WHITE NOISE ANALYSIS IN MATHEMATICAL BIOLOGY

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Abstract. We are interested in the effects of fluctuation which are observed in biological phenomena. There is a big variety in the appearance of the effects, however many of mathematical descriptions have much similarity. The most elemental and basic fluctuoation can be realized by white noise, either Gaussian or compound Poisson type. This fact leads us to discuss functions of white noise, call them white noise functionals of either Gaussian or Poisson type, which may well describe biological systems mathematically. We then consider the analysis of those functionals. The analysis in question will be the causal calculus, since the biological phenomena are to be evolutional in many cases, that is, they are developing as the time goes by.

To be more concrete, some of the following topics in mathematical biology will be discussed.

1) Applications of the Wiener expansion and of the Hellinger-Hahn theory,

2) Construction of innovations of biological, evolutional phenomena with fluctuation. It is often helped by a method of using the infinite dimensional rotation group,

3) Creation and annihilation operators applied to random evolutional phenomena, irreversibility and other properties,

4) Genralizations of the Lotka-Volterra equation with fluctuation.

5) Functionals of Poisson noise with application to biology.

6) Others.

1 Introduction Our aim is the investigation of *random evolutional complex systems* by using the white noise analysis, in particular we shall discuss those systems which are observed in biological phenomena.

For the study the following steps are in order:

$$\textbf{Reduction} \quad \longrightarrow \quad \textbf{synthesis} \quad \longrightarrow \quad \textbf{analysis}$$

The idea is that first the innovation of the random system is constructed, then functionals of the given innovation are given to express the random system in question, and finally those functionals are to be analysed.

The innovation can often be constructed by the variations of the given random evolutional phenomena, and sometimes, like in the communication theory, it is given in advance. Our attention will be focussed on the analysis of functionals of the innovations. Of course, the cases where the innovation is actually constructed are more attractive. There are other cases where the innovation is given in advance by some ways or others, and those are also important and have been well investigated.

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The main part of the analysis comes from the *white noise theory*, which is the central way of the analysis of functionals of the innovation. There naturally arises an infinite dimensional analysis.

The last step is the application of the analysis. Many applications to quantum dynamics have been known; now we are sure that application to bioscience would be the most fruitful area, although only part of it will be illustrated in what follows.

2 Background

2.1 Reduction To fix the idea let us observe the case where the random complex system is taken to be a stochastic process $X(t)$. Lévy's *stochastic infinitesima equation* for a stochastic process $X(t)$ is expressed in the form

$$\delta X(t) = \Phi(X(s), s \leq t, Y(t), t, dt),$$

where $\delta X(t)$ stands for the variation of $X(t)$ for the infinitesimal time interval $[t, t+dt)$, the Φ is a sure functional and the $Y(t)$ is the *innovation*. Intuitively speaking, the innovation is a system such that the $Y(t)$ contains the same information as that newly gained by the $X(t)$ during the infinitesimal time interval $[t, t+dt)$, hence it is independent of $X(s), s \leq t$. If such an equation is obtained, then the pair $(\Phi, Y(t))$ can completely characterize the probabilistic structure of the given process $X(t)$.

A mathematically rigorous definition can be given in terms of the sigma-fields of events determined by the $X(t)$ and $Y(t)$. There a careful treatment is required, since $Y(t)$ is not an ordinary process, but a generalized stochastic process and can be vector valued. For details we refer to the paper [1].

Under mild assumptions the innovation may be considered as the time dereivative of an additive process $Z(t)$. Further it may be assumed that the additive process has no fixed discontinuity and has stationary independent increments. Tacitly, it is assumed there is no non-random part. Then, the Lévy decomposition of such an additive process shows that

$$Z(t) = X_0(t) + X_1(t),$$

where $X_0(t)$ is Gaussian, in fact, a Brownian motion up to constant, and where $X_1(t)$ is a compound Poisson process involving mutually independent Poisson processes with various jumps.

Thus, the Brownian motion and Poisson processes with different jumps are all *elemental* additive processes.

As a generalization of the stochastic infinitesimal equation for $X(t)$, one can introduce a *stochastic variational equation* for random field $X(C)$ parameterized by an ovaloid C:

$$\delta X(C) = \Phi(X(C'), C' < C, Y(s), s \in C, C, \delta C),$$

where $C' < C$ means that C' is in the inside of C. The system $\{Y(s), s \in C\}$ is the innovation which is understood in the similar sense to the case of $X(t)$.

