

## TOWARDS DEAD TIME INCLUSION IN NEURONAL MODELING

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ABSTRACT. A mathematical description of the refractoriness period in neuronal diffusion modeling is given and its moments are explicitly obtained in a form that is suitable for quantitative evaluations. Then, for the Wiener, Ornstein-Uhlenbeck and Feller neuronal models, an analysis of the features exhibited by the mean and variance of the first passage time and of refractoriness period is performed.

**1 Introduction**

Mathematical descriptions of dead time, or refractoriness, in neuronal modeling have long traditions dating back at least to mid sixties when special attention was devoted to the description of the evolution of networks of switching elements whose behavior was meant to simulate that of physiological neurons via certain rather drastic simplifications [4]. Furthermore, the approach to neural modeling was shown to bear certain strong analogies with the stochastic description of the input-output features of radioactive particle counters. In such context, as early as 1948, W. Feller proved that under a suitable formulation all problems concerning single counters reduce to special instances of the theory of the summation of random variables. Exploiting the above mentioned analogies, the simplest neural model may be conceived as a black box possessing the following distinctive features: (i) it is a threshold element, (ii) its output response consists of pulses of constant amplitude and width and (iii) there exists a constant dead time. More accurately, one could define this dead time also as a deterministic function of certain measurable parameters, such as time or input pulse amplitude, or view it as a stochastic process.

As a first attempt towards a quantitative treatment of the dead time effects in neural modeling, we look at the input of the neuron as a randomly distributed Poisson-type pulse train. Its output is then determined by imposing the restriction that following each input pulse a dead time period is activated during which no further pulses can be produced at the output. Even for such an oversimplified instance the investigation of the role played by the dead time in determining the distribution of the output when the input is described by a given distribution is a very challenging task.

Let  $\tau$  denote this dead time, i.e. the time interval following every firing during which the neuron cannot fire again. Let us assume that the net input to the neuron in time interval  $(0, T)$  is modeled by a sequence of positive pulses of standard strength whose time of occurrences are Poisson distributed with rate  $\lambda > 0$ . We purpose to determine the distribution  $\Pi_n(T, \tau)$  of the output pulses as a function of dead time  $\tau$ . A rather cumbersome amount of computations leads one to conclude that the assumed input distribution

$$(1) \quad P_n(T) = \frac{(\lambda T)^n}{n!} e^{-\lambda T}, \quad T > 0, n = 0, 1, 2, \dots$$

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generates the following firing distribution valid for all  $n \geq 1$  (cf. [4]) :

$$(2) \quad \begin{aligned} \Pi_n(T, \tau) = \vartheta[T - (n-1)\tau] & \left\{ 1 - e^{-\lambda[T - (n-1)\tau]} \sum_{k=0}^{n-1} \frac{\lambda^k [T - (n-1)\tau]^k}{k!} \right\} \\ & - \vartheta(T - n\tau) \left[ 1 - e^{-\lambda(T - n\tau)} \sum_{k=0}^n \frac{\lambda^k (T - n\tau)^k}{k!} \right], \end{aligned}$$

where  $\vartheta(x)$  denotes the Heaviside unit step function:

$$(3) \quad \vartheta(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Although the stated problem has been the object of several investigations (see, for instance, [8]), a quantitative evaluation of the effect of dead time on the statistical parameters of the output appears to be still lacking.

In the remaining part of this paper, we shall outline a totally different approach towards the inclusion of refractoriness in the neuronal model. As in [1] and [2], we model the time course of the membrane potential by a time-homogeneous diffusion process and then assume that the firing threshold acts as some kind of elastic boundary characterized by preassigned reflection and absorption parameters. In other words, we assume that an action potential is released whenever the process first attains the firing threshold. After the firing, a period of refractoriness of random duration occurs, at the end of which the process is instantaneously reset at a fixed state. Then, the subsequent evolution of the action potential proceeds as before, until the threshold is again reached. A new firing then occurs, followed by a new period of refractoriness, and so on. Use of the above approach allows one to mimic the effects of refractoriness for the specified neuronal model.

