilon_n \downarrow 0 \text{ and } \sum_{m=n+1}^{\infty} \varepsilon_m \leq \varepsilon_n \text{ for each } n \in N, \qquad (2°)$$

and let $\varepsilon_n^*\ (n = 1, 2, \ldots)$ be the nonincreasing sequence defined by

$$\varepsilon_n^* = \min(\delta(n, \varepsilon_n/2^{n+5}), \quad \eta(n, \varepsilon_n/2^{n+5})) \text{ for each } n \in N. \tag{3°}$$

Without loss of generality, we can suppose that $\varepsilon_n^* \downarrow 0$.

Let J be an interval in the one-dimensional Eucidean space E_1 and A_n $(n = 1, 2, \dots)$ a nondecreasing sequence of closed sets in E_1 such that $\cup_{n=1}^{\infty} A_n = J$. Then, we say that a non-empty closed set F_{nm} in E_1, where $n < m$, has the property ($\mathbf{B}_1$) for $n < m$ in J associated with A_n $(n = 1, 2, \dots)$ and ε_n^* $(n = 1, 2, \dots)$ if it has the following property ($\mathbf{B}_1$):

($\mathbf{B}_1$): (1) $F_{nm} \subset J$ and $F_{nm} \subset A_m$;

(2) Denote the sequence of intervals contiguous to the set consisting of the set F_{nm} and the both end-points of J by J_j $(j = 1, 2, \dots)$. Then, J_j $(j = 1, 2, \dots)$ are classified into $m - n + 1$ parts written J_{kj} $(j = 1, 2, \dots)$(possibly empty or finite), where $k = n, n+1, n+2, \dots, m$, so that

1) $\sum_{j=1}^{\infty} |J_{kj}| < \varepsilon_k^*$;

2) $(J_{kj})^\circ \cap A_k = \emptyset$ for every $j \in N$;

3) one at least of the end-points of the interval J_{kj} belongs to A_k for each $j \in N$.

In this case, the point taken as one at least of the end-points of J_{kj} in 3) is called the characteristic point of J_{kj} and the number k is called the characteristic number of J_{kj}.

First let us apply Lemma 2 in [1, p. 72; 3, p. 2] for the interval R_0 in E_{n_0} $(n_0 > 1)$, the sequence of closed sets $\overline{M_n}$ $(n = 1, 2, \dots)$ and the sequence of positive numbers ε_n^* $(n = 1, 2, \dots)$. Then, the following statement (I) holds.

(I) There exist two increasing sequences of positive integers

$$n_i \text{ and } m_i \ (i = 1, 2, \dots) \text{ such that } i < n_i \text{ and } n_i < m_i < n_{i+1} \tag{4°}$$

and a nondecreasing sequence of non-empty closed sets

$$F_{n_i m_i} \ (i = 1, 2, \dots)$$

having the following properties (1) and (2):

(1) $F_{n_i m_i} \subset R_0$ and $F_{n_i m_i} \subset \overline{M}_{m_i}$ for every $i \in N$;

(2) Let us put

$$Y = \cup_{i=1}^{\infty} \operatorname*{proj}_{E_{n_0-1}} {}_y(F_{n_i m_i}) \text{ and } Z = \operatorname*{proj}_{E_{n_0-1}} {}_y(R_0) - Y. \tag{5°}$$

Then

(a) $\mu_{n_0-1}(Z) = 0$;

(b) for each $q \in Y$ and $i \in N$, if $(F_{n_i m_i})^q \neq \emptyset$, then the closed set $(F_{n_i m_i})^q$ has the property ($\mathbf{B}_1$) for $n_i < m_i$ in $(R_0)^q$ associated with $(\overline{M_n})^q$ $(n = 1, 2, \dots)$ and ε_n^* $(n = 1, 2, \dots)$; and

(c) $\cup_{i=1}^{\infty} (F_{n_i m_i})^q = (R_0)^q$ holds for each $q \in Y$.

Next, corresponding to each point

$$q \in Z(= \underset{E_{n_0-1}}{\text{proj}}\ {}_y(R_0) - \cup_{i=1}^{\infty} \underset{E_{n_0-1}}{\text{proj}}\ {}_y(F_{n_i m_i})),$$

let us apply Lemma 1 in [1, p. 72; 3, p. 2] for the interval $(R_0)^q$, the sequence of closed sets $(\overline{M_n})^q$ $(n = 1,\ 2,\ \dots)$ and the sequence ε_n^* $(n = 1,\ 2,\ \dots)$. Then, the following statement (II) holds.

(II) There exist two increasing sequences of positive integers
$n_i(q)$ and $m_i(q)(i = 1,\ 2,\ \dots)$ such that $i < n_i(q)$ and $n_i(q) < m_i(q) < n_{i+1}(q)$
and a nondecreasing sequence of non-empty closed sets

$$F_{n_i(q)m_i(q)}\ (i = 1,\ 2,\ \dots)$$

such that:

(1) Each $F_{n_i(q)m_i(q)}$ has the property $(\mathbf{B}_1)$ for $n_i(q) < m_i(q)$ in $(R_0)^q$ associated with $(\overline{M_n})^q$ $(n = 1,\ 2,\ \dots)$ and ε_n^* $(n = 1,\ 2,\ \dots)$; and
(2) $\cup_{i=1}^{\infty} F_{n_i(q)m_i(q)} = (R_0)^q$ holds.

We remark that, in what follows, an empty set is considered as a closed set.

Lemma 2. If $f(p)$ is a (D_0) integrable function on an interval R_0 in E_{n_0} $(n_0 > 1)$, then there exists a nondecreasing sequence of measureable sets B_h $(h = 1,\ 2,\ \dots)$(the first finite sets may be empty) such that

(1) $B_h \uparrow R_0$; and
(2) for every $h \in N$, the set $(B_h)^q$ is a closed set for each $q \in \underset{E_{n_0-1}}{\text{proj}}\ {}_y(B_h)$,

in such a way that the following statement holds:

Corresponding to $h,\ \varepsilon$ with $h \in N$ and $\varepsilon > 0$, there exists a number $\rho(h, \varepsilon) > 0$ such that:
Given a number $\varepsilon > 0$, suppose that, for some $h \in N$, a (**)-elementary system S consisting of (*)-elementary systems S_l $(l = 1,\ 2,\ \dots,\ l_0)$, where for each l

S_l is a (*)-elementary system consisting of intervals written

$$I_{lj}\ (j = 1,\ 2,\ \dots,\ j_0(l)),$$

satisfies the following conditions:

(a) For each $l \in \{1,\ 2,\ \dots,\ l_0\}$, there exists a $q_l \in \underset{E_{n_0-1}}{\text{proj}}\ {}_y(S_l) \cap \underset{E_{n_0-1}}{\text{proj}}\ {}_y(B_h)$ such that $(I_{lj})^{q_l} \cap (B_h)^{q_l} \neq \emptyset$ for every $j \in \{1,\ 2,\ \dots,\ j_0(l)\}$;
(b) $|\underset{E_{n_0-1}}{\text{proj}}\ {}_y(S)| < \rho(h, \varepsilon)$.

Then, the following inequality holds:

$$|F(S)| < \varepsilon.$$

Proof. For simplicity, we prove only for the case of $n_0 = 2$ and $R_0 = [0, 1; 0, 1]$. Denote q_l taken in the assumption (a) of the lemma by y_l.

Let

$$n_i,\ m_i \text{ and } F_{n_i m_i}\ (i = 1,\ 2,\ \dots)$$

be the two sequences of positive integers and the sequence of non-empty closed sets indicated in (I) above.

Corresponding to each $h \in N$, if there exists an m_i with $m_i \leq h$, denote

$$\text{by } i(h) \text{ the greatest integer } i \text{ for which } m_i \leq h. \tag{6°}$$

Given $k \in N$, since $m_i \leq m_k$ for $i \leq k$, we have $i(m_k) = k$ by (4°). So

$$F_{n_k m_k} = F_{n_{i(m_k)} m_{i(m_k)}}. \tag{7°}$$

Put, as in (I)

$$Z = \text{proj}_y(R_0) - \cup_{i=1}^{\infty} \text{proj}_y(F_{n_i m_i}).$$

For every $y \in Z$, let $n_i(y),\ m_i(y)$ and $F_{n_i(y) m_i(y)}$ $(i = 1,\ 2,\ \dots)$ be the two sequences of positive integers and the sequence of non-empty closed sets indicated in (II) above. Corresponding to $h \in N$, if there exists an $m_i(y)$ with $m_i(y) \leq h$, denote

$$\text{by } i(y, h) \text{ the greatest integer } i \text{ for which } m_i(y) \leq h. \tag{8°}$$

As easily seen, $i(h) \leq i(h+1)$ and $i(y, h) \leq i(y, h+1)$.

Given an $h \in N$, put

$B_h = F_{n_{i(h)} m_{i(h)}} \cup (\cup^*_{y \in Z} F_{n_{i(y,h)}(y) m_{i(y,h)}(y)})$ when $i(h)$ is definable;

$$B_h = \cup^*_{y \in Z} F_{n_{i(y,h)}(y) m_{i(y,h)}(y)} \text{ for the other case,} \tag{9°}$$

where the union $\cup^*_{y \in Z}$ is over all $y \in Z$ for which $i(y, h)$ is definable. Then, B_h $(h = 1,\ 2,\ \dots)$ is a nondecreasing sequence of measurable sets (the first finite sets may be empty) whose union is R_0 and $(B_h)^y$ is a closed set for every $y \in \text{proj}_y(B_h)$.

Now, put for $h \in N$ and $\varepsilon > 0$

$$\rho(h, \varepsilon) = \min(\delta(h, \varepsilon/2^3),\quad \eta(h, \varepsilon/2)). \tag{10°}$$

Then

$$\rho(h, \varepsilon) \geq \rho(k, \varepsilon) \text{ if } k > h \text{ and } \rho(h, \varepsilon) \geq \rho(k, \varepsilon') \text{ if } \varepsilon > \varepsilon'.$$

Given an $\varepsilon > 0$, for some $h \in N$ let S be a (**)-elementary system satisfying the conditions (a) and (b) of the lemma for B_h and $\rho(h, \varepsilon)$ defined above. For each pair $l,\ j$ with $l \in \{1,\ 2,\ \dots,\ l_0\}$ and $j \in \{1,\ 2,\ \dots,\ j_0(l)\}$, by (a) we have $I_{lj} \cap B_h \neq \emptyset$. Further, by (1) of (I) and (II), $\overline{M}_{m_{i(h)}} \cup \cup_y \overline{M}_{m_{i(y,h)}(y)} \supset B_h$. Hence, by $m_{i(h)} \leq h$ and $m_{i(y,h)}(y) \leq h$, we have $\overline{M_h} \supset B_h$. Therefore, $I_{lj} \cap \overline{M_h} \neq \emptyset$. Further, since $|\text{proj}_y(S)| < \rho(h, \varepsilon)$ by (b)

$$\sum_{l=1}^{l_0} \sum_{j=1}^{j_0(l)} |I_{lj}| < \rho(h, \varepsilon) \leq \delta(h, \varepsilon/2^3), \text{ and}$$

$$\sum_{l=1}^{l_0}\sum_{j=1}^{j_0(l)} |I_{lj}| < \rho(h,\varepsilon) \le \eta(h,\varepsilon/2).$$

Therefore, by Lemma 1

$$\left|\sum_{l=1}^{l_0}\sum_{j=1}^{j_0(l)} F(I_{lj}) - \sum_{l=1}^{l_0}\sum_{j=1}^{j_0(l)} (L)\int_{I_{lj}\cap F_h} f(p)dp\right| < 4(\varepsilon/2^3) = \varepsilon/2,$$

and so, by (1°)

$$|F(S)| = \left|\sum_{l=1}^{l_0}\sum_{j=1}^{j_0(l)} F(I_{lj})\right| < \left|\sum_{l=1}^{l_0}\sum_{j=1}^{j_0(l)} (L)\int_{I_{lj}\cap F_h} f(p)dp\right| + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

For an interval $I = [a_1,\ b_1\ ;\ a_2,\ b_2\ ;\ \dots\ ;\ a_n, b_n]$ in E_n, we denote by $\mathsf{R}_m(I)$ the family of intervals $[a_1 + (k_1(b_1-a_1))/m,\ a_1 + ((k_1+1)(b_1-a_1))/m;\ a_2 + (k_2(b_2-a_2))/m,\ a_2 + ((k_2+1)(b_2-a_2))/m;\ \dots;\ a_n + (k_n(b_n-a_n))/m,\ a_n + ((k_n+1)(b_n-a_n))/m]$, where k_i is an integer with $0 \le k_i \le m-1$ for $i = 1,\ 2,\ \dots,\ n$; by $\mathsf{H}_m(I)$ the family of intervals $[a_1,\ b_1\ ;\ a_2 + (k_2(b_2-a_2))/m,\ a_2 + ((k_2+1)(b_2-a_2))/m;\ \dots;\ a_n + (k_n(b_n-a_n))/m,\ a_n + ((k_n+1)(b_n-a_n))/m]$, where k_i is an integer with $0 \le k_i \le m-1$ for $i = 2,\ \dots,\ n$. We call an interval belonging to $\mathsf{R}_m(I)$ a **cell**.

Lemma 3. Let $f(p)$ be a (D_0) integrable function on an interval R_0 in the n_0-demensional Euclidean space E_{n_0} $(n_0 > 1)$. Given a sequence of positive numbers ε_n $(n = 1,\ 2,\ \dots)$ such that $\varepsilon_n \downarrow 0$ and $\sum_{m=n+1}^{\infty} \varepsilon_m < \varepsilon_n$ for every $n \in N$, there exist:

nondecreasing sequences of closed sets A_i $(i = 1,\ 2,\ \dots)$ and D_i $(i = 1,\ 2,\ \dots)$ such that

(1) $\mu_{n_0}(R_0 - \cup_{i=1}^{\infty} A_i) = 0$ and $\mu_{n_0}(R_0 - \cup_{i=1}^{\infty} D_i) = 0$;

(2) $A_i \supset D_i$ for every $i \in N$;

(3) $f(p)$ is Lebesgue integrable on D_i for every $i \in N$,

and a nonincreasing sequence of positive numbers κ_i^* $(i = 1,\ 2,\ \dots)$,
in such a way that the following statement (4) holds:

(4) For each $i \in N$ the following holds. If S is a (**)-elementary system in R_0 consisting of (*)-elementary systems S_l $(l = 1,\ 2,\ \dots,\ l_0)$, where for each l

S_l is a (*)-elementary system consisting of intervals written

$$I_{lj}\ (j = 1,\ 2,\ \dots,\ j_0(l)),$$

for which there exists a non-empty measurable set Y in $\operatorname{proj}_{E_{n_0-1}} {}_y(R_0)$ such that:

(4.1) $Y \subset \operatorname{proj}_{E_{n_0-1}} {}_y(S^\circ) \cap \operatorname{proj}_{E_{n_0-1}} {}_y(A_i)$;

(4.2) $\mu_{n_0-1}\left(\operatorname{proj}_{E_{n_0-1}} {}_y(S) - Y\right) < \kappa_i^*$;

(4.3) $Y \cap \operatorname*{proj}_{E_{n_0-1}} {}_y((S_l)^\circ) \neq \emptyset$ for every $l \in \{1, 2, \ldots, l_0\}$;

(4.4) for each $l \in \{1, 2, \ldots, l_0\}$, if $q \in Y \cap \operatorname*{proj}_{E_{n_0-1}} {}_y((S_l)^\circ)$, then

$$(I_{lj})^q \cap (A_i)^q \neq \emptyset \text{ for every } j \in \{1, 2, \ldots, j_0(l)\},$$

then the following inequality holds:

$$\left| F(S) - (L) \int_{S \cap D_i} f(p)dp \right| < \varepsilon_i.$$

We remark that, by the assumption (1), we have

$$\mu_{n_0-1} \left(\operatorname*{proj}_{E_{n_0-1}} {}_y(R_0) - \cup_{i=1}^{\infty} \operatorname*{proj}_{E_{n_0-1}} {}_y(D_i) \right) = 0. \tag{11°}$$

Proof of Lemma 3. For simplicity, we prove only for the case of $n_0 = 2$ and $R_0 = [0, 1; 0, 1]$. For the given sequence $\varepsilon_n (n = 1, 2, \ldots)$, we define $\varepsilon_n^* (n = 1, 2, \ldots)$ as in (3°). Let n_i, m_i and $F_{n_i m_i}$ $(i = 1, 2, \ldots)$ be the sequences of integers and the sequence of closed sets indicated in (I) above associated with $R_0, \overline{M_n}$ $(n = 1, 2, \ldots)$ and $\varepsilon_n^* (n = 1, 2, \ldots)$. Put

$$\kappa_i = (1/2)\rho(m_i, \varepsilon_i/2^4) \quad \text{for each} \quad i \in N,$$

where $\rho(h, \varepsilon)$ is the number indicated in (10°). Then, κ_i $(i = 1, 2, \ldots)$ is a nonincreasing sequence.