As for a rigorous definition of the innovation of $X(C)$, again see the paper [1].

The two equations above have only a formal significance, however we can give rigorous meaning to the equations with some additional assumptions and the interpretations to the notations introduced there. The results obtained at present are, of course, far from the general theory, however one is given a guideline of the approach to those random complex evolutional systems in line with the innovation theory and hence, with the white noise theory.

As in the case of $X(t)$ we can consider elemental random fields, or equivalently elemental noises with multi-dimensional parameter.

2.2 Synthesis The second step, synthesis, is to form a function (or a functional) of the innovation which has been obtained. Note that an innovation, say that of a stochastic process, is basically thought of as a time derivative of an additive process which, formally speaking, consists of infinitely many independent infinitesimal random variables. Since each random variable has a one-dimensional probability distribution, the innovation should be infinite dimensional. The function in question has therefore an infinite dimensional variable. This is the reason why an infinite dimensional calculus is required. Thus we come to the next step which is the analysis.

2.3 Analysis The variables of our function are random variables, although each of them is infinitesimal. Hence, it is not so easy to establish a new concept of partial derivative with respect to such a random variable. To overome this difficulty, we can appeal to the classical theory of functional analysis develped by Volterra, Hadamard, Lévy and others. The derivative in the infinitesimal random variable describes the variation of random functions when the fluctuation changes a little. Hence it is quite different from the ordinary derivatives like $\frac{d}{dt}$ or $\frac{d}{dx}$.

Actually an infinite dimensional harmonic analysis is carried out with help of the infinite dimensional rotation group. Only part of this analysis will be used in this note.

Needless to say, the final step is the application of our theory. In return, they give new questions. We therefore should study *applications of biology to mathematics.* Indeed, they are topics in theoretical applied mathematics.

3 Gaussian systems.

3.1 Gaussian processes First we discuss a Gaussian process $X(t), t \in T$, where T is an interval of R^1, say $[0, \infty)$. Assume that it is separable and has no remote past. Then, the innovation(s) can be considered explicitly in this case. The original idea came from P. Lévy (The third Berkeley Symposium paper; see [13]). Under the assumption that the process has unit multiplicity and other mild conditions like $EX(t) = 0$, a Gaussian process has innovation $\dot{B}(t)$ which is a white noise such that

$$X(t) = \int_0^t F(t,u)\dot{B}(u)du, \tag{3.1}$$

This is the so-called *canonical representation.* It might seem to be rather elementary, however such an easy understanding is, in a sense, not quite correct. The profound structure sitting behind this formula would lead us to a deep insight that is applicable to a general class of Gaussian processes and to non Gaussian case, too.

Take a Brownian motion $B(t)$ and a kernel function $G(t,u)$ of Volterra type. Define a Gaussian process $X(t)$ by

$$X(t) = \int_0^t G(t,u)\dot{B}(u)du.$$

Now we assume that $G(t,u)$ is a smooth function on the domain $0 \le u \le t < \infty$ and $G(t,t)$ never vanishes. Then we have

Theorem 3.1. The variation $\delta X(t)$ of the process $X(t)$ is defined and it is given by

$$\delta X(t) = G(t,t)\dot{B}(t)dt + dt \int_0^t G_t(t,u)\dot{B}(u)du,$$

where $G_t(t,u) = \frac{\partial}{\partial t}G(t,u)$. The $\dot{B}(t)$ is the innovation of $X(t)$ if and only if G(t,u) is the canonical kernel.

Proof. The formula for the variation of $X(t)$ is easily obtained. If G is not a canonical kernel, then the sigma field $\mathbf{B}_t(X)$ is strictly smaller than $\mathbf{B}(\dot{B})$, in particular the $\dot{B}(t)$ is not really a function of $X(s), s \leq t+0$.

Note that if, in particular, $G(t,u)$ is of the form $f(t)g(u)$, then $X(t)$ is a Markov process and there is always given a canonical representation. Hence $\dot{B}(t)$ is the innovation.

Remark. In the variational equation, the two terms in the right hand side are of different order as dt tend to zero, so that two terms seem to be discrimiated. But in reality, the problem like that is not so simple.