In order to be able to apply the specified paradigm to the description of neuronal models in the presence of refractoriness, an investigation of certain general features of diffusion processes in the presence of an elastic boundary is necessary. This task will be accomplished in Section 2, where we shall analyze the features of the moments of the random variable modeling the neuron's intrinsic refractoriness. In Section 3, a specific analysis will be provided of three neuronal models based on the Wiener, Ornstein-Uhlenbeck and Feller diffusion processes, and a comparative discussion of the refractoriness features exhibited by these models will be performed.

## 2 Effect of Refractoriness

Let  $\{X(t), t \geq 0\}$  be a regular, time-homogeneous diffusion process, defined over the interval  $I = (r_1, r_2)$ , characterized by drift and infinitesimal variance  $A_1(x)$  and  $A_2(x)$ , respectively. Throughout, we shall assume that Feller conditions on these functions are fulfilled [3]. Let  $h(x)$  and  $k(x)$  denote scale function and speed density of  $X(t)$ :

$$h(x) = \exp\left\{-2 \int^x \frac{A_1(z)}{A_2(z)} dz\right\}, \quad k(x) = \frac{2}{A_2(x) h(x)}$$

and

$$H(r_1, y) = \int_{r_1}^y h(z) dz, \quad K(r_1, y) = \int_{r_1}^y k(z) dz$$

scale and speed measures, respectively.

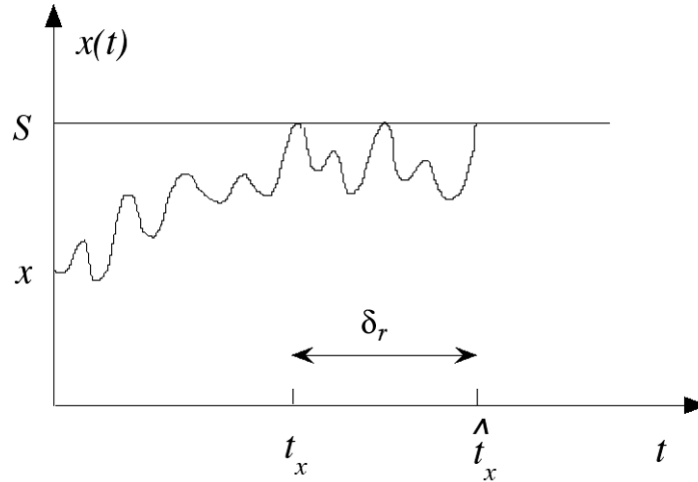


Figure 1: Illustrating first passage time  $t_x$  through  $S$  (i.e. firing time), neuron's refractoriness period  $\delta_r$  and first exit time  $\hat{t}_x$  for a single sample path  $x(t)$  of  $X(t)$ . By  $t_x$ ,  $\hat{t}_x$  and  $\delta_r$  we have indicated the appropriate values of  $T_x$ ,  $\hat{T}_x$  and  $T_r$ , respectively.

We define the random variable “first passage time” (FPT) of  $X(t)$  through  $S$  ( $S \in I$ ) with  $X(0) = x < S$ :

$$(4) \quad T_x = \inf_{t \geq 0} \{t : X(t) \geq S\}, \quad X(0) = x < S.$$

Then,

$$(5) \quad g(S, t|x) = \frac{\partial}{\partial t} P(T < t), \quad x < S$$

is the FPT pdf of  $X(t)$  through  $S$  conditional upon  $X(0) = x$ .

In the neuronal modeling context the state  $S$  represents the neuron's firing threshold and  $g(S, t|x)$  the firing pdf.