For each $i \in N$, take an $h(i) \in N$ so that

$$h(i) > i, \ h(j) > h(i) \text{ for } j > i \text{ and } \mu_2(F_{n_i m_i} - F_{m_{h(i)}}) < \kappa_i. \tag{12°}$$

Put

$$A_i = F_{n_i m_i} \text{ and } D_i = F_{n_i m_i} \cap F_{m_{h(i)}} \text{ for each } i \in N. \tag{13°}$$

Then

$$A_i \supset D_i \text{ and } \mu_2(A_i - D_i) < \kappa_i \text{ for each } i \in N.$$

Put

$$\kappa_i^* = (1/2)\min(\kappa_i, \eta(m_{h(i)}, \varepsilon_{h(i)}/2^5)) \text{ for each } i \in N. \tag{14°}$$

It is clear that D_i $(i = 1, 2, \ldots)$ and A_i $(i = 1, 2, \ldots)$ are nondecreasing sequences of closed sets satisfying (1), (2) and (3) of the lemma, and κ_i^* $(i = 1, 2, \ldots)$ is a nonincreasing sequence. Next, we shall prove that the statement (4) holds for them. The proof requires three steps.

Take an $i \in N$ and fix. Under the assumption of (4) of the lemma:

(i) The case when $\mu_1(Y \cap \operatorname{proj}_y((S_l)^\circ)) > 0$ for $l = 1, 2, \ldots, l_0$; Since, by (4.3) and (4.4), $I_{lj} \cap A_i$ is a non-empty closed set for each pair l, j with $l \in \{1, 2, \ldots, l_0\}$ and $j \in \{1, 2, \ldots, j_0(l)\}$, there exists an $m_0(i)$ with

$$m_0(i) > m_i$$

such that: for each pair $l,\ j$ with $l \in \{1,\ 2,\ \dots,\ l_0\}$ and $j \in \{1,\ 2,\ \dots,\ j_0(l)\}$, there exists a non-empty family of cells belonging to $\mathsf{R}_{m_0(i)}(I_{lj})$, denoted by

$$R_{ljs}\ (s = 1,\ 2,\ \dots,\ s_0(l,j)),$$

such that:

1) $R_{ljs} \cap A_i \neq \emptyset$ for $s = 1,\ 2,\ \dots,\ s_0(l,j)$;

2) $R \cap A_i = \emptyset$ for the other cells R belonging to $\mathsf{R}_{m_0(i)}(I_{lj})$;

3) $\mu_2(\cup_{s=1}^{s_0(l,j)} R_{ljs} - A_i) < \kappa_{h(i)}/\sum_{l=1}^{l_0} j_0(l)$;

and, further, when we denote the family of R_{ljs} for which

$$R_{ljs} \cap D_i \neq \emptyset,\ \text{where } s \in \{1,\ 2,\ \dots,\ s_0(l,j)\},$$

by $R_{ljs}(s = 1,\ 2,\ \dots,\ s_1(l,j))$(possibly empty), where $s_1(l,j) \leq s_0(l,j)$ (without loss of generality, such expression is possible), we have

4) $\mu_2(\cup_{s=1}^{s_1(l,j)} R_{ljs} - D_i) < \kappa_{h(i)}/\sum_{l=1}^{l_0} j_0(l)$.

In this case, $\cup_{s=1}^{s_0(l,j)} R_{ljs} \supset I_{lj} \cap A_i$ and $\cup_{s=1}^{s_1(l,j)} R_{ljs} \supset I_{lj} \cap D_i$.

Denote, by E_{lj}, the set $\cup(\mathrm{proj}_y(R) - \mathrm{proj}_y(R^\circ))$, where the union $\cup$ is over all cells R belonging to $\mathsf{R}_{m_0(i)}(I_{lj})$. Then, $E_{lj} = E_{lj'}$ for $j,\ j' \in \{1,\ 2,\ \dots,\ j_0(l)\}$. Denote the common set by E_l.

Fix an $l \in \{1,\ 2,\ \dots,\ l_0\}$:

(i,a) Let $y \in (Y - E_l) \cap \mathrm{proj}_y((S_l)^\circ)$ for which there exist a $j \in \{1,\ 2,\ \dots,\ j_0(l)\}$ and a cell $R \in \mathsf{R}_{m_0(i)}(I_{lj})$ such that $y \in \mathrm{proj}_y(R)$ and $(R)^y \cap (A_i)^y = \emptyset$. For each such y, consider the family of cells

$$\{R:\ R \in \cup_{j=1}^{j_0(l)} \mathsf{R}_{m_0(i)}(I_{lj}),\ y \in \mathrm{proj}\,_y(R) \text{ and } (R)^y \cap (A_i)^y = \emptyset\},$$

and denote the family by $Q_{lk}(y)(k = 1,\ 2,\ \dots,\ k_0(l,y))$. In this case, $\mathrm{proj}_y(Q_{lk}(y))(k = 1,\ 2,\ ,\dots,k_0(l,y))$ are equal. Next,consider the family of intervals, written $G_{lt}(y)(t = 1,\ 2,\ \dots,\ t_0(l,y))$, taken nuiquely to be that: they are mutually disjoint; the union is equal to $\cup_{k=1}^{k_0(l,y)} Q_{lk}(y)$; and $\mathrm{proj}_y(G_{lt}(y)) = \mathrm{proj}_y(G_{lt'}(y))$ (which is equal to $\mathrm{proj}_y\, Q_{lk}(y)$) for every pair t, t' with $t, t' \in \{1,\ 2,\ \dots,\ t_0(l,y)\}$. Put $g_{lt}(y) = (G_{lt}(y))^y$ for $t = 1,\ 2,\ \dots,t_0(l,y)$. Clearly we have $g_{lt}(y) \cap A_i = \emptyset$. Since $g_{lt}(y)$ and A_i are closed and $y \notin E_l$, there exist two dimensional intervals $K_{lt}(y)(t = 1,\ 2,\ \dots,t_0(l,y))$ such that:

$$(K_{lt}(y))^y = g_{lt}(y) \text{ and } K_{lt}(y) \cap A_i = \emptyset;$$

$$\mathrm{proj}\,_y(K_{lt}(y)) = \mathrm{proj}\,_y(K_{lt'}(y)) \text{ for every pair } t, t' \text{ with } t, t' \in \{1,\ 2,\ \dots,t_0(l,y)\};$$

$$y \in \mathrm{proj}\,_y((K_{lt}(y))^\circ) \subset \mathrm{proj}\,_y(I^\circ)(\subset \mathrm{proj}\,_y((S_l)^\circ)),$$

where I is an interval belonging to $\mathsf{H}_{m_0(i)}(I_{lj})$ such that $y \in \mathrm{proj}\,_y(I)$. Associsted with y, take a one-dimensional interval $J^*(y)$ for which

$$y \in (J^*(y))^\circ \subset \mathrm{proj}_y((K_{lt}(y))^\circ) \subset \mathrm{proj}_y(I^\circ) \text{ for } t \in \{1, 2, \dots, t_0(l, y)\}. \tag{15°}$$

(i, b) Let $y \in (Y - E_l) \cap \mathrm{proj}_y((S_l)^\circ)$ for which $(R)^y \cap (A_i)^y \neq \emptyset$ for every cell $R \in \cup_{j=1}^{j_0(l)} \mathsf{R}_{m_0(i)}(I_{lj})$ with $y \in \mathrm{proj}_y(R)$. For each such y, take a one-dimensional interval $J^*(y)$ so that

$$y \in (J^*(y))^\circ \subset \mathrm{proj}_y(I^\circ)(\subset \mathrm{proj}_y((S_l)^\circ)), \tag{16°}$$

where I is an interval belonging to $\mathsf{H}_{m_0(i)}(I_{lj})$ such that $y \in \mathrm{proj}_y(I)$.

Now put $Y_l = (Y - E_l) \cap \mathrm{proj}_y((S_l)^\circ)$. We remark that

$$Y_l \cap Y_{l'} = \emptyset \text{ for } l, l' \in \{1, 2, \dots, l_0\} \text{ with } l \neq l';$$

$$\cup_{l=1}^{l_0} Y_l \subset Y; \text{ and } \mu_1(Y - \cup_{l=1}^{l_0} Y_l) = 0.$$

For every point y of density for Y_l with $y \in Y_l$, take a sequence $J_\lambda(y)(\lambda = 1, 2, \dots)$ of one-dimensional intervals tending to y such that $J^*(y) \supset J_\lambda(y)$, $y \in (J_\lambda(y))^\circ$ and the both end-points of $J_\lambda(y)$ belong to Y_l. Then, the family of intervals $\{J_\lambda(y) : y \in Y_l,\ y$ is a point of density for Y_l and $\lambda = 1, 2, \dots\}$ covers almost all points of Y_l in the sense of Vitai. Hence, by virtue of Vitali's covering theorem, for Y_l there exists a finite sequence of intervals in $\mathrm{proj}_y(R_0)$:

$$J_{\lambda(l,v)}(y_v^l), \text{ simply written } J(y_v^l),\ v = 1, 2, \dots, v_0(l),$$

having the following properties:

1*) $y_v^l \in Y_l$ and y_v^l is a point of density for Y_l;

2*) $y_v^l \in (J(y_v^l))^\circ$;

3*) $\cup_{v=1}^{v_0(l)} J(y_v^l) \subset \mathrm{proj}_y((S_l)^\circ)$;

4*) $\mu_1(Y_l - \cup_{v=1}^{v_0(l)} J(y_v^l)) < \kappa_i^*/l_0$;

5*) $J(y_v^l) \cap J(y_{v'}^l) = \emptyset$ for $v \neq v'$;

6*) the both end-points of $J(y_v^l)$ belong to Y_l.

(Refer to Remark 1, (1) below for the case of $n_0 - 1 \geq 2$.)

Next, put

$$I_v^l = \mathrm{proj}_x(R_0) \times J(y_v^l) \text{ for } v = 1, 2, \dots, v_0(l).$$

For each pair l, v with $l \in \{1, 2, \dots, l_0\}$ and $v \in \{1, 2, \dots, v_0(l)\}$, consider the family of intervals:

$$\{I_v^l \cap R_{ljs} : (R_{ljs})^{y_v^l} \cap (A_i)^{y_v^l} \neq \emptyset, \text{ where } j = 1, 2, \dots, j_0(l) \text{ and } s = 1, 2, \dots, s_0(l, j)\}, \tag{17°}$$

which is a non-empty family. Then, by (15°) and (16°) for each $I_v^l \cap R_{ljs}$ belonging to the family, we have $\mathrm{proj}_y(I_v^l \cap R_{ljs}) = J(y_v^l)$. Put

$$I^l_{vj} = \text{proj}_x(I_{lj}) \times J(y^l_v) \text{ for } j = 1,\ 2,\ \dots,\ j_0(l).$$

Next, for $l = 1,\ 2,\ \dots, l_0,\ v = 1,\ 2,\ \dots,\ v_0(l)$ and $j = 1,\ 2,\ \dots, j_0(l)$, denote by

$$L^l_{vjz}\ (z = 1,\ 2,\ \dots,\ z_0(l, v, j))$$

the family of two-dimensional intervals contained in I^l_{vj}, contiguous to the closed set consisting of the set $\cup(R_{ljs} \cap I^l_v)$, where the union $\cup$ is over all $R_{ljs},\ s = 1,\ 2.\ \dots, s_0(l, j)$, with $(R_{ljs})^{y^l_v} \cap (A_i)^{y^l_v} \neq \emptyset$, and the sides parallel to y-axis of two-dimensional interval I^l_{vj}. Denote, simply, for each pair l, v with $l \in \{1,\ 2,\ \dots, l_0\}$ and $v \in \{1,\ 2,\ \dots, v_0(l)\}$ the family

$$L^l_{vjz}\ (j = 1,\ 2,\ \dots,\ j_0(l),\ z = 1,\ 2,\ \dots,\ z_0(l, v, j))$$

by

$$L^l_{vw}\ (w = 1,\ 2,\ \dots,\ w_0(l, v)).$$

By considering the definition of $J^*(y)$ in (15°) and (16°) and the definition of $J(y^l_v)$, we have

$$L^l_{vw} \cap A_i = \emptyset \text{ for } w = 1,\ 2,\ \dots,\ w_0(l, v). \qquad (18^\circ)$$

And L^l_{vw} $(l = 1,\ 2,\ \dots,\ l_0,\ v = 1,\ 2,\ \dots,\ v_0(l), w = 1,\ 2,\ \dots,\ w_0(l, v))$ are mutually disjoint.

Next, for each $l \in \{1,\ 2,\ \dots,\ l_0\}$, denote the family of intervals contiguous to the closed set consisting of $\cup_{v=1}^{v_0(l)} J(y^l_v)$ and the both end-points of $\text{proj}_y(S_l)$ by

$$J^{*l}_u\ (u = 1,\ 2,\ \dots, u_0(l)).$$

(Refer to Remark 1, (2) below for the case of $n_0 - 1 \geq 2$.)

Put

$$I^{*l}_u = \text{proj}_x(R_0) \times J^{*l}_u \text{ for } u = 1,\ 2,\ \dots,\ u_0(l),$$

and put, for $l = 1,\ 2,\ \dots,\ l_0$ and $u = 1,\ 2,\ \dots,\ u_0(l)$

$$I^{*l}_{uj} = I^{*l}_u \cap I_{lj} \text{ for } j = 1,\ 2,\ \dots, j_0(l). \qquad (19^\circ)$$

(i,1) For L^l_{vw} $(v = 1,\ 2,\ \dots, v_0(l),\ w = 1,\ 2,\ \dots,\ w_0(l, v))$: Corresponding to each two-dimensional interval L^l_{vw}, consider the one-dimensional interval, named J^l_{vw}, determined uniquely by the following four conditions, in virtue of the assumption of (4,4) of the lemma:

1°) J^l_{vw} is contained in an interval, say J^*, which is one of the intervals contiguous to the closed set consisting of the set $(A_i)^{y^l_v}$, i.e., $(F_{n_i m_i})^{y^l_v}$ and the both end-points of the interval $(R_0)^{y^l_v}$. In this case, an end-point of J^* is the point called the characteristic point of J^* as in the property $(\mathbf{B}_1)$;

2°) one end-point of J^l_{vw} is one of the end-points of $(L^l_{vw})^{y^l_v}$;

3°) the other end-point of J_{vw}^l is the characteristic point of J^*, named p_{vw}^l;

4°) $J_{vw}^l \supset (L_{vw}^l)^{y_v^l}$.