Having obtained the innovation, we can define the partial derivative denoted by ∂_t and expressed in the form

$$\partial_t = \frac{\partial}{\partial \dot{B}(t)},$$

which is hard to compare $\frac{d}{dt}$ or $\frac{\partial}{\partial x}$.

Note that ∂_t is definen by the knowledge of the original process $X(s), s \leq t$. Thus, the canonical kernel is obtained by

$$F(t,u) = \partial_u X(t), u < t.$$

The adjoint opereator ∂^* is the creation operator.

3.2 Nonlinear white noise functionals The analysis of noninear functionals of white noise $\dot{B}(t), t \in R$, has been established. The collection of those functionals with finite variance forms a Hilbert space (L^2), the direct sum decomposition of which into the spaces $\mathcal{H}_n, n \geq 0$, is obtained:

$$(L^2) = \bigoplus \mathcal{H}_n.$$

The decomposition stands for the Fock space. The $\mathcal{H}_n$ is called the space of homogeneous chaos of degree n.

The time propagation is particularly important. It is expressed as a one-parameter unitary group $U_t, t \in R$, acting on (L^2) determined by

$$U_t\dot{B}(s) = \dot{B}(s+t).$$

Appealing to the Hellinger-Hahn theorem, it is shown that U_t has

1) simple multiplicity on $\mathcal{H}_1$, and

2) countable multiplicity on $\mathcal{H}_n, n \geq 2$.

In addition we can associate a symmetric $L^2(R^n)$-function with a functional φ in $\mathcal{H}_n$:

$$\varphi \ \longleftrightarrow \ F \in \widehat{L^2(R^n)},$$

where $\widehat{}$ means symmetric. Such an isomorphism can be obtained by the S-transform:

$$(S\varphi)(\xi) = \exp[-\frac{1}{2}\|\xi\|^2] \int_{(S)^*} \exp[< x, \xi >]\varphi(x) d\mu(x), \ \ \varphi \in (L^2),$$

which is an infinite dimensional analogue of the Laplace transform.

This property has been applied to biological problems, in particular

K. Naka's method of identifying the function of the retina.

$$\begin{array}{lcccl} White\ noise\ input & \longrightarrow & retina & \longrightarrow \otimes \longrightarrow & output \\ & \searrow & & \uparrow & \\ & & known\ circuit \longrightarrow & U_t & \end{array}$$

In the above diagram U_t denotes the time shift and $\otimes$ is the multiplier.

We can compute the correlationfunction and hence the spectgrum. The same processs is repeated for many different known circuits to identify the retina. Actual data and his interesting results can be seen in [9] and [23].

Such a method of analysis is called the Wiener expansion. In reality, the Hellinger-Hahn theory is applied to identify the specral density function associated to each cyclic subspace or to each circuit, since the spectrum can well determine the structure of a cyclic subspace.

Now one may think of its generalizations. If one wishes to carry on the so-called *causal analysis*, where time propagation is expressed explicitly in terms of $\dot{B}(t)$'s (without smearing), then he is led to get the kernel function by applying differential opereators $\frac{\partial}{\partial \dot{B}(t)}$ as well as their powers to the given random phenomena.

It is noted that, to realize this idea, enough tools fromn analysis are provided for this purpose. Actually, the space $(S)^*$ of generalized white noise functionals in such a way that

$$(S) \ \subset \ (L^2) \ \subset \ (S)^*,$$

where (S) is like an infinite dimensional analogue of the Schwartz space of test functions and $(S)^*$ is the dual space. The canonical bilinear form that connects (S) and $(S)^*$ is denoted by $\langle , \rangle$.

The isomorphism between $\mathcal{H}_n$ and $L^2(R^n)$ can be generalized by the generalized S-transform, which is well defined since $\exp[< x, \xi >]$ belongs to (S).

Functions of the differential operators including exponential functions, and their adjoint operators play important roles in our analysis.

Having established the generalization of white noise analysis, we have the following theorem which will be useful in applictions. Suppose a given random complex system is expressed as a generalized white noise functional ψ. The question is to identify the system in terms of the kernel functions (generalized functions). A generalization of what was explained in the last subsection to obtain the canonical kernel is given by the following theorem.