More realistically then in past approaches, here we shall assume that after each firing a period of refractoriness of random duration occurs, during which either the neuron is completely unable to respond, or it only partially responds to the received stimulations. To this end, we look at the threshold  $S$  as an elastic barrier being ‘partially transparent’, in the sense that its behavior is intermediate between total absorption and total reflection. The degree of elasticity of the boundary depends on the choice of two parameters,  $\alpha$  (absorbing coefficient) and  $\beta$  (reflecting coefficient), with  $\alpha > 0$  and  $\beta \geq 0$ . Hence,  $p_R := \beta/(\alpha + \beta)$  denotes the reflecting probability at the boundary  $S$ , and  $1 - p_R = \alpha/(\alpha + \beta)$  the absorption probability at  $S$ . We denote by  $\hat{T}_x$  the random variable describing the “first exit time” (FET) of  $X(t)$  through  $S$  if  $X(0) = x < S$ , and by  $g_e(S, t|x)$  its pdf. The random variable  $T_r$  will denote the “refractoriness period” and  $g_r(S, t|S)$  its pdf. Since  $\hat{T}_x$  can be viewed as the sum of random variable  $T_x$  describing the first passage time through  $S$  (firing time) and of  $T_r$  (see Figure 1) one has:

$$(6) \quad g_e(S, t|x) = \int_0^t g(S, \tau|x) g_r(S, t|S, \tau) d\tau.$$

In the sequel we assume that one of the following cases holds:

- (i)  $r_1$  is a natural nonattracting boundary and  $K(r_1, y] < +\infty$ ;
- (ii)  $r_1$  is a reflecting boundary or it is an entrance boundary.

Under such assumptions, if  $x < S$  the first passage probability  $P(S|x)$  from  $x$  to  $S$  is unity and the FPT moments  $t_n(S|x_0) \equiv E(T_x^n)$  are finite and can be iteratively calculated as

$$(7) \quad t_n(S|x) := \int_0^\infty t^n g(S, t|x) dt = n \int_x^S h(z) dz \int_{r_1}^z k(u) t_{n-1}(S|u) du \quad (n = 1, 2, \dots),$$

where  $t_0(S|x) = P(S|x) = 1$  (cf., for instance, [7]).

**Theorem 2.1** *Under the assumption (i) or (ii), if  $\alpha > 0$  the first exit time probability*

$$\widehat{P}(S|x) := \int_0^{+\infty} g_e(S, t|x) dt \quad (x < S)$$

*is unity.*

**Proof.** We consider separately the cases (i) and (ii).

*Case (i)* Let  $\widehat{P}(S_1, S|x)$  ( $r_1 < S_1 < x < S$ ) be the first exit time probability through the elastic boundary  $S$  in the presence of an absorbing boundary  $S_1$ . This is solution of the differential equation

$$(8) \quad A_1(x) \frac{d\psi_0(x)}{dx} + \frac{A_2(x)}{2} \frac{d^2\psi_0(x)}{dx^2} = 0$$

subject to conditions

$$(9) \quad \lim_{x \downarrow S_1} \psi_0(x) = 0, \quad \alpha \lim_{x \uparrow S} [1 - \psi_0(x)] - \beta \lim_{x \uparrow S} \left\{ h^{-1}(x) \frac{d\psi_0(x)}{dx} \right\} = 0.$$

Since

$$(10) \quad A_1(x) \frac{d\psi_0(x)}{dx} + \frac{A_2(x)}{2} \frac{d^2\psi_0(x)}{dx^2} \equiv \frac{1}{k(x)} \frac{d}{dx} \left[ \frac{1}{h(x)} \frac{d\psi_0(x)}{dx} \right]$$

from (8) one has

$$(11) \quad \psi_0(x) = A + B \int_{r_1}^x h(z) dz,$$

where  $A$  and  $B$  are arbitrary real constants. By imposing boundary conditions (9), one obtains

$$(12) \quad \widehat{P}(S_1, S|x) = \frac{\alpha \int_{S_1}^x h(z) dz}{\alpha \int_{S_1}^S h(z) dz + \beta}.$$

Since  $r_1$  is a natural nonattracting boundary one has  $H(r_1, x] = +\infty$ ; hence, making use of (12), one has

$$\widehat{P}(S|x) := \lim_{S_1 \downarrow r_1} \widehat{P}(S_1, S|x) = 1,$$

where the last equality follows by exploiting l'Hospital's rule.