In this case, J_{vw}^l $(w = 1,\ 2,\ \dots,\ w_0(l,v))$ are classified into two parts: J_{vw}^{l1} $(w = 1,\ 2,\ \dots,\ w_1(l,v))$ and J_{vw}^{l2} $(w = 1,\ 2,\ \dots,\ w_2(l,v))$ so that each family consists of mutually disjoint intervals. Denote the characteristic point and number associated with J_{vw}^{l1} by p_{vw}^{l1} and h_{vw}^{l1}, and the characteristic point and number associated with J_{vw}^{l2} by p_{vw}^{l2} and h_{vw}^{l2}, respectively.

Put

$$H_{vw}^{l1} = J_{vw}^{l1} \times J(y_v^l) \text{ for } v = 1,\ 2,\ \dots,\ v_0(l) \text{ and } w = 1,\ 2,\ \dots,\ w_1(l,v).$$

For each $k \in N$ with $n_i \le k \le m_i$, denote by

$$J_{vw}^{l1k}\ (w = 1,\ 2,\ \dots,\ w_1\ (l,v,k)) \text{ and } H_{vw}^{l1k}\ (w = 1,\ 2,\ \dots,\ w_1\ (l,v,k))$$

the families of all J_{vw}^{l1} and H_{vw}^{l1} for which the characteristic number h_{vw}^{l1} of J_{vw}^{l1} is k, respectively. Then, H_{vw}^{l1k} $(l = 1,\ 2,\ \dots,\ l_0,\ v = 1,\ 2,\ \dots,\ v_0(l),\ w = 1,\ 2,\ \dots,\ w_1(l,v,k))$ is an elementary system in R_0 such that:

(1) $H_{vw}^{l1k} \cap \overline{M_k} \neq \emptyset$;

(2) $\mu_2(\cup_{l=1}^{l_0} \cup_{v=1}^{v_0(l)} \cup_{w=1}^{w_1(l,v,k)} H_{vw}^{l1k}) = \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \left(\sum_{w=1}^{w_1(l,v,k)} |J_{vw}^{l1k}| \right) \times |J(y_v^l)|$
$< \varepsilon_k^* \times \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} |J(y_v^l)| \le \varepsilon_k^* \times |\operatorname{proj}_y(R_0)| = \varepsilon_k^* \le \delta(k, \varepsilon_k/2^{k+5})$,

where the first inequality follows from the fact that, by (b) of (I), the set $(F_{n_i m_i})^{y_v^l}$ has the property ($\mathbf{B}_1$) for $n_i < m_i$ associated with $(\overline{M_n})^{y_v^l}(n = 1,\ 2,\ \dots)$ and ε_n^* $(n = 1,\ 2,\ \dots)$. Hence, by Lemma 1 we have:

$$\left| \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v,k)} F(H_{vw}^{l1k}) - \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v,k)} (L) \iint_{H_{vw}^{l1k} \cap F_k} f(x,y)d(x,y) \right|$$
$$< 4(\varepsilon_k/2^{k+5}) = \varepsilon_k/2^{k+3}.$$

Further, since $\mu_2(\cup_{l=1}^{l_0} \cup_{v=1}^{v_0(l)} \cup_{w=1}^{w_1(l,v,k)} H_{vw}^{l1k}) < \varepsilon_k^* \le \eta(k, \varepsilon_k/2^{k+5})$ by (3°),

$$\left| \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v,k)} (L) \iint_{H_{vw}^{l1k} \cap F_k} f(x,y)d(x,y) \right| < \varepsilon_k/2^{k+5}.$$

Therefore $\left| \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v,k)} F(H_{vw}^{l1k}) \right| < \varepsilon_k/2^{k+2}$, and so

$$\left| \sum_{k=n_i}^{m_i} \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v,k)} F(H_{vw}^{l1k}) \right| < \sum_{k=n_i}^{m_i} \varepsilon_k/2^{k+2} \le \varepsilon_{n_i}/8 \le \varepsilon_i/8.$$

Hence

$$\left| \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v)} F(H_{vw}^{l1}) \right| < \varepsilon_i/8.$$

Similarly

$$\left|\sum_{l=1}^{l_0}\sum_{v=1}^{v_0(l)}\sum_{w=1}^{w_1(l,v)} F(H^{l1}_{vw}-L^{l1}_{vw})\right| < \varepsilon_i/8,$$

where possibly the set $H^{l1}_{vw}-L^{l1}_{vw}$ is empty. Hence

$$\left|\sum_{l=1}^{l_0}\sum_{v=1}^{v_0(l)}\sum_{w=1}^{w_1(l,v)} F(L^{l1}_{vw})\right| < \varepsilon_i/4.$$

On the other hand, since $L^{l1}_{vw}\cap A_i=\emptyset$ by (18°) and so $L^{l1}_{vw}\cap D_i=\emptyset$. Thus

$$\left|\sum_{l=1}^{l_0}\sum_{v=1}^{v_0(l)}\sum_{w=1}^{w_1(l,v)} \Big(F(L^{l1}_{vw})-(L)\iint_{L^{l1}_{vw}\cap D_i} f(x,y)d(x,y)\Big)\right| < \varepsilon_i/4.$$

Similarly

$$\left|\sum_{l=1}^{l_0}\sum_{v=1}^{v_0(l)}\sum_{w=1}^{w_2(l,v)} \Big(F(L^{l2}_{vw})-(L)\iint_{L^{l2}_{vw}\cap D_i} f(x,y)d(x,y)\Big)\right| < \varepsilon_i/4.$$

Therefore

$$\left|\sum_{l=1}^{l_0}\sum_{v=1}^{v_0(l)}\sum_{w=1}^{w_0(l,v)} \Big(F(L^{l}_{vw})-(L)\iint_{L^{l}_{vw}\cap D_i} f(x,y)d(x,y)\Big)\right| < \varepsilon_i/2.$$

(i, 2) Denote the family of intervals

$$\{I^l_v\cap R_{ljs} : (R_{ljs})^{y^l_v}\cap (A_i)^{y^l_v}\neq\emptyset,\ l\in\{1,\ 2,\ \dots,\ l_0\},\ v\in\{1,\ 2,\ \dots, v_0(l)\},$$
$$j\in\{1,\ 2,\ \dots, j_0(l)\} \text{ and } s\in\{1,\ 2,\ \dots,\ s_0(l,j)\}\}$$

considered in (17°) by $R_s (s=1,\ 2,\ \dots,\ s_0)$. In this case, without loss of generality we can suppose that

$$R_s\cap D_i\neq\emptyset \text{ for } s\in\{1,\ 2,\ \dots,\ s_1\};\ \text{and } R_s\cap D_i=\emptyset \text{ for } s\in\{s_1+1,\ \dots,\ s_0\},$$

where $0\le s_1\le s_0$ (if $s_1=0$, then the former is empty; if $s_1=s_0$, then the latter is empty). We remark that $R_s\cap A_i\neq\emptyset$ for $s=1,\ 2,\ \dots,\ s_0$.

First, consider for $R_s(s=1,\ 2,\ \dots,\ s_1)$. Then, we have:

(1) $R_s\cap M_{m_{h(i)}}\neq\emptyset$ for $s=1,\ 2,\ \dots,\ s_1$. Because, $R_s\cap D_i\neq\emptyset$, and so $R_s\cap F_{m_{h(i)}}\neq\emptyset$ by (13°), and moreover $M_{m_{h(i)}}\supset F_{m_{h(i)}}$;

(2)$\mu_2(\cup_{s=1}^{s_1}R_s-M_{m_{h(i)}})<\delta(m_{h(i)},\varepsilon_{h(i)}/2^7)$. Because, $\mu_2(\cup_{s=1}^{s_1}R_s-M_{m_{h(i)}})\le\mu_2(\cup_{s=1}^{s_1}R_s-F_{m_{h(i)}})\le\mu_2(\cup_{s=1}^{s_1}R_s-D_i)$. Since, further, $\cup_{s=1}^{s_1}R_s$ is contained in the union of R_{lsj} for which we have $R_{lsj}\cap D_i\neq\emptyset$, by 4) above

$$\mu_2(\cup_{s=1}^{s_1} R_s - D_i) < \left(\kappa_{h(i)} / \prod_{l=1}^{l_0} j_0(l)\right) \times \prod_{l=1}^{l_0} j_0(l) = \kappa_{h(i)} < \delta(m_{h(i)}, \varepsilon_{h(i)}/2^7).$$

Hence, by the definition of (D_0) integral

$$\left| \sum_{s=1}^{s_1} F(R_s) - \sum_{s=1}^{s_1} (L) \iint_{R_s \cap F_{m_{h(i)}}} f(x,y) d(x,y) \right| < \varepsilon_{h(i)}/2^7.$$

On the other hand, we have

$$\mu_2(\cup_{s=1}^{s_1} R_s \cap (F_{m_{h(i)}} - D_i)) \le \mu_2(\cup_{s=1}^{s_1} R_s - D_i) < \kappa_{h(i)} < \eta(m_{h(i)}, \varepsilon_{h(i)}/2^5).$$

Hence

$$\left| \sum_{s=1}^{s_1} (L) \iint_{R_s \cap \left(F_{m_{h(i)}} - D_i\right)} f(x,y) d(x,y) \right| < \varepsilon_{h(i)}/2^5.$$

Therefore

$$\left| \sum_{s=1}^{s_1} F(R_s) - \sum_{s=1}^{s_1} (L) \iint_{R_s \cap D_i} f(x,y) d(x,y) \right| < \varepsilon_{h(i)}/2^7 + \varepsilon_{h(i)}/2^5 \le \varepsilon_i/2^7 + \varepsilon_i/2^5.$$

Next, consider for R_s ($s = s_1 + 1, \ \dots, \ s_0$). Then, we have:

(1) $R_s \cap A_i \ne \emptyset$ for $s = s_1 + 1, \ \dots, \ s_0$. Hence, $R_s \cap F_{n_i m_i} \ne \emptyset$, and so $R_s \cap \overline{M}_{m_i} \ne \emptyset$ for $s = s_1 + 1, \ \dots, \ s_0$.

(2)$\mu_2(\cup_{s=s_1+1}^{s_0} R_s) = \mu_2(\cup_{s=s_1+1}^{s_0} R_s - A_i) + \mu_2(\cup_{s=s_1+1}^{s_0} R_s \cap A_i)$. On the other hand, $(\cup_{s=s_1+1}^{s_0} R_s) \cap D_i = \emptyset$, and so $\cup_{s=s_1+1}^{s_0} R_s \cap A_i = (\cup_{s=s_1+1}^{s_0} R_s) \cap (A_i - D_i) \subset A_i - D_i$. Hence

$$\mu_2(\cup_{s=s_1+1}^{s_0} R_s) \le \mu_2(\cup_{s=s_1+1}^{s_0} R_s - A_i) + \mu_2(A_i - D_i).$$

Further, since $\cup_{s=s_1+1}^{s_0} R_s$ is contained in the union of R_{ljs} with $R_{ljs} \cap A_i \ne \emptyset$, $\mu_2(\cup_{s=s_1+1}^{s_0} R_s - A_i) < \kappa_{h(i)}$ by 3) above, and $\mu_2(A_i - D_i) < \kappa_i$ by (12°) and (13°), and so

$$\mu_2(\cup_{s=s_1+1}^{s_0} R_s) < 2\kappa_i \le \delta(m_i, \varepsilon_i/2^7).$$

Hence, by Lemma 1 we have

$$\left| \sum_{s=s_1+1}^{s_0} F(R_s) - \sum_{s=s_1+1}^{s_0} (L) \iint_{R_s \cap F_{m_i}} f(x,y) d(x,y) \right| < 4\varepsilon_i/2^7 = \varepsilon_i/2^5.$$

On the other hand, $\mu_2(\cup_{s=s_1+1}^{s_0} R_s) < 2\kappa_i \le \eta(m_i, \varepsilon_i/2^5)$. So

$$\left| \sum_{s=s_1+1}^{s_0} F(R_s) \right| < \varepsilon_i/2^5 + \varepsilon_i/2^5.$$

Further, since $R_s \cap D_i = \emptyset$ for $s = s_1 + 1, \ldots , s_0$

$$\left| \sum_{s=s_1+1} F(R_s) - \sum_{s=s_1+1} (L) \iint_{R_s \cap D_i} f(x,y)d(x,y) \right| < \varepsilon_i/2^5 + \varepsilon_i/2^5.$$

Consequently

$$\left| \sum_{s=1} F(R_s) - \sum_{s=1} (L) \iint_{R_s \cap D_i} f(x,y)d(x,y) \right| < \varepsilon_i/2^7 + 3\varepsilon_i/2^5.$$

(i,3): For I^{*l}_{uj} ($l = 1, 2, \ldots , l_0$, $u = 1, 2, \ldots , u_0(l)$ and $j = 1, 2, \ldots , j_0(l)$) indicated in (19°): For each pair l, u with $l \in \{1, 2, \ldots , l_0\}$ and $u \in \{1, 2, \ldots , u_0(l)\}$, denote by S^l_u the (*)-elementary system:

$$I^{*l}_{uj} \ (j = 1, 2, \ldots , j_0(l)),$$

and consider the (**)-elementary system consisting of (*)-elementary systems

$$S^l_u \ (l = 1, 2, \ldots , l_0, \ u = 1, 2, \ldots , u_0(l)).$$

(Refer to Remark 1, (3) below for the case of $n_0 - 1 \geq 2$.)

For each pair l, u with $l \in \{1, 2, \ldots , l_0\}$ and $u \in \{1, 2, \ldots , u_0(l)\}$, by 6*) above and the definition of J^{*l}_u, there exists a $y_{lu} \in Y_l \cap J^{*l}_u$. Since then $y_{lu} \in Y_l$ and so $y_{lu} \in Y \cap \mathrm{proj}_y((S_l)^\circ)$, by (4,4) $(I_{lj})^{y_{lu}} \cap (F_{n_i m_i})^{y_{lu}} \neq \emptyset$ for $j = 1, 2, \ldots , j_0(l)$. Moreover, since $y_{lu} \in J^{*l}_u$, $(I^{*l}_{uj})^{y_{lu}} = (I_{lj})^{y_{lu}}$ by (19°). Hence $(I^{*l}_{uj})^{y_{lu}} \cap (F_{n_i m_i})^{y_{lu}} \neq \emptyset$, and so, since $B_{m_i} \supset F_{n_{i(m_i)} m_{i(m_i)}} = F_{n_i m_i}$ by (9°) and (7°), $(I^{*l}_{uj})^{y_{lu}} \cap (B_{m_i})^{y_{lu}} \neq \emptyset$ for $j = 1, 2, \ldots , j_0(l)$. Needless to say, $y_{lu} \in \mathrm{proj}_y(S^l_u) \cap \mathrm{proj}_y(B_{m_i})$. Further

$$\sum_{l=1}^{l_0} \sum_{u=1}^{u_0(l)} |\,\mathrm{proj}_y(S^l_u)|$$

$$= \mu_1(\mathrm{proj}_y(S) - \cup_{l=1}^{l_0} \cup_{v=1}^{v_0(l)} J(y^l_v))$$

$$= \mu_1((\mathrm{proj}_y(S) - Y) \cup Y) - \cup_{l=1}^{l_0} \cup_{v=1}^{v_0(l)} J(y^l_v))$$

$$\leq \mu_1((\mathrm{proj}_y(S) - Y) \cup (Y - \cup_{l=1}^{l_0} \cup_{v=1}^{v_0(l)} J(y^l_v)))$$

$$\leq \mu_1(\mathrm{proj}_y(S) - Y) + \mu_1(\cup_{l=1}^{l_0} Y_l - \cup_{l=1}^{l_0} \cup_{v=1}^{v_0(l)} J(y^l_v)) < \kappa^*_i + \kappa^*_i \leq \kappa_i \quad \text{(by (4.2) and 4*))}$$

$$< \rho(m_i, \varepsilon_i/2^4).$$

Therefore, by Lemma 2

$$\left| \sum_{l=1}^{l_0} \sum_{u=1}^{u_0(l)} \sum_{j=1}^{j_0(l)} F(I^{*l}_{uj}) \right| = \left| \sum_{l=1}^{l_0} \sum_{u=1}^{u_0(l)} F(S^l_u) \right| < \varepsilon_i/2^4.$$