Theorem 3.2. The kernel function (generalized function) F_n of degree n associated with $\psi \in (S)^*$ can be obtained by the formula

$$\langle \partial_{t_1} \partial_{t_2} \cdots \partial_{t_n} \psi, 1 \rangle = F(t_1, t_2, \ldots, t_n).$$

Next application is obtained by a randomization of the Lotka-Volterra equation. The attempt towards this direction is wide and numerous, however we take a particular case, still giving us a suggestion. (see e.g. [18])

Consider a simple example and start with a Lagrangian form which is randomized. With the notation established by Volterra, set N_r be the size of the population r, $\int N_r dt = X_r$, and

$$\chi = \sum_r \beta_r X_r' \log X_r', \quad 1 \le r \le n,$$

the demographic potential

$$P = \sum_r \beta_r \epsilon_r X_r + \frac{1}{2} \sum_{r,s} c_{r,s} X_r X_s,$$

and a bilinear form

$$Z = \sum_{r,s} a_{r,s} X_r' X_s.$$

Then, the Lagrangian form L, which is a functional of X_r's, is given by

$$L = \chi + \frac{1}{2} Z + P.$$

By the usual manner of the variational calculus to find the stationary point of the demographic action U:

$$U = \int_o^t L dt,$$

the Euler eaquation is given:

$$\frac{d}{dt} \frac{\partial L}{\partial X_r'} - \frac{\partial}{\partial X_r} = 0.$$

Thus, we have

$$\frac{dN_r}{dt} = (\epsilon_r + \frac{1}{\beta_r} \sum_s a_{s,r} N_s) N_r.$$

Now we come to a random environment, namely the coefficients of the linear term of the demographic potential is taken to be a random variable. A good example is introduced by replacing ϵ_r with $\epsilon_r + \dot{B}_r(t)$. Namely,

$$\frac{dN_r}{dt} = f_r(N) + \partial_t^* N_r$$

where $(\epsilon_r + \frac{1}{\beta_r}\sum_s a_{s,r}N_s)N_r$ is simply written by $f_r(N)$.

The solution to this equation can be obtained as is done in [3], however we only note the significance of this direction, specifically noting the idea to randomize the Lagrangian form.

Another direction of applying the $\{\partial_t, \partial_t^*\}$ calculus to biological science can be seen in the study of fluctuation in living cells, for example in the Oosawa equation ([12]): The probability $p(t)$ that molecules at some state satifies

$$\frac{dp(t)}{dt} = k_+ \exp[-\beta E(t)]p(t) + k_- \exp[\beta E(t)](1 - p(t)),$$

where $k_+, k_- > 0, \beta > 0$ are constants.

Before coming to our understanding the equations of this kind, some operators are prepared. Set

$$A(t)^* = \int^t F(t-u)\partial_u^* du,$$

where the kernel F is now taken to be a function only of $t - u$, since stationarity and causality in time for the action is assumed in most interesting cases. Note that ∂_u^* is a creation operator that stands for the action that coming from the fluctuation involved in the phenomena.

Note that there are many choices of the kernel function F. One extreme is the canonical one F^-, and another extreme is the backward canonical F^+. The F^- is concerned with the past, while the F^+ is related to the future. Asymmetry, in a sense, can be seen from the relationship between F^- and F^+.

As in the representation of Gaussian processes, the kernel F may or may not be canonical. This is seen when we form a Gaussian process

$$X(t) = A(t)^* 1.$$

When F is taken to be canonical, then the operator is fitting for the prediction. In contrast with this case, there is a choice of the kernel F so as to be fitting for the backward operations, that is future values will determine the innovation.

If the exponentials of $A(t)^*$ is introduced, it is interesting to see coherent families of random variables parameterized by t :

$$\{\exp[A(s)^*] \cdot 1, s \ \leq t\}.$$

the family is fitting for the study of nonlinear predictions or symmetry (asymmetry) of the evolution.

3.3 Gaussian random fields. There are various Leitmotive to discuss random fields, among others [12] and [17] in our case.

To fix the idea we consider a Gaussian random field $X(C)$ parameterized by a smooth convex contour in R^2 that runs through a certain class $\mathbf{C}$ which is topologized by the usual method using the Euclidean metric. Denote by $W(u), u \in R^2$, a two dimensional parameter white noise. Let (C) denote the domain enclosed by the contour C.