Case (ii) Let  $\widehat{P}(S|x)$  ( $r_1 < x < S$ ) the first exit time probability through the elastic boundary  $S$  in the presence of a reflecting boundary or of an entrance boundary  $r_1$ . This is solution of the differential equation (8) subject to conditions:

$$(13) \quad \lim_{x \downarrow r_1} \left\{ h^{-1}(x) \frac{d\psi_0(x)}{dx} \right\} = 0, \quad \alpha \lim_{x \uparrow S} [1 - \psi_0(x)] - \beta \lim_{x \uparrow S} \left\{ h^{-1}(x) \frac{d\psi_0(x)}{dx} \right\} = 0.$$

Since (10) holds, from (8) one obtains again the general solution (11). By imposing boundary conditions (13), one finally has  $\widehat{P}(S|x) = 1$ .

**Theorem 2.2** Under the assumption (i) or (ii), if  $\alpha > 0$  the first exit time moments  $\widehat{t}_n(S|x) \equiv E(\widehat{T}_x^n)$  can be iteratively calculated as

$$(14) \quad \widehat{t}_n(S|x) := \int_0^\infty t^n g_e(S, t|x) dt = n \left\{ \int_x^S h(z) dz \int_{r_1}^z k(u) \widehat{t}_{n-1}(S|u) du + \frac{\beta}{\alpha} \int_{r_1}^S k(u) \widehat{t}_{n-1}(S|u) du \right\} \quad (n = 1, 2, \dots; x < S),$$

where  $\widehat{t}_0(S|x) = \widehat{P}(S|x) = 1$ .

**Proof.** A derivation of (14) follows from the properties of elastic boundaries. We consider again separately the cases (i) and (ii).

Case (i) Let  $\widehat{t}_n(S_1, S|x)$  ( $r_1 < S_1 < x < S$ ) be the first exit time moments through the elastic boundary  $S$  in the presence of an absorbing boundary  $S_1$ . This is solution of the differential equation

$$(15) \quad A_1(x) \frac{d\psi_n(x)}{dx} + \frac{A_2(x)}{2} \frac{d^2\psi_n(x)}{dx^2} = -n \psi_{n-1}(x)$$

subject to conditions

$$(16) \quad \lim_{x \downarrow S_1} \psi_n(x) = 0, \quad \alpha \lim_{x \uparrow S} \psi_n(x) + \beta \lim_{x \uparrow S} \left\{ h^{-1}(x) \frac{d\psi_n(x)}{dx} \right\} = 0.$$

The general solution of (15) is

$$(17) \quad \psi_n(x) = A + B \int^x h(z) dz - n \int^x h(z) dz \int^z k(u) \psi_{n-1}(u) du,$$

where  $A$  and  $B$  are arbitrary real constants. By imposing boundary conditions (16), one has

$$(18) \quad \widehat{t}_n(S_1, S|x) = \frac{n}{\alpha \int_{S_1}^S h(z) dz + \beta} \left\{ \beta \int_{S_1}^x h(z) dz \int_z^S k(y) \widehat{t}_{n-1}(S_1, S|y) dy + \alpha \left[ \int_{S_1}^x h(u) du \cdot \int_x^S h(z) dz \int_x^z k(y) \widehat{t}_{n-1}(S_1, S|y) dy + \int_x^S h(u) du \cdot \int_{S_1}^x h(z) dz \int_z^x k(y) \widehat{t}_{n-1}(S_1, S|y) dy \right] \right\}.$$