On the other hand, since $\sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)}\sum_{j=1}^{j_0(l)}|I^{*l}_{uj}| \le \sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)}|\mathrm{proj}_y(S^l_u)| < 2\kappa^*_i \le \eta(m_{h(i)},\varepsilon_{h(i)}/2^5)$ and $D_i \subset F_{m_{h(i)}}$, we have

$$\left|\sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)}\sum_{j=1}^{j_0(l)}(L)\iint_{I^{*l}_{uj}\cap D_i} f(x,y)d(x,y)\right| < \varepsilon_{h(i)}/2^5 \le \varepsilon_i/2^5 \text{ by } h(i) > i.$$

Hence

$$\left|\sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)}\sum_{j=1}^{j_0(l)} F(I^{*l}_{uj}) - \sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)}\sum_{j=1}^{j_0(l)}(L)\iint_{I^{*l}_{uj}\cap D_i} f(x,y)d(x,y)\right| < \varepsilon_i/2^4 + \varepsilon_i/2^5.$$

By (i,1), (i,2) and (i,3)

$$\left|F(S) - (L)\iint_{S\cap D_i} f(x,y)d(x,y)\right| < \varepsilon_i/2 + \varepsilon_i/2^7 + 3\varepsilon_i/2^5 + \varepsilon_i/2^4 + \varepsilon_i/2^5.$$

(ii) The case when $\mu_1(Y \cap \mathrm{proj}_y((S_l)^\circ)) = 0$ for $l = 1, 2, \ldots, l_0$: For every $l \in \{1, 2, \ldots, l_0\}$, by (4,3) and (4,4) there exists a y_l such that

$$y_l \in Y \cap \mathrm{proj}_y((S_l)^\circ) \text{ and } (I_{lj})^{y_l} \cap (F_{n_i m_i})^{y_l} \ne \emptyset \text{ for } j = 1, 2, \ldots, j_0(l).$$

Therefore, since $B_{m_i} \supset F_{n_i m_i}$, $(I_{lj})^{y_l} \cap (B_{m_i})^{y_l} \ne \emptyset$ for $j = 1, 2, \ldots, j_0(l)$. Needless to say, $y_l \in \mathrm{proj}_y(S_l) \cap \mathrm{proj}_y(B_{m_i})$. Further

$$\begin{aligned}\sum_{l=1}^{l_0}|\mathrm{proj}_y(S_l)| &= \mu_1(\mathrm{proj}_y(S)) \\ &= \mu_1(\mathrm{proj}_y(S) - Y) + \mu_1(Y) \\ &< \kappa^*_i + \sum_{l=1}^{l_0}\mu_1(Y \cap \mathrm{proj}_y((S_l)^\circ)) \text{ (by (4.2) and (4.1))} \\ &= \kappa^*_i < \kappa_i < \rho(m_i, \varepsilon_i/2^4).\end{aligned}$$

Hence, by Lemma 2

$$\left|\sum_{l=1}^{l_0} F(S_l)\right| < \varepsilon_i/2^4.$$

Since $\mu_2(S) \le \mu_1(\mathrm{proj}_y(S)) < \kappa^*_i < \eta(m_{h(i)}, \varepsilon_{h(i)}/2^5)$ and $D_i \subset F_{m_{h(i)}}$

$$\left|\sum_{l=1}^{l_0}\iint_{S_l\cap D_i} f(x,y)d(x,y)\right| \le (L)\iint_{S\cap F_{m_{h(i)}}}|f(x,y)|d(x,y) < \varepsilon_{h(i)}/2^5 \le \varepsilon_i/2^5.$$

Therefore

$$\left| \sum_{l=1}^{l_0} F(S_l) - \sum_{l=1}^{l_0} (L) \iint_{S_l \cap D_i} f(x,y)d(x,y) \right| < \varepsilon_i/2^4 + \varepsilon_i/2^5.$$

(iii) The case when $\mu_1(Y \cap \operatorname{proj}_y((S_l)^\circ)) > 0$ for some $l \in \{1, 2, \dots, l_0\}$ and $\mu_1(Y \cap \operatorname{proj}_y((S_l)^\circ)) = 0$ for some $l \in \{1, 2, \dots, l_0\}$: This case follows from the results for the cases (i) and (ii).

By (i), (ii) and (iii), the proof is complete.

Remark 1. (1): In general, for the case when $n_0 - 1 \geq 2$, by the density theorem there exists a sub-set X_l of Y_l with $\mu_{n_0-1}(X_l) = 0$ such that the set $Y_l - X_l$ is all points of density for Y_l in Y_l. Further, since a point $p \in Y_l - X_l$ is a density point for Y_l, there exists a regular sequence of intervals $I_j(p)$ $(j = 1, 2, \dots)$ in $\underset{E_{n_0-1}}{\operatorname{proj}_y}(R_0)$ tending to p such that for every $j \in N$, $p \in (I_j)^\circ$ and every vertex of the interval $I_j(p)$ belongs to Y_l. Since then the family $\{I_j(p); p \in Y_l - X_l$ and $j \in N\}$ covers the set $Y_l - X_l$ in the sense of Vitali, there exists, by Vitali's covering theorem, a finite sequence of intervals in $\underset{E_{n_0-1}}{\operatorname{proj}_y}(R_0) : J(p_v^l)(v = 1, 2, \dots, v_0(l))$ satisfying the following condition 6^{**}) in addition to the conditions 1^*)- 5^*) replaced y_v^l with p_v^l:

6^{**}) Every vertex of interval $J(p_v^l)$ belongs to Y_l for $v = 1, 2, \dots, v_0(l)$.

(2): By virtue of 6^{**}), for each $l \in \{1, 2, \dots, l_0\}$ the complement of the set $\cup_{v=1}^{v_0(l)} J(y_v^l)$ for $\underset{E_{n_0-1}}{\operatorname{proj}_y}(S_l)$ is covered by finite intervals J_u^{*l} $(u = 1, 2, \dots, u_0(l))$ such that

1) $J_u^{*l} \cap Y_l \neq \emptyset$ for every $u \in \{1, 2, \dots, u_0(l)\}$;

2) $(J_u^{*l})^\circ \cap (J_{u'}^{*l})^\circ = \emptyset$ for every pair $u \neq u'$ with $u, u' \in \{1, 2, \dots, u_0(l)\}$.

(3): We can classify the intervals J_u^{*l} $(l = 1, 2, \dots, l_0,\ u = 1, 2, \dots, u_0(l))$ into 2^{n_0-1} parts at most to be that the intervals belonging to each part are mutually disjoint.

Lemma 4. Let I_0 be an interval in the one-dimensional Euclidean space E_1 and D_n $(n = 1, 2, \dots)$ a nondecreasing sequence of non-empty measurable sets contained in I_0 such that $\mu_1(I_0 - \cup_{n=1}^\infty D_n) = 0$. Let $f(x)$ be a function defined on I_0 which is Lebesgue integrable on D_n for each $n \in N$. Suppose that the function $f(x)$ has the following property (*):

(*): there exists an interval $I \subset I_0$ for which the limit $\lim_{n\to\infty}(L)\int_{I\cap D_n} f(x)dx$ does not exist.

Then, there exist a number $h_0 > 0$ and a sub-sequence m_i $(i = 1, 2, \dots)$ of $\{1, 2, \dots\}$ having the following property(**):

(**) Given a number $\eta > 0$, for each $i \in N$ there exist an elementary system I_j $(j = 1, 2, \dots, j_0)$ and an integer m_i^* with $m_i^* > m_i$ such that:

1) the both end-points of the interval I_j are rational points for $j = 1, 2, \dots, j_0$;

2) $I_j \cap D_{m_i} \neq \emptyset$ for $j = 1, 2, \dots, j_0$;

3) $\left|\sum_{j=1}^{j_0} (L)\int_{I_j\cap(D_{m_i^*} - D_{m_i})} f(x)dx\right| > h_0$;

4) $\left|\sum_{j=1}^{j_0} (L)\int_{I_j\cap D_{m_i}} f(x)dx\right| < \eta$.

Proof. By the assumption of the lemma, there exists an interval $I \subset I_0$ for which $\lim_{n\to\infty}(L)\int_{I\cap D_n} f(x)dx$ does not exist. Hence, there exist an $h_0 > 0$ and a positive integer m_0 such that: $I \cap D_{m_0} \neq \emptyset$ and for every integer $m > m_0$ there exists an integer m^* with $m^* > m$ such that

$$\left|(L)\int_{I\cap(D_{m^*}-D_m)} f(x)dx\right| > h_0.$$

Put $\left|(L)\int_{I\cap(D_{m^*}-D_m)} f(x)dx\right| = \alpha(m)$. Take an $\varepsilon > 0$ with $\varepsilon < \min(\eta, \alpha(m) - h_0)$, where η is the positive number given in (**). Since $f(x)$ is Lebesgue integrable on D_{m^*}, there exists a number $\pi(\varepsilon, m^*) > 0$ such that

$$(L)\int_{E\cap D_{m^*}} |f(x)|dx < \varepsilon/3 \text{ for any set } E \subset I_0 \text{ with } \mu_1(E) < \pi(\varepsilon, m^*).$$

In the interval I, by Vitali's covering theorem there exist mutually disjoint intervals J_j $(j = 1, 2, \ldots, j_0 - 1)$ such that $J_j \subset I^\circ$, the both end-points of J_j belong to D_m, $\mu_1(\cup_{j=1}^{j_0-1} J_j - (I \cap D_m)) < \pi(\varepsilon, m^*)$ and $\mu_1((I \cap D_m) - \cup_{j=1}^{j_0-1} J_j) < \pi(\varepsilon, m^*)$.

Let I_j^* $(j = 1, 2, \ldots, j_0)$ be the intervals contiguous to the closed set consisting of the set $\cup_{j=1}^{j_0-1} J_j$ and the both end-points of I. Associated with these intervals I_j^*, take an elementary system I_j $(j = 1, 2, \ldots, j_0)$ whose end-points are rational points and such that:

$$I_j \supset I_j^* \text{ for } j = 1, 2, \ldots, j_0;$$

$$\sum_{j=1}^{j_0} \mu_1(I_j - I_j^*) < \pi(\varepsilon, m^*) \text{ and } \mu_1(\cup_{j=1}^{j_0} I_j - I) < \pi(\varepsilon, m^*).$$

Then, I_j $(j = 1, 2, \ldots, j_0)$ has the properties desired in (**) for h_0 taken above and $m^* > m$. Indeed, we first have $I_j \cap D_m \neq \emptyset$ for $j = 1, 2, \ldots, j_0$. Next, since

$\cup_{j=1}^{j_0} I_j = (I - (\cup_{j=1}^{j_0-1} J_j - \cup_{j=1}^{j_0} I_j)) \cup (\cup_{j=1}^{j_0} I_j - I)$;

$(\cup_{j=1}^{j_0-1} J_j - \cup_{j=1}^{j_0} I_j) \cap (D_{m^*} - D_m) \subset (\cup_{j=1}^{j_0-1} J_j) \cap (I - D_m) = (\cup_{j=1}^{j_0-1} J_j) - (I \cap D_m)$ and so $\mu_1((\cup_{j=1}^{j_0-1} J_j - \cup_{j=1}^{j_0} I_j) \cap (D_{m^*} - D_m)) < \pi(\varepsilon, m^*)$; and

$\mu_1(\cup_{j=1}^{j_0} I_j - I) < \pi(\varepsilon, m^*)$,

we have

$$\begin{aligned}
&\left|\sum_{j=1}^{j_0} (L)\int_{I_j\cap(D_{m^*}-D_m)} f(x)dx\right| \\
&\quad \geq \left|(L)\int_{I\cap(D_{m^*}-D_m)} f(x)dx\right| - \left|(L)\int_{(\cup_{j=1}^{j_0-1} J_j - \cup_{j=1}^{j_0} I_j)\cap(D_{m^*}-D_m)} f(x)dx\right| \\
&\qquad - \left|(L)\int_{(\cup_{j=1}^{j_0} I_j - I)\cap(D_{m^*}-D_m)} f(x)dx\right| \\
&\quad > \alpha(m) - \varepsilon/3 - \varepsilon/3 > h_0.
\end{aligned}$$

Further, since

$\cup_{j=1}^{j_0} I_j = (\cup_{j=1}^{j_0} I_j^*) \cup (\cup_{j=1}^{j_0-1} J_j \cap \cup_{j=1}^{j_0} I_j) \cup (\cup_{j=1}^{j_0} I_j - I)$;

$\mu_1((\cup_{j=1}^{j_0} I_j^*) \cap D_m) = \mu_1((I - \cup_{j=1}^{j_0-1} J_j) \cap D_m) = \mu_1((I \cap D_m) - \cup_{j=1}^{j_0-1} J_j)$ and so

$\mu_1((\cup_{j=1}^{j_0} I_j^*) \cap D_m) < \pi(\varepsilon, m^*)$;

$\mu_1((\cup_{j=1}^{j_0-1} J_j)\cap(\cup_{j=1}^{j_0} I_j)) \le \mu_1(\cup_{j=1}^{j_0} (I_j - I_j^*))$ and so $\mu_1(\cup_{j=1}^{j_0-1} J_j \cap \cup_{j=1}^{j_0} I_j) < \pi(\varepsilon, m^*)$;

$\mu_1(\cup_{j=1}^{j_0} I_j - I) < \pi(\varepsilon, m^*)$; and $D_m \subset D_{m^*}$,

we have

$$\left|\sum_{j=1}^{j_0} (L)\int_{I_j \cap D_m} f(x)dx\right| \le \left|(L)\int_{(\cup_{j=1}^{j_0} I_j^*)\cap D_m} f(x)dx\right| + \left|(L)\int_{((\cup_{j=1}^{j_0-1} J_j)\cap(\cup_{j=1}^{j_0} I_j))\cap D_m} f(x)dx\right| + \left|(L)\int_{(\cup_{j=1}^{j_0} I_j - I)\cap D_m} f(x)dx\right|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon < \eta.$$

Thus, we obtain the desired result by putting $m_i = m_0 + i$ and $m_i^* = (m_0 + i)^*$ for $i = 1, 2, \ldots$.

Lemma 5. Let I_0 be an interval in the one-dimensional Euclidean space E_1, A_n $(n = 1, 2, \ldots)$ a nondecreasing sequence of non-empty measurable sets such that $\cup_{n=1}^{\infty} A_n = I_0$, and D_n $(n = 1, 2, \ldots)$ a nondecreasing sequence of non-empty closed sets such that $D_n \subset A_n$ for each $n \in N$ and $\mu_1(I_0 - \cup_{n=1}^{\infty} D_n) = 0$. Let $f(x)$ be a function defined on I_0 which is Lebesgue integrable on D_n for each $n \in N$. Suppose that the function $f(x)$ has the following property (*):

(*):(1) $\lim_{n\to\infty}(L)\int_{I\cap D_n} f(x)dx$ exists for every interval $I \subset I_0$;
but the following statement (2) does not hold.