Assume that a Gaussian random field $X(C)$ is exressed as a stochastic integral of the form:

$$X(C) = \int_{(C)} F(C,u)W(u)du,$$

where $F(C,u)$ be a kernel function which is locally square integrable in u. For convenience we assume that $F(C,u)$ is smooth in (C,u). The integral is a *causal* representation of the $X(C)$. The canonical property can be defined as a generalization to a random field as in the case of a Gaussian process.

The stochastic variational equation for this $X(C)$ is of the form

$$\delta X(C) = \int_C F(C,s)\delta n(s) W(s) ds + \int_{(C)} \delta F(C,u) W(u) du.$$

In a similar manner to the case of a process $X(t)$, but somewhat complicated manner, we can form the innovation $\{W(s), s \in C\}$, parameterized by a point s running through the one-dimensional parameter set C.

Example. A variational equation of Langevin type.

Given a stochastic variational equation

$$\delta X(C) = -X(C) \int_C k\delta n(s) ds + X_0 \int_C v(s)\partial_s^* \delta n(s) ds, C \in \mathbf{C},$$

where $\mathbf{C}$ is taken to be a class of concentric circles, v is a given continuous function and it is reminded that ∂_s^* is the adjoint operator of the differential operator ∂_s.

Applying the S-ransform to the equation, we can solve the transformed equation by appealing to the classical theory of functional analysis. Then, applying the inverse transform S^{-1}, the solution is given:

$$X(C) = X_0 \int_{(C)} \exp[-k\rho(C,u)] \partial_u^* v(u) du,$$

where ρ is the Euclidean distance.

Once the innovation is obtained, the above example suggests that one can think of possibility of application of the theory to the biological systems, where $X(C)$ is a mathematical model of random phenomena that varies as C changes in a space-time region being interfered with by fluctuation that occurs at every point in (C). For example, the kernel functions are obtained by the same idea used in Theorem 2.

As for the question on how to obtain the innovation from more general random fields may be discussed by refering to the papers [1] and [10].

4 Functionals of Poisson noise

4.1 Poisson noise Having been suggested by the Lévy decomposition of an additive process, a Poisson process $P(t)$ comes after Brownian motion. Poisson process is another kind of elemental additive process. Taking its time derivative $\dot{P}(t)$ we have a *Poisson white noise*. It is a generalized stationary stochastic process with independent value at every point. For convenience we may assume that t runs through the whole real line. In fact, it is easy to define such a noise. The characteristic functional of the Poisson white noise is of the form

$$C_P(\xi) = \exp[\lambda \int_{-\infty}^{\infty} (e^{i\xi(t)} - 1) dt],$$

where $\xi \in E$.

There is the associated measure space (E^*, μ_P), and the Hilbert space $L^2(E^*, \mu_P) = (L^2)_P$ is defined.

We now pause to write some Leitmotive of the analysis of Poisson functionals. The first one goes back to 1940's. N. Wiener and his collaborators discussed functionals of Poisson process motivated by the research of Biological objects. See [19], [20], etc. The present author has discussed with physicians, and he had opportunity to attend the colloquiem at Nagoya in 1994, where L Ricciardi gave a stimulating talk on the functionals of noise (in our terminology) under the title "On some computational problems for diffusion processes in biological modeling"[15].

There have been, of course, many attempts in this area and we have found many interesting applications of white noise analysis. Indeed, various results of the analysis on $(L^2)_P$ have been obtained, however most of them have been studied by analogy with the Gaussian case or its modifications, so far as the construction of the space of generalized functionals and their analysis are concerned. Here, we only note that the $(L^2)_P$ admits the direct sum decomposition of the form

$$(L^2)_P = \bigoplus_n H_{P,n}.$$

The subspace $H_{P,n}$ is formed by the Poisson Charlier polynomials of degree n which are defined as follows.