Since  $r_1$  is a natural nonattracting boundary and  $K(r_1, y] < +\infty$ , making use of (18) and by applying l'Hospital's rule, one has

$$\begin{aligned} \widehat{t}_n(S|x) := \lim_{S_1 \downarrow r_1} \widehat{t}_n(S_1, S|x) &= n \left\{ \frac{\beta}{\alpha} \int_{r_1}^S k(y) \widehat{t}_{n-1}(S|y) dy \right. \\ &\left. + \int_x^S h(z) dz \int_x^z k(y) \widehat{t}_{n-1}(S|y) dy + \int_x^S h(u) du \cdot \int_{r_1}^x k(y) \widehat{t}_{n-1}(S|y) dy \right\}, \end{aligned}$$

that identifies with the the right-hand side of (14).

*Case (ii)* Let  $\widehat{t}_n(S|x)$  ( $r_1 < x < S$ ) be the first exit time moments through the elastic boundary  $S$  in the presence of a reflecting boundary or of an entrance boundary  $r_1$ . This is solution of the differential equation (15) subject to conditions

$$(19) \quad \lim_{x \downarrow r_1} h^{-1}(x) \frac{d\psi_n(x)}{dx} = 0, \quad \alpha \lim_{x \uparrow S} \psi_n(x) + \beta \lim_{x \uparrow S} \left\{ h^{-1}(x) \frac{d\psi_n(x)}{dx} \right\} = 0.$$

From (15) one obtains again the general solution (17). By imposing boundary conditions (19), one finally is led to (14).

Note that in the absence of refractoriness, (14) are in agreement with (7). Indeed, if  $\beta = 0$  one has  $\widehat{t}_n(S|x) = t_n(S|x)$ . The following remark shows that FET moments  $\widehat{t}_n(S|x)$  are related to the FPT moments  $t_n(S|x)$ .

**Remark 2.1** *Under the assumption (i) or (ii), if  $\alpha > 0$  one has*

$$(20) \quad \widehat{t}_n(S|x) = t_n(S|x) + n \frac{\beta}{\alpha} \sum_{j=0}^{n-1} \binom{n-1}{j} t_{n-1-j}(S|x) \int_{r_1}^S k(u) \widehat{t}_j(S|u) du,$$

$$(21) \quad \widehat{t}_n(S|x) = t_n(S|x) + n \frac{\beta}{\alpha} \sum_{j=0}^{n-1} \binom{n-1}{j} \widehat{t}_j(S|x) \int_{r_1}^S k(u) t_{n-1-j}(S|u) du.$$

**Proof.** Making use of (14), relations (20) and (21) immediately follow by induction.

Setting  $n = 1$  in (20) or in (21), one can see that the mean of first exit time is given by

$$(22) \quad \widehat{t}_1(S|x) = t_1(S|x) + \frac{\beta}{\alpha} \int_{r_1}^S k(u) du \quad (x < S).$$

Furthermore, setting  $n = 2$  in (20) and in (21), one can obtain two equivalent expressions for the second order moment of first exit time:

$$\begin{aligned} \widehat{t}_2(S|x) &= t_2(S|x) + 2 \frac{\beta}{\alpha} \widehat{t}_1(S|x) \int_{r_1}^S k(u) du + 2 \frac{\beta}{\alpha} \int_{r_1}^S k(u) t_1(S|u) du \\ (23) \quad &= t_2(S|x) + 2 \frac{\beta}{\alpha} t_1(S|x) \int_{r_1}^S k(u) du + 2 \frac{\beta}{\alpha} \int_{r_1}^S k(u) \widehat{t}_1(S|u) du. \end{aligned}$$

Hence, the variance  $\widehat{V}(S|x)$  of the first exit time is given by

$$(24) \quad \widehat{V}(S|x) = V(S|x) + \left( \frac{\beta}{\alpha} \int_{r_1}^S k(u) du \right)^2 + 2 \frac{\beta}{\alpha} \int_{r_1}^S k(u) t_1(S|u) du,$$

where  $V(S|x)$  denotes the FPT variance.