(2) Put $F(I) = \lim_{n\to\infty}(L)\int_{I\cap D_n} f(x)dx$ for every interval $I \subset I_0$. Then, given $\varepsilon_n \downarrow 0$, for every $n \in N$ there exists an integer $m \ge n$ such that if I_j $(j = 1, 2, \ldots, j_0)$ is an elementary system in I_0 such that $I_j \cap A_m \ne \emptyset$ for $j = 1, 2, \ldots, j_0$, then

$$\left|\sum_{j=1}^{j_0} F(I_j) - \sum_{j=1}^{j_0} (L)\int_{I_j\cap D_m} f(x)dx\right| < \varepsilon_n.$$

Then, there exist a number $h_0 > 0$ and a sub-sequence m_i $(i = 1, 2, \ldots)$ of $\{1, 2, \ldots\}$ having the following property (**):

(**) Given a number $\eta > 0$, for each $i \in N$ there exist an elementary system I_j $(j = 1, 2, \ldots, j_0)$ and an integer m_i^* with $m_i^* > m_i$ such that:

1) the both end-points of the interval I_j are rational points for $j = 1, 2, \ldots, j_0$;

2) $I_j \cap A_{m_i} \ne \emptyset$ for $j = 1, 2, \ldots, j_0$;

3) $\left|\sum_{j=1}^{j_0} (L)\int_{I_j\cap(D_{m_i^*} - D_{m_i})} f(x)dx\right| > h_0$;

4) $\left|\sum_{j=1}^{j_0} (L)\int_{I_j\cap D_{m_i}} f(x)dx\right| < \eta.$

Proof. By the assumption of the lemma there exists an n_0 such that for every integer $m \geq n_0$ there exists an elementary system I_j $(j = 1, 2, \ldots, j_0)$ in I_0 depending on m, such that:

a) $I_j \cap A_m \neq \emptyset$ for every $j = 1, 2, \ldots, j_0$; but

b) $\left|\sum_{j=1}^{j_0} F(I_j) - \sum_{j=1}^{j_0} (L)\int_{I_j\cap D_m} f(x)dx\right| > \varepsilon_{n_0}$.

In this case, since $F(I_j) = \lim_{n\to\infty}(L)\int_{I_j\cap D_n} f(x)dx$, for every $m \geq n_0$ there exists an integer m^* with $m^* > m$ such that

c) $\left|\sum_{j=1}^{j_0} (L)\int_{I_j\cap(D_{m^*}-D_m)} f(x)dx\right| > \varepsilon_{n_0}$.

Put $h_0 = \varepsilon_{n_0}/2$. We now remark that the intervals I_j $(j = 1, 2, \ldots, j_0)$ obtained above are classified into the two parts so that: one part is, for every interval I_j belonging to the part, we have $I_j \cap D_m = \emptyset$; and the other is, for every interval I_j belonging to the part, we have $I_j \cap D_m \neq \emptyset$. Then, for one part at least of these we have

c*) $\left|\sum_{j} (L)\int_{I_j\cap(D_{m^*}-D_m)} f(x)dx\right| > h_0$,

where $\sum_j$ is over the part chosen.

Hence, without loss of generality we can suppose that one at least of the following statements (†) and (††) holds.

(†) There exists a sub-sequence m_i $(i = 1, 2, \ldots)$ of the sequence $\{n_0, n_0 + 1, \ldots\}$ such that, for each $i \in N$, there exist an elementary system I_j $(j = 1, 2, \ldots, j_0)$ and an integer m_i^* with $m_i^* > m_i$ such that

a*) $I_j \cap A_{m_i} \neq \emptyset$ for $j = 1, 2, \ldots, j_0$;

c*) $\left|\sum_{j=1}^{j_0} (L)\int_{I_j\cap(D_{m_i^*}-D_{m_i})} f(x)dx\right| > h_0$; and

d*) $I_j \cap D_{m_i} = \emptyset$ for $j = 1, 2, \ldots, j_0$.

(††) There exists a sub-sequence m_i $(i = 1, 2, \ldots)$ of the sequence $\{n_0, n_0 + 1, \ldots\}$ such that, for each $i \in N$, there exist an elementary system I_j $(j = 1, 2, \ldots, j_0)$ and an integer m_i^* with $m_i^* > m_i$ such that, in addition to a*) and c*) above, the following holds:

e*) $I_j \cap D_{m_i} \neq \emptyset$ for $j = 1, 2, \ldots, j_0$.

For the case of (†): 2), 3) and 4) in (**) of the lemma clearly hold. In this case, as easily seen, since D_{m_i} is a closed set, we can choose I_j $(j = 1, 2, \ldots, j_0)$ so that the both end-points of I_j are rational for $j = 1, 2, \ldots, j_0$.

Next, for the case of (††) : Given $\eta > 0$, for each $i \in N$ let I_j $(j = 1, 2, \ldots, j_0)$, m_i and m_i^* be the elementary system and the integers indicated in (††). Putting

$$\left|\sum_{j=1}^{j_0} (L)\int_{I_j\cap(D_{m_i^*}-D_{m_i})} f(x)dx\right| = \alpha(m_i),$$

take an $\varepsilon > 0$ with $\varepsilon < \min(\eta, \alpha(m_i) - h_0)$. This is possible by c*). Since $f(x)$ is Lebesgue integrable on $D_{m_i^*}$, there exists a number $\pi(\varepsilon, m_i^*) > 0$ such that

$$(L)\int_{E\cap D_{m_i^*}} |f(x)|dx < \varepsilon/3 \text{ for every set } E \text{ with } \mu_1(E) < \pi(\varepsilon, m_i^*).$$

In I_j ($j \in \{1, 2, \dots, j_0\}$), by Vitali's covering theorem there exist mutually disjoint intervals J_k^j ($k = 1, 2, \dots, k_0(j) - 1$) such that: $J_k^j \subset (I_j)^\circ$, the both end-ponts of J_k^j belong to D_{m_i}, $\mu_1(\cup_{j=1}^{j_0} \cup_{k=1}^{k_0(j)-1} J_k^j - (\cup_{j=1}^{j_0} I_j \cap D_{m_i})) < \pi(\varepsilon, m_i^*)$, and $\mu_1((\cup_{j=1}^{j_0} I_j \cap D_{m_i}) - \cup_{j=1}^{j_0} \cup_{k=1}^{k_0(j)-1} J_k^j) < \pi(\varepsilon, m_i^*)$.

For each $j \in \{1, 2, \dots, j_0\}$, let I_k^{j*} ($k = 1, 2, \dots, k_0(j)$) be the intervals contiguous to the closed set consisting of the set $\cup_{k=1}^{k_0(j)-1} J_k^j$ and the both end-points of I_j. Associated with these intervals I_k^{j*} ($j = 1, 2, \dots, j_0$, $k = 1, 2, \dots, k_0(j)$), take an elementary system I_k^j ($j = 1, 2, \dots, j_0$, $k = 1, 2, \dots, k_0(j)$) whose end-points are rational points and such that:

$$I_k^j \supset I_k^{j*} \text{ for } j = 1, 2, \dots, j_0 \text{ and } k = 1, ,2, \dots, k_0(j);$$

$$\sum_{j=1}^{j_0}\sum_{k=1}^{k_0(j)} \mu_1(I_k^j - I_k^{j*}) < \pi(\varepsilon, m_i^*); \text{ and } \sum_{j=1}^{j_0} \mu_1(\cup_{k=1}^{k_0(j)} I_k^j - I_j) < \pi(\varepsilon, m_i^*).$$

Then, the family I_k^j ($j = 1, 2, \dots, j_0$, $k = 1, 2, \dots, k_0(j)$) has the properties desired in (**) for h_0 taken above and $m_i^* > m_i$. Indeed :

$I_k^j \cap A_{m_i} \neq \emptyset$ for $j = 1, 2, \dots, j_0$ and $k = 1, 2, \dots, k_0(j)$, because $I_k^j \cap D_{m_i} \neq \emptyset$ and $D_{m_i} \subset A_{m_i}$; by the consideration similar to the case of Lemma 4, we have:

$$\left|\sum_{j=1}^{j_0}\sum_{k=1}^{k_0(j)} (L)\int_{I_k^j \cap (D_{m_i^*} - D_{m_i})} f(x)dx\right|$$

$$\geq \left|(L)\int_{\cup_{j=1}^{j_0} I_j \cap (D_{m_i^*} - D_{m_i})} f(x)dx\right| - \left|(L)\int_{(\cup_{j=1}^{j_0}\cup_{k=1}^{k_0(j)-1} J_k^j - \cup_{j=1}^{j_0}\cup_{k=1}^{k_0(j)} I_k^j)\cap(D_{m_i^*} - D_{m_i})} f(x)dx\right|$$
$$- \left|(L)\int_{(\cup_{j=1}^{j_0}(\cup_{k=1}^{k_0(j)} I_k^j - I_j))\cap(D_{m_i^*} - D_{m_i})} f(x)dx\right| \geq \alpha(m_i) - \varepsilon/3 - \varepsilon/3 > h_0;$$

$$\left|\sum_{j=1}^{j_0}\sum_{k=1}^{k_0(j)} (L)\int_{I_k^j \cap D_{m_i}} f(x)dx\right|$$

$$\leq \left|(L)\int_{(\cup_{j=1}^{j_0}\cup_{k=1}^{k_0(j)} I_k^{j*})\cap D_{m_i}} f(x)dx\right| + \left|(L)\int_{(\cup_{j=1}^{j_0}\cup_{k=1}^{k_0(j)-1} J_k^j \cap \cup_{j=1}^{j_0}\cup_{k=1}^{k_0(j)} I_k^j)\cap D_{m_i}} f(x)dx\right|$$
$$+ \left|(L)\int_{(\cup_{j=1}^{j_0}(\cup_{k=1}^{k_0(j)} I_k^j - I_j))\cap D_{m_i}} f(x)dx\right| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon < \eta.$$

Thus, J_k^j $(j = 1,\ 2,\ \dots,\ j_0,\ k = 1,\ 2,\ ,\dots,\ k_0(j))$ is an elementary system desired in (**).

The proof is complete.

Remark 2. Let A_n $(n = 1,\ 2,\ \dots)$ and D_n $(n = 1,\ 2,\ \dots)$ be the sequences of sets given in Lemma 5. In this case, for the function $f(x)$ on I_0 given in Lemma 5, if the statements (1) and (2) in Lemma 5 hold, then, when we put $F(I) = \lim_{n\to\infty}(L)\int_{I\cap D_n} f(x)dx$ for every interval $I \subset I_0$, the function $F(I)$ is a finitely additive interval function on I_0, and further:

Given $\varepsilon_n \downarrow 0$, there exists a sub-sequence m_n $(n = 1,\ 2,\ \dots)$ of $\{1,\ 2,\ \dots\}$ such that if I_j $(j = 1,\ 2,\ \dots,\ j_0)$ is an elementary system in I_0 such that $I_j \cap A_{m_n} \neq \emptyset$ for $j = 1,\ 2,\ \dots,\ j_0$, then

$$\left|\sum_{j=1}^{j_0} F(I_j) - \sum_{j=1}^{j_0} (L)\int_{I_j\cap D_{m_n}} f(x)dx\right| < \varepsilon_n.$$

Hence, by [1, Theorem 5, p. 84] (or [2, Main Theorem, p. 229]) if, for the function $f(x)$ given in Lemma 5, the statements (1) and (2) in Lemma 5 are true, then the function $f(x)$ is special Denjoy integrable on I_0, and so by Proposition 4, it is (D_0) integrable on I_0. In this case, $F(I_0)$ is the (D_0) integral of $f(x)$ on I_0.

Theorem 1. Let $f(x_1,\ x_2,\ \dots,\ x_{n_0})$ be a (D_0) integrable function on an interval $R_0 = [a_1,\ b_1;\ a_2,\ b_2;\ \dots;\ a_{n_0},\ b_{n_0}]$ in the n_0-dimensional Euclidean space E_{n_0}. Then, the following two statements hold.

(1) Given any $n \in \{1,\ 2,\ \dots,\ n_0\}$, for almost all $(x_1,\ x_2,\ \dots,\ x_{n-1},\ x_{n+1},\ \dots,\ x_{n_0})$ in the (n_0-1)-dimensional interval $[a_1,\ b_1;\ a_2,\ b_2;\ \dots;\ a_{n-1},\ b_{n-1};\ a_{n+1},\ b_{n+1};\ \dots;\ a_{n_0},\ b_{n_0}]$ the function $f(x_1, x_2,\ \dots,\ x_{n_0})$ considered as a function of x_n in the one-dimensional interval $[a_n,\ b_n]$ is (D_0) integrable on $[a_n,\ b_n]$.

(2) Corresponding to each $n \in \{1,\ 2,\ \dots,\ n_0\}$, there exists a nondecreasing sequence of closed sets D_i $(i = 1,\ 2,\ \dots)$ in R_0 such that μ_{n_0} $(R_0 - \cup_{i=1}^{\infty} D_i) = 0$ and

$$\begin{aligned}(D_0)\int_{a_n}^{b_n} & f(x_1,\ x_2,\ \dots,\ x_n,\ \dots,\ x_{n_0})dx_n \\ &= \lim_{i\to\infty}(L)\int_{(D_i)^q} f(x_1,\ x_2,\ \dots,\ x_n,\ \dots,\ x_{n_0})dx_n\end{aligned}$$

for almost all $q = (x_1,\ x_2,\ \dots,\ x_{n-1},\ x_{n+1},\ \dots,\ x_{n_0})$ in the (n_0-1)-dimensional interval $[a_1,\ b_1;\ a_2,\ b_2;\ \dots;\ a_{n-1},\ b_{n-1};\ a_{n+1},\ b_{n+1};\ \dots;\ a_{n_0},\ b_{n_0}]$.

Proof. For simplicity, we prove only for the case of $n_0 = 2$ and $R_0 = [0,\ 1;\ 0,\ 1]$. Let ε_n $(n = 1,\ 2,\ \dots)$ be the sequence of positive numbers given in (2°) such that $\varepsilon_n \downarrow 0$ and $\sum_{m=n+1}^{\infty} \varepsilon_m < \varepsilon_n$ for every $n \in N$. For the sequence, let

$$A_i = F_{n_i m_i} \text{ and } D_i = F_{n_i m_i} \cap F_{m_{h(i)}}\ (i = 1,\ 2,\ \dots)$$

be the nondecreasing sequences of closed sets defined as in (13°) and $\kappa_i^*(i = 1,\ 2,\ \dots)$ the nonincreasing sequence of positive numbers defined as in (14°). Let $Z = \mathrm{proj}_y(R_0) - \cup_{i=1}^{\infty} \mathrm{proj}_y(F_{n_i m_i})$ as in (I), defined in (5°).

Then, as seen in (c) of (I), for every $y \in \mathrm{proj}_y(R_0) - Z$ we have:

(a) $(A_i)^y$ $(i = 1, 2, \dots)$ is a nondecreasing sequence of closed sets whose union is $(R_0)^y$.

Since, further in the definition of (D_0) integral for $f(x,y)$, $F_n \uparrow$, $\mu_2(R_0 - \cup_{n=1}^{\infty} F_n) = 0$ and $f(x,y)$ is Lebesgue integrable on F_n for every $n \in N$, there exists a set

$$X_0 \subset \operatorname{proj}_y(R_0) \tag{20°}$$

such that $X_0 \supset Z$, $\mu_1(X_0) = 0$, and for every $y \in \operatorname{proj}_y(R_0) - X_0$

(b) $(D_i)^y (i = 1, 2, \dots)$ is a nondecreasing sequence of closed sets such that $(D_i)^y \subset (A_i)^y$, $\mu_1((R_0)^y - \cup_{i=1}^{\infty}(D_i)^y) = 0$, and $f(x,y)$ is Lebesgue integrable on $(D_i)^y$ as a function of x for every $i \in N$.

Hence, for every $y \in \operatorname{proj}_y(R_0) - X_0$, (a) and (b) hold. Therefore, by Remark 2 above in order that the function $f(x,y)$ is (D_0) integrable on $[0, 1]$ as a function of x for almost all $y \in \operatorname{proj}_y(R_0) - X_0$, it suffices to prove that: when we denote the set of all $y \in \operatorname{proj}_y(R_0) - X_0$ for which one at least of the statements (1) and (2) in Lemma 5 is not true by Y^*, we have $\mu_1(Y^*) = 0$. We have $Y^* \subset \operatorname{proj}_y(R_0) - X_0$. By Lemmas 4 and 5, we know that if $y \in Y^*$, then the following statement (*) holds.