Let $p(k,\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}, k = 0, 1, 2, ...$ be the Poisson distribution with intensity λ. Then

$$p_n(k,\lambda) = \frac{\lambda^{n/2}}{n}(-1)^n \frac{\Delta^n p(k,\lambda)}{p(k,\lambda)},$$

where $\Delta f(k) = f(k) - f(k-1)$. Then, we have

$$\sum_k p_n(k,\lambda) p_m(k,\lambda) p(k,\lambda) = \delta_{n,m}$$

and the addition formula

$$\sqrt{\frac{(a+b)^n}{n!}} p_n(j+k+1, a+b) = \sum_{m=0}^{n} \sqrt{\frac{(-a)^m(-b)^{n-m}}{m!(n-m)!}} p_m(j,a) p_{n-m} k, b).$$

There might occur a misunderstanding regarding the functionals of Poisson noise, even in the case of linear functional. The following example would illustrate this fact (see e.g. [1]).

Let a stochastic process $X(t)$ be given by an integral

$$X(t) = \int_0^t F(t,u)\dot{P}(u)du.$$

It seems to be simply a linear functional of $P(t)$, however there are two ways of understanding the meaning of the integral; one is defined

i) taking $\dot{P}(t)dt$ to be a random measure, the integral is defined in the Hilbert space, in particular the topology is defined by Hilbert space norm.

Another way is to define the integral :

ii) for each sample function of $P(t)$ (the path-wise integral). This can be done if the kernel is a smooth function of u over the interval $[0, t]$.

Assume that $F(t, t)$ never vanishes and that it is not a canonical kernel, that is, it is not a kernel function of an invertible integral operator. Then, we can claim that for the integral in the first sense $X(t)$ has less information compared to $P(t)$. Because there is a linear function of $P(s), s \leq t$ which is orthogonal to $X(s), s \leq t$. On the other hand, if $X(t)$ is defined in the second sense, we can prove

Proposition. Under the assumptions stated above, if the $X(t)$ above is defined sample function-wise, we have the following equality for sigma-fields:

$$\mathbf{B}_t(X) = \mathbf{B}_t(P), t \geq 0.$$

Proof. By assumption it is eqasy to see that $X(t)$ and $P(t)$ share the jump points, which means the information is fully transfered from $P(t)$ to $X(t)$.This proves the equality

The above argument tells us that we are led to introduce a space $(\mathbf{P})$ of random variables that come from separable stochastic processes for which existence of variance is not expected. This sounds to be a vague statement, however we can rigorously defined by using a Lebesgue space without atoms, and others. There the topology is defined by either the almost sure convergent or the convergence in probability, and there is no need to think of mean square topology. On the space $(\mathbf{P})$ filtering and prediction for strictly stationary process can naturally be discussed. For further idea we may refer to the literatures [6] and [11], where one can see profound idea of N. Wiener in his paper [20], "Generalized harmonic analysis", although in a classical description.

4.2 Multi-dimensional parameter Poisson noise It is quite natural for us to come to an introduction of a multi-parameter Poisson white noise, denoted by $\{V(u)\}$, which is a generalization of $\{\dot{P}(t)\}$.

Start with the charcateristic functional $C_P(\xi)$ which is to be the expectation of $\exp[i < V, \xi >]$:

$$C_P(\xi) = \exp[\lambda \int_{R^d} (e^{i\xi(t)} - 1) dt^d],$$

where $\xi \in E$ with a nuclear space $E \subset L^2(R^d)$. A probability measure μ_P defined on the space E^*.

It can be shown that a stochastic bilinear form $< x, f >$ with $x \in E^*, f \in L^2(R^d)$ can be defined a.e. μ_P. In particular, if f is taken to be the indicator function I_D of a domain D, then the charcateristic functional shows that $< x, I_D >$ is a random variable on the probability space (E^*, μ_P) and is subject to the Poisson distribution with intensity $\lambda|D|$, where $|D|$ denotes the volume of D.

To fix the idea, consider the case $d = 2$. The paper [21] by Wiener and Roswnblueth gave a Leitmotiv for the following observation.

Set $X = X(x) =< x, I_D >$.

Theorem 4.1. 1) The $X(x)$ expresses the number of the singular points each of which a delta function is associated as a sample function of V.

2) Under the condition that $X(x)$ involves delta functions as many as n, then those points are equally distributed over D.

Remark. More precise meaning of 2) can be given.

Proof. The characteristic functional $\varphi(z)$ of $X(x)$ is given by

$$\varphi(z) = \exp[\lambda|D|(e^{iz} - 1)].$$

The expression of the characteristic functional shows that disjoint domains are associated to independent Poisson processes, and the intensity is additive with respect to the domain. Thus the probability distribution is spacially homogeneous.