**Theorem 2.3** *Under the assumption (i) or (ii), if  $\alpha > 0$  the refractoriness period is doomed to end with certainty and its moments can be iteratively calculated as*

$$(25) \quad E(T_r^n) := \int_0^\infty t^n g_r(S, t|S) dt = n \frac{\beta}{\alpha} \int_{\tau_1}^S k(u) \widehat{t}_{n-1}(S|u) du \quad (n = 1, 2, \dots).$$

**Proof.** Integrating both sides of (6) in  $(0, +\infty)$  one has

$$\int_0^{+\infty} g_r(S, t|S) dt = 1,$$

implying that the refractoriness period is doomed to end with certainty. Furthermore, from (6) we also have:

$$\begin{aligned} \widehat{t}_n(S|x) &:= \int_0^{+\infty} t^n g_e(S, t|x) dt = \int_0^{+\infty} dt t^n \int_0^t g(S, \tau|x) g_r(S, t|S, \tau) d\tau \\ &= \int_0^{+\infty} d\tau g(S, \tau|x) \int_\tau^{+\infty} t^n g_r(S, t|S, \tau) dt \\ &= \int_0^{+\infty} d\tau g(S, \tau|x) \int_0^{+\infty} (\tau + \vartheta)^n g_r(S, \vartheta|S) d\vartheta \\ &= \sum_{j=0}^n \binom{n}{j} t_{n-j}(S|x) E(T_r^j). \quad (n = 1, 2, \dots). \end{aligned}$$

Hence,

$$(26) \quad E(T_r^n) = \widehat{t}_n(S|x) - \sum_{j=0}^{n-1} \binom{n}{j} t_{n-j}(S|x) E(T_r^j) \quad (n = 1, 2, \dots).$$

We now proceed by induction. Setting  $n = 1$  in (26) one sees that  $E(T_r) = \widehat{t}_1(S|x) - t_1(S|x)$ . Hence, on account of (22), (25) holds for  $n = 1$ . Furthermore, assuming that (26) hold for  $j = 1, 2, \dots, n$ , the right-hand side of (26) for  $n + 1$  becomes:

$$\begin{aligned} &\widehat{t}_{n+1}(S|x) - \sum_{j=0}^n \binom{n+1}{j} t_{n+1-j}(S|x) E(T_r^j) \\ &= \widehat{t}_{n+1}(S|x) - t_{n+1}(S|x) - (n+1) \frac{\beta}{\alpha} \sum_{j=0}^{n-1} \binom{n}{j} t_{n-j}(S|x) \int_{\tau_1}^S k(u) \widehat{t}_j(S|u) du \\ (27) \quad &= (n+1) \frac{\beta}{\alpha} \int_{\tau_1}^S k(u) \widehat{t}_n(S|u) du, \end{aligned}$$

where the last equality follows from (20). From (26) we note that the left-hand side of (27) is equal to  $E(T_r^{n+1})$ . Hence, if (25) holds for an arbitrarily fixed  $n$ , it also holds for  $n + 1$ , which completes the proof.

Comparing (14) and (25) we note that

$$E(T_r^n) \equiv \lim_{x \uparrow S} \widehat{t}_n(S|x).$$

In particular, from (25) the first two moments and the variance of the refractoriness period are seen to be:

$$\begin{aligned}
 E(T_r) &= \frac{\beta}{\alpha} \int_{r_1}^S k(u) \, du \\
 E(T_r^2) &= 2 \frac{\beta}{\alpha} \int_{r_1}^S k(u) t_1(S|u) \, du + 2 \left( \frac{\beta}{\alpha} \int_{r_1}^S k(u) \, du \right)^2 \\
 V(T_r) &= 2 \frac{\beta}{\alpha} \int_{r_1}^S k(u) t_1(S|u) \, du + \left( \frac{\beta}{\alpha} \int_{r_1}^S k(u) \, du \right)^2.
 \end{aligned}
 \tag{28}$$