(*) There exist a number $h_0(y) > 0$ and a sub-sequence $i_j(y)(j = 1, 2, \dots)$ of $\{1, 2, \dots\}$ for which the following statement (**) holds:

(**) Given a number $\eta > 0$, for each $j \in N$ there exist a one-dimensional elementary system $J_t(y)(= J_t(j,\eta,y))(t = 1, 2, \dots, t_0(y)(= t_0(j,\eta,y)))$ on $\operatorname{proj}_x(R_0)$ and an integer $i'_j(y)(= i'_j(\eta,y))$ with $i'_j(y) > i_j(y)(\geq j)$ such that:

1) the both end-points of $J_t(y)$ are rational points for $t = 1, 2, \dots, t_0(y)$;

2) $(J_t(y) \times \{y\}) \cap (A_{i_j(y)})^y \neq \emptyset$ for $t = 1, 2, \dots, t_0(y)$;

3) $\left|\sum_{t=1}^{t_0(y)} (L)\int_{(J_t(y)\times\{y\})\cap((D_{i'_j(y)})^y-(D_{i_j(y)})^y)} f(x,y)dx\right| > h_0(y)$;

4) $\left|\sum_{t=1}^{t_0(y)} (L)\int_{(J_t(y)\times\{y\})\cap(D_{i_j(y)})^y} f(x,y)dx\right| < \eta$.

Suppose that the outer measure of Y^* is positive, we shall lead a contradition. Take an $h_0 > 0$ so that the one-dimensional outer measure of the set Y^{**} consisting of all $y \in Y^*$ for which $h_0(y) \geq h_0$ is $2k_0$ for some $k_0 > 0$. Take indices i_0, i_1 and i_2 so that $\varepsilon_{i_0} + \varepsilon_{i_1} < (h_0, k_0)/8$, $\varepsilon_{i_1} < \varepsilon_{i_0}$ and $\sum_{i=i_2}^{\infty} \varepsilon_i < \varepsilon_{i_0}$.

Let us consider the statement (**) above for $\eta = \varepsilon_{i_1}/2$ and $j = i_2$. Then, for every $y \in Y^{**}$ there exist a one-dimensional elementary system $J_t(y)(t = 1, 2, \dots, t_0(y))$ and $i(y)$ and $i'(y)$ with $i'(y) > i(y) \geq i_2$ such that:

α) the both end-points of $J_t(y)$ are rational points for $t = 1, 2, \dots, t_0(y)$;

β) $(J_t(y) \times \{y\}) \cap (A_{i(y)})^y \neq \emptyset$ for $t = 1, 2, \dots, t_0(y)$;

γ) $\left|\sum_{t=1}^{t_0(y)} (L)\int_{(J_t(y)\times\{y\})\cap((D_{i'(y)})^y-(D_{i(y)})^y)} f(x,y)dx\right| > h_0$;

δ) $\left|\sum_{t=1}^{t_0(y)} (L)\int_{(J_t(y)\times\{y\})\cap(D_{i(y)})^y} f(x,y)dx\right| < \varepsilon_{i_1}/2$.

In this case, we suppose that there exists a subset Y' of Y^{**}whose outer measure is $\geq k_0$ and such that for every $y \in Y'$ we have

γ') $\sum_{t=1}^{t_0(y)} (L)\int_{(J_t(y)\times\{y\})\cap((D_{i'(y)})^y-(D_{i(y)})^y)} f(x,y)dx > h_0$

instead of γ). Because, for the opposite case: $\sum_{t=1}^{t_0(y)}(L)\int_{(J_t(y)\times\{y\})\cap((D_{i'(y)})^y-(D_{i(y)})^y)} f(x,y)dx < -h_0$ it is sufficient to consider the function $-f(x,y)$ by Proposition 3.

Now, for every integer i^* with $i^* > i_2$, denote by Y_{i^*} the set of all $y \in (\mathrm{proj}_y(R_0) - X_0)$ for which there exist

$$\text{a one-dimensional elementary system } J_t\ (t = 1,\ 2,\ \dots,\ t_0) \text{ on } \mathrm{proj}_y(R_0) \text{ and}$$

$$i' \text{ and } i \text{ with } i^* \geq i' > i \geq i_2,$$

satisfying the following $\alpha^*)$, $\beta^*)$, $\gamma^*)$ and $\delta^*)$:

$\alpha^*)$ the both end-points of J_t are rational points for $t = 1,\ 2,\ \dots,\ t_0$;

$\beta^*)$ $(J_t \times \{y\}) \cap (A_i)^y \neq \emptyset$ for $t = 1,\ 2,\ \dots,\ t_0$;

$\gamma^*)$ $\sum_{t=1}^{t_0}(L)\int_{(J_t\times\{y\})\cap((D_{i'})^y-(D_i)^y)} f(x,y)dx > h_0$;

$\delta^*)$ $\left|\sum_{t=1}^{t_0(y)}(L)\int_{(J_t\times\{y\})\cap(D_i)^y} f(x,y)dx\right| < \varepsilon_{i_1}/2$.

Then, we have

$$\cup_{i^*=i_2+1}^{\infty} Y_{i^*} \supset Y',\ \text{each } Y_{i^*} \text{ is measurable and } Y_{i^*} \uparrow \text{ as } i^* \to \infty.$$

Hence, there exists an integer $i^* > i_2$ such that

$$\mu_1(Y_{i^*}) > (3/4)k_0. \tag{21°}$$

Fix such an i^*.

Next, we consider the set of all combinations (S, i', i) such that S is a one-dimensional elementary system on $\mathrm{proj}_x(R_0)$ consisting of intervals whose end-points are rational; and $i', i \in N$ with $i^* \geq i' > i \geq i_2$. Then, the set is countable. Therefore, we can denote the set by

$$C_s = (S_s, i'(s), i(s))\ (s \in N),$$

where S_s is a one-dimensional elementary system on $\mathrm{proj}_x(R_0)$ written

$$S_s : \{J_j^s\ (j = 1,\ 2,\ \dots,\ j_0(s))\} \text{ and}$$

$$i^* \geq i'(s) > i(s) \geq i_2.$$

In this case, needless to say

$\alpha^{**})$ the both end-points of J_j^s are rational points for $j = 1,\ 2,\ \dots,\ j_0(s)$.

Associate with each $C_s (s \in N)$, denote by Y_s' the set of all $y \in Y_{i^*}$ satisfying the following :

$\beta^{**})$ $(J_j^s \times \{y\}) \cap (A_{i(s)})^y \neq \emptyset$ for $j = 1,\ 2,\ \dots,\ j_0(s)$;

$\gamma^{**})$ $\sum_{j=1}^{j_0(s)}(L)\int_{(J_j^s\times\{y\})\cap((D_{i'(s)})^y-(D_{i(s)})^y)} f(x,y)dx > h_0$;

$\delta^{**})$ $\left|\sum_{j=1}^{j_0(s)}(L)\int_{(J_j^s\times\{y\})\cap(D_{i(s)})^y} f(x,y)dx\right| < \varepsilon_{i_1}/2$.

Then, Y'_s is measurable. Put

$$Z_1 = Y'_1,\ Z_s = Y'_s - \cup_{t=1}^{s-1} Y'_t \text{ (possible empty) for } s = 2,\ 3,\ \ldots .$$

Clearly, $Y_{i^*} = \cup_{s=1}^{\infty} Z_s$ and $Z_s (s = 1,\ 2,\ \ldots)$ are measurable and mutually disjoint. Now, we take an s_0 so that

$$\sum_{s=1} \mu_1(Z_s) > (3/4)k_0 \qquad (22^\circ)$$

and fix. In what follows, $s \in^* \{1,\ 2,\ \ldots,\ s_0\}$ means that $s \in \{1,\ 2,\ \ldots,\ s_0\}$ and $\mu_1(Z_s) \neq 0$. For each Z_s, where $s \in^* \{1,\ 2,\ \ldots,\ s_0\}$, consider a one-dimensional elementary system $K_l^s (l = 1,\ 2,\ \ldots,\ l_0(s))$ on $\text{proj}_y(R_0)$ such that

$$(K_l^s)^\circ \cap Z_s \neq \emptyset \text{ for } l = 1,\ 2,\ \ldots,\ l_0(s); \qquad (23^\circ)$$

$$\mu_1(Z_s - \cup_{l=1}^{l_0(s)} K_l^s) < k_0/2s_0 \text{ and } \mu_1(\cup_{l=1}^{l_0(s)} K_l^s - Z_s) < (1/s_0)(\min(\delta, \kappa^*_{i'(s)})), \qquad (24^\circ)$$

where δ is a positive number such that if $\mu_2(E) < \delta$, then $(L)\iint_{E\cap D_{i^*}} |f(x,y)|d(x,y) < \varepsilon_{i_1}/2$. The existence of such elementary system $\{K_l^s\}$ follows from Vitali's covering theorem. (See Remark 3 below.) As easily seen, we can choose $K_l^s (s \in^* \{1,\ 2,\ \ldots,\ s_0\}$ and $l \in \{1,\ 2,\ \ldots,\ l_0(s)\})$ to be $K_l^s \cap K_{l'}^{s'} = \emptyset$ for $s \neq s'$, $l \in \{1,\ 2,\ \ldots,\ l_0(s)\}$ and $l' \in \{1,\ 2,\ \ldots,\ l_0(s')\}$. Put

$$I_{lj}^s = J_j^s \times K_l^s$$

for $s \in^* \{1,\ 2,\ \ldots,\ s_0\}, l \in \{1,\ 2,\ \ldots,\ l_0(s)\}$ and $j \in \{1,\ 2,\ \ldots,\ j_0(s)\}$. Consider, for each pair s, l with $s \in^* \{1,\ 2,\ \ldots,\ s_0\}$ and $l \in \{1,\ 2,\ \ldots,\ l_0(s)\}$, a two-dimensional (*)-elemetary system

$$S_l^s : \{I_{lj}^s (j = 1,\ 2,\ \ldots,\ j_0(s))\}.$$

In this case, $\text{proj}_y(S_l^s) = K_l^s$ and

$$\text{proj}_y(\cup_{l=1}^{l_0(s)}(S_l^s)) \cap \text{proj}_y(\cup_{l=1}^{l_0(s')}(S_l^{s'})) = \emptyset \text{ for } s \neq s'. \qquad (25^\circ)$$

For each $s \in^* \{1,\ 2,\ \ldots,\ s_0\}$, put

$$Y_s^* = \text{proj}_y(\cup_{l=1}^{l_0(s)}(S_l^s)^\circ) \cap Z_s = \cup_{l=1}^{l_0(s)}(K_l^s)^\circ \cap Z_s. \qquad (26^\circ)$$

Then

$$Y_s^* \subset Z_s \subset Y'_s \text{ and } Y_s^* \cap Y_{s'}^* = \emptyset \text{ for } s \neq s'.$$

Now, we consider the two-dimensional (**)-elementary system S consisting of two-dimensional (*)-elementary systems

$$\{S_l^s\ :\ s \in^* \{1,\ 2,\ \ldots,\ s_0\} \text{ and } l \in \{1,\ 2,\ \ldots,\ l_0(s)\}\}.$$

For the consideration,

(A) first, we consider an i with $i^* > i \geq i_2$. Put

$$\triangle_i = \{s : s \in^* \{1,\ 2,\ \dots,\ s_0\} \text{ and } i(s) = i\},$$

and we consider the two-dimensional (**)-elementary system associated with $\triangle_i$:
$$\{S_l^s\ :\ l \in \{1,\ 2,\ \dots,\ l_0(s)\},\ \text{where } s \text{ is taken over } \triangle_i\}.$$

In this case

$$\cup_{i_2 \leq i < i^*} \triangle_i = \{s :\ s \in^* \{1,\ 2,\ \dots,\ s_0\}\} \text{ and } \triangle_i\ (i_2 \leq i < i^*) \text{ are mutually disjoint.}$$

Put

$$Y^*_{\triangle_i} = \cup_{s \in \triangle_i} Y^*_s.$$

Then, for $s \in^* \{1,\ 2,\ \dots,\ s_0\}$ $Y^*_s \subset Y'_s$, so by $\beta^{**}) Y^*_s \subset \mathrm{proj}_y(A_{i(s)})$. Further, $Y^*_s \subset \mathrm{proj}_y(\cup_{l=1}^{l_0(s)} (S_l^s)^\circ)$ by (26°). Hence, $Y^*_s \subset \mathrm{proj}_y(\cup_{l=1}^{l_0(s)} (S_l^s)^\circ) \cap \mathrm{proj}_y(A_{i(s)})$. Therefore, for each $s \in \triangle_i$ we have $Y^*_s \subset \mathrm{proj}_y(\cup_{l=1}^{l_0(s)} (S_l^s)^\circ) \cap \mathrm{proj}_y(A_i)$, and so

(4.1) $Y^*_{\triangle_i} \subset \mathrm{proj}_y(\cup_{s \in \triangle_i} \cup_{l=1}^{l_0(s)} (S_l^s)^\circ) \cap \mathrm{proj}_y(A_i)$.

Further, by (26°) and (24°), for each $s \in \triangle_i$

$$\begin{aligned}\mu_1(\mathrm{proj}\,_y(\cup_{l=1}^{l_0(s)} S_l^s) - Y^*_s) &= \mu_1(\cup_{l=1}^{l_0(s)} K_l^s - Z_s) < (1/s_0)(\min(\delta,\ \kappa^*_{i'(s)})) \\ &\leq (1/s_0)(\min(\delta, \kappa^*_{i(s)})) = (1/s_0)(\min(\delta, \kappa^*_i))\end{aligned}$$

and so, by (25°) and (26°)

(4.2) $\mu_1(\mathrm{proj}\,_y(\cup_{s \in \triangle_i} \cup_{l=1}^{l_0(s)} S_l^s) - Y^*_{\triangle_i}) < (1/s_0)(\min(\delta,\ \kappa^*_i)) \times s_0 \leq \kappa^*_i$.