These observations prove the assertions.

We then come to a study of linear functionals of a Poisson noise. In view of the biological applications alluded to in the Introduction, it is of fundamental importance to have a functional of Poisson noise parameterized by a point on the surface describing the observed results. The Wiener-Rosenblueth's paper (see [21]) is of historical interest. Namely, they discussed the fibrillation of heart and proposed a mathematical study of anastomosing fibers, in connection with a cardiac muscle. There Poisson noise appeared and its functionals were discussed. Quite recently, a doctor at medical school asked to study functionals of Poisson noise which is a model of random impulse.

Thus, it becomes popular to know the significance of the analysis of Poisson functionals.

Theorem 4. 2. Let a random field $X(C)$ parameterized by a contour C be given by a stochastic integral

$$X(C) = \int_{(C)} G(C, u)V(u)du,$$

where the kernel $G(C, u)$ is continuous in (C, u). Assume that $G(C, s)$ never vanishes on C for every C. Then, the $V(u)$ is the innovation.

Proof. The variation $\delta X(C)$ exists and it involves the term

$$\int_C G(C, s)\delta n(s)V(s)ds,$$

where $\{\delta n(s)\}$ determines the variation δC of C. Here is used the same technique as in the case of [10], so that the values $V(s), s \in C$, are determined by taking various δC's. This shows that the $V(s)$ is obtained by the $X(C)$ according to the infinitesimal change of C. Hence $V(s)$ is the innovation.

Here are two remarks worth to be noted.

Remark. 1) Suppose one is permitted to take single variation, then it is impossible to form $V(s)$, but one may use conformal transformations acting on C to have the values of the innovation $V(s)$.

2) One can see a significant difference between Poisson fields and Gaussian fields when we try to get the innovation. However, if one is permitted to use some operations acting on the given process, it is possible to form the innovation from a non-canonical representation

of a Gaussian process ([1], [17]), although the proof needs a profound property of a Brownian motion.

4.3 Compound Poisson noise. As soon as we come to a compound Poisson process, which is a more general innovation, the second order moment may not exist, so that we have to come to the space (**P**). The Lévy decomposition of an additive process, with which we are now concerned, is expressed in he form

$$Z(t) = \int (uP_{du}(t) - \frac{tu}{1+u^2} dn(u)) + \sigma B(t),$$

where $P_{du}(t)$ is a random measure of the set of Poisson processes, and where $dn(u)$ is the Lévy measure such that

$$\int \frac{u^2}{1+u^2} dn(u) < \infty.$$

The decomposition of a Compound Poisson process into the individual elemental Poisson

processes with different jumps can be carried out in the space (**P**) with the use of the quasi-convergence (see [12, Chapt.V]). We are keen to discuss the analysis acting on sample functions of a compound Poisson process, since we can see many applications in biology, although mathematical theory of these problems is not so much developed.

A generalization of the assertions in the last two subsections to the case of compound Poisson white noise is not difficult in a formal way without paying much attention. However, we wish to pause at this moment to consider carefully about how to find a jump point of $Z(t)$ with the height u designated in adavance. This question is heavily depending on the computability or measurement problem in quantum dynamics.

5 Concluding remarks A Brownian motion and each Poisson process which is one of the the component of the compound Poisson process seem to be elemental. Indeed, this is true in a sense. On the other hand, there is another aspect. Indeed, we know that the inverse function of the Maximum of a Brownian motion is a stable process, which is a compound Poisson process (see. [12]). There the L^2-technique is no more available.

There is another surprising result. A Poisson noise can eventually be derived from a Browniann motion (Cochran-Kuo-Sengupta), certainly not by the L^2 method. This may be illustrated in the following manner. In terms of the probability distribution, it is shown that a certain generalized (Gaussian) white noise functional has the same distribution as that of a Poisson white noise. There arises a question on how to find concrete operations (variational calculus may be involved there) acting on the sample functions of $\dot{B}(t)$'s to have a Poisson white noise. We need some more examples to propose a problem to give a good interpretation to such phenomena.

6 Addenda

1. Between two kinds of noises Gaussian and Poisson type, there are many similarities, but dissimilarities are important and interesting. Thus, it is worth mentioning the significance of the properties enjoyed by Poisson noise.

2. Measurement problem occurs in the case of compound Poisson process.

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