By (25°), (26°) and (23°), we have:

(4.3) for $l \in \{1,\ 2,\ \dots,\ l_0(s)\}$, where $s \in \triangle_i$, $Y^*_{\triangle_i} \cap \mathrm{proj}_y((S_l^s)^\circ) = Y^*_s \cap (K_l^s)^\circ = ((\cup_{l'=1}^{l_0(s)} (K_{l'}^s)^\circ \cap Z_s) \cap (K_l^s)^\circ = Z_s \cap (K_l^s)^\circ \neq \emptyset$; and

(4.4) for $l \in \{1,\ 2,\ \dots,\ l_0(s)\}$, where $s \in \triangle_i$, if $y \in Y^*_{\triangle_i} \cap \mathrm{proj}_y((S_l^s)^\circ)$, then $y \in \mathrm{proj}_y((S_l^s)^\circ)$, so $(I_{lj}^s)^y = J_j^s \times \{y\}$ for $j = 1,\ 2,\ \dots,\ j_0(s)$. Further, by (25°) and (26°) $y \in Y^*_s$ and so $y \in Y'_s$. Hence, for $l \in \{1,\ 2,\ \dots,\ l_0(s)\}\}$, where $s \in \triangle_i$, by β^{**})

$$(I_{lj}^s)^y \cap (A_i)^y = (J_j^s \times \{y\}) \cap (A_{i(s)})^y \neq \emptyset \text{ for } j = 1,\ ,2\ \dots,\ j_0(s).$$

Consequently, by Lemma 3 for the (**)-elementary system $\{S_l^s : l \in \{1,\ 2,\ \dots,\ l_0(s)\}$, where s is taken over $\triangle_i\}$, the following inequality holds:

$$\left| \sum_{s \in \triangle_i} \sum_{l=1}^{l_0(s)} F(S_l^s) - (L) \iint_{(\cup_{s \in \triangle_i} \cup_{l=1}^{l_0(s)} S_l^s) \cap D_i} f(x,y) d(x,y) \right| < \varepsilon_i.$$

Since the above inequality holds for every integer i with $i^* > i \geq i_2$

$$\left|\sum_{i=i_2}^{i^*-1}\sum_{s\in\triangle_i}\sum_{l=1}^{l_0(s)} F(S_l^s) - \sum_{i=i_2}^{i^*-1}\sum_{s\in\triangle_i}\sum_{l=1}^{l_0(s)} (L)\iint_{S_l^s\cap D_i} f(x,y)d(x,y)\right| < \sum_{i=i_2}^{i^*-1} \varepsilon_i < \varepsilon_{i_0}.$$

Thus

$$\left|F(S) - \sum_{s\in^*\{1,\ 2,\ \dots,\ s_0\}}\sum_{l=1}^{l_0(s)} (L)\iint_{S_l^s\cap D_i(s)} f(x,y)d(x,y)\right| < \varepsilon_{i_0},$$

and so

$$|F(S)| < \varepsilon_{i_0} + \left|\sum_{s\in^*\{1,2,\dots,s_0\}}\sum_{l=1}^{l_0(s)} (L)\iint_{S_l^s\cap D_{i(s)}} f(x,y)d(x,y)\right|. \qquad (27^\circ)$$

Further, since, for $s \in^* \{1,\ 2,\ \dots,\ s_0\}$,

$$S_l^s = \cup_{j=1}^{j_0(s)} J_j^s \times K_l^s \text{ and } Y_s^* \subset \cup_{l=1}^{l_0(s)} K_l^s \text{ by } (22^\circ),$$

$$\begin{aligned}
&\left|\sum_{s\in^*\{1,2,\dots,s_0\}}\sum_{l=1}^{l_0(s)} (L)\iint_{S_l^s\cap D_{i(s)}} f(x,y)d(x,y)\right| \\
&\quad\le \left|\sum_{s\in^*\{1,2,\dots,s_0\}} (L)\iint_{((\cup_{j=1}^{j_0(s)} J_j^s)\times Y_s^*)\cap D_{i(s)}} f(x,y)d(x,y)\right| \\
&\quad\quad + \left|\sum_{s\in^*\{1,2,\dots,s_0\}} (L)\iint_{((\cup_{j=1}^{j_0(s)} J_j^s)\times(\cup_{l=1}^{l_0(s)} K_l^s - Y_s^*))\cap D_{i(s)}} f(x,y)d(x,y)\right|.
\end{aligned}$$

Since $Y_s^* \subset Y_s'$ and $Y_s^* \cap Y_{s'}^* = \emptyset$ for $s \neq s'$, by δ^{**})

$$\left|\sum_{s\in^*\{1,2,\dots,s_0\}} (L)\iint_{((\cup_{j=1}^{j_0(s)} J_j^s)\times Y_s^*)\cap D_{i(s)}} f(x,y)d(x,y)\right| < \varepsilon_{i_1}/2.$$

By (25°), (26°) and (24°), $\sum_{s\in^*\{1,2,\dots,s_0\}} \mu_1(\cup_{l=1}^{l_0(s)} K_l^s - Y_s^*) < \sum_{s\in^*\{1,2,\dots,s_0\}} (1/s_0)(\min(\delta, \kappa_{i'(s)}^*)) \le \delta$, and further, since $i(s) < i^*$, $D_{i(s)} \subset D_{i^*}$. Hence, by the definition of δ

$$\left|\sum_{s\in^*\{1,2,\dots,s_0\}} (L)\iint_{((\cup_{j=1}^{j_0(s)} J_j^s)\times(\cup_{l=1}^{l_0(s)} K_l^s - Y_s^*))\cap D_{i(s)}} f(x,y)d(x,y)\right| < \varepsilon_i/2.$$

Thus

$$\left| \sum_{s\in^*\{1,2,\ldots,s_0\}} \sum_{l=1}^{l_0(s)} (L)\iint_{S_l^s\cap D_{i(s)}} f(x,y)d(x,y)\right| < \varepsilon_{i_1}/2+\varepsilon_{i_1}/2.$$

Consequently

$$|F(S)| < \varepsilon_{i_0} + \varepsilon_{i_1}, \text{ and so } |F(S)| < (h_0\ k_0)/8. \tag{28°}$$

(B) In a quite similar way as in the case of $i(s)$, for the case of $i'(s)$ we obtain

$$\left| F(S) - \sum_{s=1}^{s_0} \sum_{l=1}^{l_0(s)} (L)\iint_{S_l^s\cap D_{i'(s)}} f(x,y)d(x,y)\right| < \varepsilon_{i_0}.$$

Indeed, first consider an i with $i^* \geq i > i_2$. Put

$$\Lambda_i = \{s : s \in^* \{1,\ 2,\ \ldots,\ s_0\} \text{ and } i'(s) = i\}.$$

Fix an i with $i^* \geq i > i_2$. Next, consider the two-dimensional (**)-elementary system associated with Λ_i :

$$\{S_l^s : l \in \{1,\ 2,\ \ldots,\ l_0(s)\},\ \text{where } s \text{ is taken over } \Lambda_i\}.$$

Then, the (**)-elemetary system has the following properties as in the case (A), putting $Y^*_{\Lambda_i} = \cup_{s\in\Lambda_i} Y^*_s$.

First, for each $s \in \Lambda_i$, since $i'(s) > i(s)$, $A_{i'(s)} \supset A_{i(s)}$, and so, as in the case of (A), $Y^*_s \subset \mathrm{proj}_y(\cup_{l=1}^{l_0(s)} (S_l^s)^\circ) \cap \mathrm{proj}_y(A_{i'(s)}) = \mathrm{proj}_y(\cup_{l=1}^{l_0(s)} (S_l^s)^\circ) \cap \mathrm{proj}_y(A_i)$. Hence

(4.1) $Y^*_{\Lambda_i} \subset \mathrm{proj}_y(\cup_{s\in\Lambda_i} \cup_{l=1}^{l_0(s)} (S_l^s)^\circ) \cap \mathrm{proj}_y(A_i)$.

Since, for each $s \in \Lambda_i$, $\mu_1(\mathrm{proj}_y(\cup_{l=1}^{l_0(s)} S_l^s) - Y^*_s) < (1/s_0)(\min(\delta, \kappa^*_{i'(s)})) = (1/s_0)(\min(\delta, \kappa^*_i))$, we have

(4.2) $\mu_1(\mathrm{proj}_y(\cup_{s\in\Lambda_i} \cup_{l=1}^{l_0(s)} S_l^s) - Y^*_{\Lambda_i}) < \kappa^*_i$.

Further, as in the case of (A), we obtain:

(4.3) For $l = 1,\ ,2,\ \ldots,\ l_0(s)$, where $s \in \Lambda_i$, $Y^*_{\Lambda_i} \cap \mathrm{proj}_y((S_l^s)^\circ) = Z_s \cap (K_l^s)^\circ \neq \emptyset$; and, since $A_{i'(s)} \supset A_{i(s)}$,

(4.4) for $l = 1,\ ,2,\ \ldots,\ l_0(s)$, where $s \in \Lambda_i$, if $y \in Y^*_{\Lambda_i} \cap \mathrm{proj}_y((S_l^s)^\circ)$, then

$$(I^s_{lj})^y \cap (A_i)^y = (J^s_j \times \{y\}) \cap (A_{i'(s)})^y \neq \emptyset \text{ for } j = 1,\ 2,\ \ldots,\ j_0(s).$$

Consequently, by Lemma 3 for the (**)-elementary system $\{S_l^s : l = 1,\ 2,\ \ldots,\ l_0(s)$, where s is taken over $\Lambda_i\}$

$$\left| \sum_{s\in\Lambda_i} \sum_{l=1}^{l_0(s)} F(S_l^s) - (L)\iint_{(\cup_{s\in\Lambda_i}\cup_{l=1}^{l_0(s)} S_l^s)\cap D_i} f(x,y)d(x,y)\right| < \varepsilon_i.$$

Since the above inequality holds for every i with $i^* \geq i > i_2$

$$\left| \sum_{i=i_2+1}^{i^*} \sum_{s\in\Lambda_i} \sum_{l=1}^{l_0(s)} F(S_l^s) - \sum_{i=i_2+1}^{i^*} \sum_{s\in\Lambda_i} \sum_{l=1}^{l_0(s)} (L) \iint_{S_l^s \cap D_i} f(x,y)d(x,y) \right| < \sum_{i=i_2+1}^{i^*} \varepsilon_i < \varepsilon_{i_0}.$$

Therefore

$$\left| F(S) - \sum_{s\in^*\{1,2,\dots,s_0\}} \sum_{l=1}^{l_0(s)} (L) \iint_{S_l^s \cap D_{i'(s)}} f(x,y)d(x,y) \right| < \varepsilon_{i_0}.$$

From this inequality, it follows that

$$|F(S)| > \sum_{s\in^*\{1,2,\dots,s_0\}} \sum_{l=1}^{l_0(s)} (L) \iint_{S_l^s \cap D_{i'(s)}} f(x,y)d(x,y) - \varepsilon_{i_0}.$$

Furthermore

$$\begin{aligned} &\sum_{s\in^*\{1,2,\dots,s_0\}} \sum_{l=1}^{l_0(s)} (L) \iint_{S_l^s \cap D_{i'(s)}} f(x,y)d(x,y) \\ &\qquad \geq \sum_{s\in^*\{1,2,\dots,s_0\}} (L) \iint_{((\cup_{j=1}^{j_0(s)} J_j^s)\times Y_s^*)\cap(D_{i'(s)}-D_{i(s)})} f(x,y)d(x,y) \\ &\qquad - \left| \sum_{s\in^*\{1,2,\dots,s_0\}} (L) \iint_{((\cup_{j=1}^{j_0(s)} J_j^s)\times Y_s^*)\cap D_{i(s)}} f(x,y)d(x,y) \right| \\ &\qquad - \left| \sum_{s\in^*\{1,2,\dots,s_0\}} (L) \iint_{((\cup_{j=1}^{j_0(s)} J_j^s)\times(\cup_{l=1}^{l_0(s)} K_l^s - Y_s^*))\cap D_{i'(s)}} f(x,y)d(x,y) \right| \\ &\qquad > (h_0\ k_0)/4 - \varepsilon_{i_1}/2 - \varepsilon_{i_1}/2 = (h_0\ k_0)/4 - \varepsilon_{i_1}. \end{aligned}$$

Because, we first have, by (24°)

$$\sum_{s\in^*\{1,2,\dots,s_0\}} \mu_1(Z_s - \cup_{l=1}^{l_0(s)} K_l^s) < (k_0/2s_0) \times s_0 = k_0/2.$$

Hence, by (26°) and (22°)

$$\begin{aligned} \sum_{s\in^*\{1,2,\dots,s_0\}} \mu_1(Y_s^*) &= \sum_{s=\in^*\{1,2,\dots,s_0\}} \mu_1(Z_s) - \sum_{s=\in^*\{1,2,\dots,s_0\}} \mu_1(Z_s - \cup_{l=1}^{l_0(s)} K_l^s) \\ &> \sum_{s=1}^{s_0} \mu_1(Z_s) - k_0/2 > (3/4)k_0 - k_0/2 = k_0/4. \end{aligned}$$

Further, for every $y \in Y_s^*(s \in^* \{1,\ 2,\ \dots,\ s_0\})$, we have, by $Y_s^* \subset Y_s'$ and γ^{**}),

$$(L)\int_{((\cup_{j=1}^{j_0(s)} J_j^s)\times\{y\})\cap((D_{i'(s)})^y-(D_{i(s)})^y)} f(x,y)dx > h_0,$$

and $Y_s^* \cap Y_{s'}^* = \emptyset$ for $s \neq s'$. Hence

$$\sum_{s\in^*\{1,2,\dots,s_0\}} (L)\iint_{((\cup_{j=1}^{j_0(s)} J_j^s)\times Y_s^*)\cap(D_{i'(s)}-D_{i(s)})} f(x,y)d(x,y) > (h_0,k_0)/4.$$

Further, by δ^{**})

$$\left|\sum_{s\in^*\{1,2,\dots,s_0\}} (L)\iint_{((\cup_{j=1}^{j_0(s)} J_j^s)\times Y_s^*)\cap D_{i(s)}} f(x,y)d(x,y)\right| < \varepsilon_{i_1}/2.$$

Finally, since, by (26°), $\mu_1(\cup_{l=1}^{l_0(s)} K_l^s - Y_s^*) = \mu_1(\cup_{l=1}^{l_0(s)} K_l^s - Z_s)$,

$$\sum_{s\in^*\{1,2,\dots,s_0\}} \mu_1(\cup_{l=1}^{l_0(s)} K_l^s - Y_s^*) < (1/s_0)(\min(\delta,\kappa_{i'(s)}^*)) \times s_0 \leq \delta.$$

And further $i'(s) \leq i^*$. Hence, by the definition of δ

$$\left|\sum_{s\in^*\{1,2,\dots,s_0\}} (L)\iint_{((\cup_{j=1}^{j_0(s)} J_j^s)\times(\cup_{l=1}^{l_0(s)} K_l^s - Y_s^*))\cap D_{i'(s)}} f(x,y)d(x,y)\right| < \varepsilon_{i_1}/2.$$

Consequently

$$|F(S)| > (h_0\ k_0)/4 - (\varepsilon_{i_1} + \varepsilon_{i_0}) > (h_0\ k_0)/4 - (h_0\ k_0)/8 = (h_0\ k_0)/8,$$

which contradicts $|F(S)| < (h_0\ k_0)/8$ shown in (28°).

Consequently, the outer measure of the set Y^* must be zero.

Now, put

$$W_0 = X_0 \cup Y^*. \tag{29°}$$

Then, $\mu_1(W_0) = 0$, and for every $y \in \mathrm{proj}_y(R_0) - W_0$, $f(x,y)$ is (D_0) integrable as a function of x and the limit

$$\lim_{n\to\infty} (L)\int_{(D_n)^y} f(x,y)dx$$

exists, and the limit coincides with the (D_0) integral of $f(x,y)$ considered as a function of x on $\mathrm{proj}_y(R_0)$. Thus, the proof of Theorem 1 is complete.

Remark 3. The family of intervals in $\underset{E_{n_0-1}}{\mathrm{proj}}\ {}_y(R_0)$ covers the set Z_s in the sense of Vitali. Hence, there exists an elementary system in $\underset{E_{n_0-1}}{\mathrm{proj}}\ {}_y(R_0) : K_l^s(l = 1,\ 2,\ \dots,\ l_0(s))$

satisfying (24°) and such that $K_l^s \cap Z_s \neq \emptyset$ for $l = 1, 2, \dots, l_0(s)$. By a slight modification, we can obtain $K_l^s (l = 1, 2, \dots, l_0(s))$ to satisfy (23°) and (24°).

The result of Theorem 1 leads us to the following theorem.

Theorem 2. Let $f(x_1, x_2, \dots, x_{n_0})$ be a (D$_0$) integrable function on an interval $R_0 = [a_1, b_1; a_2, b_2; \dots; a_{n_0}, b_{n_0}]$ in the n_0-dimensional Euclidean space E_{n_0}. Then

(1) for each $n \in \{1, 2, \dots, n_0\}$, the function $(D_0)\int_{a_n}^{b_n} f(x_1, x_2, \dots, x_{n_0})dx_n$ defined for almost all $(x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_{n_0})$ in the $(n_0 - 1)$-dimensional interval $R_{n_0-1} = [a_1, b_1; a_2, b_2; \dots; a_{n-1}, b_{n-1}; a_{n+1}, b_{n+1}; \dots; a_{n_0}, b_{n_0}]$ is (D$_0$) integrable on the interval R_{n_0-1},

and the following equalities hold:

(2) $(D_0)\int\int\cdots\int_{R_0} f(x_1, x_2, \dots, x_{n_0})d(x_1, x_2, \dots, x_{n_0})$
$= (D_0)\int\int\cdots\int_{R_{n_0-1}} \left((D_0)\int_{a_n}^{b_n} f(x_1, x_2, \dots, x_{n_0})dx_n\right) d(x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_{n_0})$;

(3) $(D_0)\int\int\cdots\int_{R_0} f(x_1, x_2, \dots, x_{n_0})d(x_1, x_2, \dots, x_{n_0})$
$= (D_0)\int_{a_{n_0}}^{b_{n_0}} \cdots \left((D_0)\int_{a_{n_2}}^{b_{n_2}} \left((D_0)\int_{a_{n_1}}^{b_{n_1}} f(x_1, x_2, \dots, x_{n_0})dx_{n_1}\right) dx_{n_2}\right) \cdots dx_{n_{n_0}}$

for every sequence $n_1, n_2, \dots, n_{n_0}$ consisting of $1, 2, \dots, n_0$.

Proof. For simplicity, we prove only for the case when $n_0 = 2$ and $R_0 = [0, 1; 0, 1]$. We first prove (1) and (2) for the case when $x_n = x$, putting $x_1 = x$ and $x_2 = y$. We will omit the proof of the other case, because the proof is similar. Put

$$f_i(y) = (L)\int_{(D_i)^y} f(x, y)dx \text{ for every } y \in \operatorname{proj}_y(R_0) - W_0, \text{ and}$$
$$= 0 \qquad \text{for every } y \in W_0;$$

$$f(y) = (D_0)\int_0^l f(x, y)dx \text{ for every } y \in \operatorname{proj}_y(R_0) - W_0, \text{ and}$$
$$= 0 \qquad \text{for every } y \in W_0,$$

where $D_i (i = 1, 2, \dots)$ is the sequence of closed sets indicated in (2) of Theorem 1, and W_0 is the set of μ_1-measure zero defined in (29°).

Since then $f(y) = \lim_{i\to\infty} f_i(y)$ on $\operatorname{proj}_y(R_0)$ by Theorem 1 and $f_i(y)$ is measurable for each $i \in N$, by the Egoroff's theorem there exists a sequence of measurable sets $M_k^* (k = 0, 1, \dots)$ on $\operatorname{proj}_y(R_0)$ such that: $\cup_{k=0}^\infty M_k^* = \operatorname{proj}_y(R_0)$; $\mu_1(M_0^*) = 0$; and for each $k \in N$, $M_k^* \cap M_0^* = \emptyset$, $M_{k+1}^* \supset M_k^*$, M_k^* is a closed set, and $f_i(y)$ converges uniformly to $f(y)$ on M_k^*. In this case, since $f_i(y)$ is Lebesgue integrable on $\operatorname{proj}_y(R_0)$ for each $i \in N$, $f(y)$ is Lebesgue integrable on M_k^*.

Let $B_k (k = 1, 2, \dots)$ be the sequence of measurable sets indicated in Lemma 2, defined in (9°). Put

$Z_0 = \operatorname{proj}_y(R_0) - \cup_{k=1}^\infty \operatorname{proj}_y(D_k)$;

$L_k = ((\operatorname{proj}_y(B_k) \cap Z_0) \cup \operatorname{proj}_y(D_k)) \cap (M_0^* \cup M_k^*)$ for $k = 1, 2, \dots$;

$N_k = \operatorname{proj}_y(D_k) \cap M_k^*$ for $k = 1, 2, \dots$.

Then, $\mu_1(Z_0) = 0$ by (11°), $L_k (k = 1, 2, \dots)$ is a nondecreasing sequence of measurable sets whose union is $\mathrm{proj}_y(R_0)$ and $N_k (k = 1, 2, \dots)$ is a nondecreasing sequence of closed sets such that $L_k \supset N_k$ and $\mu_1(\mathrm{proj}_y(R_0) - \cup_{k=1}^{\infty} N_k) = 0$. Further, $f(y)$ is Lebesgue integrable on N_k for each $k \in N$.

Let $\varepsilon_i (i = 1, 2, \dots)$ be the sequence of positive numbers given in (2°). Given a $k \in N$ and a number $\varepsilon > 0$, take an $i_0(k, \varepsilon)$ so that $i_0(k, \varepsilon) \geq k$, $\varepsilon_{i_0(k,\varepsilon)} < \varepsilon/7$ and $|f(y) - f_{i_0(k,\varepsilon)}(y)| < \varepsilon/7$ for every $y \in M_k^*$. Let $\lambda(k, \varepsilon)$ be a positive number such that if $\mu_2(E) < \lambda(k, \varepsilon)$, then $\left|(L)\iint_{E\cap D_{i_0(k,\varepsilon)}} f(x, y)d(x, y)\right| < \varepsilon/7$. Let $\lambda^*(k, \varepsilon)$ be a positive number such that if $\mu_1(E) < \lambda^*(k, \varepsilon)$, then $\left|(L)\int_{E\cap N_k} f(y)dy\right| < \varepsilon$. Put

$$\delta^*(k, \varepsilon) = (1/2) \min(\kappa^*_{i_0(k,\varepsilon)}, \rho(k, \varepsilon/7), \lambda(k, \varepsilon), \lambda^*(k, \varepsilon/7)).$$

Put $E^* = \mathrm{proj}_x(R_0) \times E$ for a set $E \subset \mathrm{proj}_y(R_0)$.

Now, in order to prove (1) and (2) of the theorem it suffices to prove that:

Let $I_t(t = 1, 2, \dots, t_0)$ be an elementary system in $\mathrm{proj}_y(R_0)$ such that $I_t \cap L_k \neq \emptyset$ for $t = 1, 2, \dots, t_0$ and $\mu_1(\cup_{t=1}^{t_0} I_t - L_k) < \delta^*(k, \varepsilon)$. Then

$$\left|\sum_{t=1} G(I_t) - \sum_{t=1} (L)\int_{I_t\cap N_k} f(y)dy\right| < \varepsilon,$$

where $G(I) = F(I^*)$ for an interval I in $\mathrm{proj}_y(R_0)$.

Denote by $I_{1t}(t = 1, 2, \dots, t_1)$ the family of all intervals I_t for which $I_t \cap \mathrm{proj}_y(D_k) \neq \emptyset$, where $t = 1, 2, \dots, t_0$; and by $I_{2t}(t = 1, 2, \dots, t_2)$ the others. Then, $I_{2t} \cap \mathrm{proj}_y(D_k) = \emptyset$ for $t = 1, 2, \dots, t_2$.

(i) For $I_{1t}(t = 1, 2, \dots, t_1)$: By Proposition 5, there exists an elementary system $H_{1t}(t = 1, 2, \dots, t_1)$ such that $(H_{1t})^\circ \supset I_{1t}$ for $t = 1, 2, \dots, t_1$; $\mu_1(\cup_{t=1}^{t_1} H_{1t} - \cup_{t=1}^{t_1} I_{1t}) < \delta^*(k, \varepsilon)$; and

$$\left|\sum_{t=1} G(H_{1t}) - \sum_{t=1} G(I_{1t})\right| = \left|\sum_{t=1} F((H_{1t})^*) - \sum_{t=1} F((I_{1t})^*)\right| < \varepsilon/7.$$

For $H_{1t}(t = 1, 2, \dots, t_1)$, we have:

$$\begin{aligned}&\left|\sum_{t=1} G(H_{1t}) - \sum_{t=1} (L)\int_{H_{1t}\cap N_k} f(y)dy\right| \\ &\leq \left|\sum_{t=1} G(H_{1t}) - \sum_{t=1} (L)\int_{H_{1t}} f_{i_0(k,\varepsilon)}(y)dy\right| + \left|\sum_{t=1} (L)\int_{H_{1t}\cap N_k} (f(y) - f_{i_0(k,\varepsilon)}(y))dy\right| \\ &\quad + \left|\sum_{t=1} (L)\int_{H_{1t}-N_k} f_{i_0(k,\varepsilon)}(y)dy\right|.\end{aligned}$$

Since $i_0(k, \varepsilon) \geq k$ and $A_k \supset D_k$, we have $A_{i_0(k,\varepsilon)} \supset D_k$. Further, since $(H_{1t})^\circ \cap \mathrm{proj}_y(D_k) \supset I_{1t} \cap \mathrm{proj}_y(D_k) \neq \emptyset$, we have $(H_{1t})^\circ \cap \mathrm{proj}_y(A_{i_0(k,\varepsilon)}) \neq \emptyset$ for $t = 1, 2, \dots, t_1$; and

$$\mu_1(\cup_{t=1}^{t_1} H_{1t} - (\cup_{t=1}^{t_1}(H_{1t})^{\circ} \cap \operatorname{proj}_y(A_{i_0(k,\varepsilon)})))$$

$$\leq \mu_1(\cup_{t=1}^{t_1} H_{1t} - (\cup_{t=1}^{t_1} I_{1t} \cap \operatorname{proj}_y(D_k)))$$

$$\leq \mu_1(\cup_{t=1}^{t_1} H_{1t} - \cup_{t=1}^{t_1} I_{1t}) + \mu_1(\cup_{t=1}^{t_1} I_{1t} - (\cup_{t=1}^{t_1} I_{1t} \cap \operatorname{proj}_y(D_k)))$$

$$\leq \delta^*(k,\varepsilon) + \mu_1(\cup_{t=1}^{t_1} I_{1t} - L_k) < 2\delta^*(k,\varepsilon) \leq \kappa^*_{i_0(k,\varepsilon)}.$$

Hence, by Lemma 3

$$\left| \sum_{t=1}^{t_1} F((H_{1t})^*) - \sum_{t=1}^{t_1} (L) \iint_{(H_{1t})^* \cap D_{i_0(k,\varepsilon)}} f(x,y)d(x,y) \right| < \varepsilon_{i_0(k,\varepsilon)} < \varepsilon/7.$$

And so

$$\left| \sum_{t=1}^{t_1} G(H_{1t}) - \sum_{t=1}^{t_1} (L) \int_{H_{1t}} f_{i_0(k,\varepsilon)}(y)dy \right| < \varepsilon/7.$$

Further, since $|f(y) - f_{i_0(k,\varepsilon)}(y)| < \varepsilon/7$ for every $y \in N_k$

$$\left| \sum_{t=1}^{t_1} (L) \int_{H_{1t} \cap N_k} (f(y) - f_{i_0(k,\varepsilon)}(y))dy \right| < \varepsilon/7.$$

Finally, since $\mu_2(\cup_{t=1}^{t_1}(H_{1t})^* - (N_k)^*) = \mu_2(\cup_{t=1}^{t_1}(H_{1t})^* - (L_k)^*) \leq \mu_1(\cup_{t=1}^{t_1} H_{1t} - \cup_{t=1}^{t_1} I_{1t}) + \mu_1(\cup_{t=1}^{t_1} I_{1t} - L_k) < 2\delta^*(k,\varepsilon) \leq \lambda(k,\varepsilon)$,

$$\left| \sum_{t=1}^{t_1} (L) \int_{H_{1t} - N_k} f_{i_0(k,\varepsilon)}(y)dy \right| = \left| \sum_{t=1}^{t_1} (L) \iint_{(H_{1t} - N_k)^* \cap D_{i_0(k,\varepsilon)}} f(x,y)d(x,y) \right| < \varepsilon/7.$$

Therefore

$$\left| \sum_{t=1}^{t_1} G(H_{1t}) - \sum_{t=1}^{t_1} (L) \int_{H_{1t} \cap N_k} f(y)dy \right| < 3(\varepsilon/7).$$

Consequently, since $\mu_1(\cup_{t=1}^{t_1} H_{1t} - \cup_{t=1}^{t_1} I_{1t}) < \delta^*(k,\varepsilon) < \lambda^*(k,\varepsilon/7)$,

$$\left| \sum_{t=1}^{t_1} G(I_{1t}) - \sum_{t=1}^{t_1} (L) \int_{I_{1t} \cap N_k} f(y)dy \right| \leq \left| \sum_{t=1}^{t_1} G(I_{1t}) - \sum_{t=1}^{t_1} G(H_{1t}) \right|$$

$$+ \left| \sum_{t=1}^{t_1} G(H_{1t}) - \sum_{t=1}^{t_1} (L) \int_{H_{1t} \cap N_k} f(y)dy \right| + \left| \sum_{t=1}^{t_1} (L) \int_{(H_{1t} - I_{1t}) \cap N_k} f(y)dy \right|$$

$$< \varepsilon/7 + 3(\varepsilon/7) + \varepsilon/7 = 5\varepsilon/7.$$

(ii) For $I_{2t}(t = 1, 2, \ldots, t_2)$: Since $I_{2t} \cap L_k \neq \emptyset$ and $I_{2t} \cap \text{proj}_y(D_k) = \emptyset$, we have $I_{2t} \cap \text{proj}_y(B_k) \neq \emptyset$, and so $(I_{2t})^* \cap B_k \neq \emptyset$ for $t = 1, 2, \ldots, t_2$. Further, since $\mu_1((\text{proj}_y(B_k) \cap Z_0) \cap (M_0^* \cup M_k^*)) = 0$ and $I_{2t} \cap \text{proj}_y(D_k) = \emptyset$ for $t = 1, 2, \ldots, t_2$, we have $\mu_1(\text{proj}_y(\cup_{t=1}^{t_2}(I_{2t})^*)) = \mu_1(\cup_{t=1}^{t_2} I_{2t}) = \mu_1(\cup_{t=1}^{t_2} I_{2t} - (\text{proj}_y(B_k) \cap Z_0) \cap (M_0^* \cup M_k^*)) = \mu_1(\cup_{t=1}^{t_2} I_{2t} - L_k) < \delta^*(k, \varepsilon) < \rho(k, \varepsilon/7)$. Hence, by Lemma 2

$$\left| \sum_{t=1} G(I_{2t}) \right| = \left| \sum_{t=1} (F(I_{2t})^*) \right| < \varepsilon/7.$$

Further, since $\mu_1(\cup_{t=1}^{t_2} I_{2t}) < \delta^*(k, \varepsilon) < \lambda^*(k, \varepsilon/7)$

$$\left| \sum_{t=1} (L) \int_{I_{2t} \cap N_k} f(y)dy \right| < \varepsilon/7.$$

Therefore

$$\left| \sum_{t=1} G(I_{2t}) - \sum_{t=1} (L) \int_{I_{2t} \cap N_k} f(y)dy \right| < 2(\varepsilon/7) = 2\varepsilon/7.$$

Thus, by (i) and (ii)

$$\left| \sum_{t=1} G(I_t) - \sum_{t=1} (L) \int_{I_t \cap N_k} f(y)dy \right| < \varepsilon.$$

(3) of the theorem is an immediate consequence of (1) and (2) of the theorem.

References

[1] S. Nakanishi (formerly S. Enomoto): Sur une Totalisation dans les Espaces de Plusieurs Dimensions, I, II, Osaka Math. J., 7(1955), 69–102. 157–178.

[2] S. Nakanishi: A new definition of the Denjoy's special integral by the method of successive approximation, Math. Japonica, 41(1995), 217–230.

[3] S. Nakanishi: A structure of an increasing sequence of closed sets whose union is a multidimensional interval, Scientiae Mathematicae Japonicae, 55(2002), 1–15.

[4] S. Saks: Theory of the Integral, Warsaw (1937).

4-15 Jonanteramachi, Tennoji-ku, Osaka, 543-0017 Japan