Comparing the first and last of (28) with (22) and (24), we have

$$\hat{t}_1(S|x) = t_1(S|x) + E(T_r), \quad \hat{V}(S|x) = V(S|x) + V(T_r),
 \tag{29}$$

i.e. the mean (variance) of first exit time through  $S$  starting from  $x$  is the sum of the mean (variance) of first passage time through  $S$  starting from  $x$  and of the mean (variance) of the refractoriness period.

### 3 Analysis of three neuronal models

In order to embody some physiological features of real neurons, several alternative models have been proposed in the literature (cf, for instance, [5], [6] and references therein). In this Section we shall investigate the behavior of the refractoriness period for the Wiener, Ornstein-Uhlenbeck (OU) and Feller neuronal models. We assume that all three neuronal models are restricted to the same diffusion interval  $I = [\nu, +\infty)$ , having set  $r_1 = \nu$ .

$\sigma^2$	$t_1(S \rho)$	$E(T_r)$	$E(T_r)$	$E(T_r)$	$E(T_r)$
		$p_R = 0.1$	$p_R = 0.5$	$p_R = 0.9$	$p_R = 0.99$
10.	3.073451 E+2	6.294544 E+2	5.665090 E+3	5.098581 E+4	5.608439 E+5
20.	7.331871 E+1	9.425701 E+0	8.483131 E+1	7.634818 E+2	8.398300 E+3
30.	3.936016 E+1	2.021650 E+0	1.819485 E+1	1.637537 E+2	1.801290 E+3
40.	2.663797 E+1	8.663807 E-1	7.797426 E+0	7.017684 E+1	7.719452 E+2
50.	2.007160 E+1	4.966112 E-1	4.469501 E+0	4.022551 E+1	4.424806 E+2
100.	8.937578 E+0	1.281821 E-1	1.153639 E+0	1.038275 E+1	1.142103 E+2
200.	4.225259 E+0	4.617762 E-2	4.155986 E-1	3.740387 E+0	4.114426 E+1
300.	2.765483 E+0	2.760995 E-2	2.484895 E-1	2.236406 E+0	2.460046 E+1
400.	2.055224 E+0	1.961207 E-2	1.765086 E-1	1.588577 E+0	1.747435 E+1
500.	1.635207 E+0	1.518666 E-2	1.366799 E-1	1.230119 E+0	1.353131 E+1

Table 1: Wiener model with  $\mu = -0.5$  and  $\sigma^2 = 10 \cdot i, 100 \cdot i$  ( $i = 1, 2, \dots, 5$ ), restricted to  $I = [\nu, +\infty)$  with  $\nu = -80$ . In the second column we have listed the FPT mean  $t_1(S|\rho)$  with  $S = -50$  and  $\rho = -70$ . Instead, in columns three, four, five and six we have respectively listed the mean of refractoriness period for  $p_R = 0.1, 0.5, 0.9, 0.99$ .

**3.1 Wiener model** The Wiener neuronal model is defined as the diffusion process  $X(t)$  characterized by the following drift and infinitesimal variance:

$$A_1(x) = \mu \quad A_2(x) = \sigma^2, \quad (\mu \in \mathbf{R}, \sigma > 0),
 \tag{30}$$

restricted to  $I = [\nu, +\infty)$ , where on the regular boundary  $x = \nu$  a reflecting condition is imposed. For such process scale and speed functions are

$$h(x) = \exp\left\{-\frac{2\mu x}{\sigma^2}\right\}, \quad k(x) = \frac{2}{\sigma^2} \exp\left\{\frac{2\mu x}{\sigma^2}\right\}.$$



$\sigma^2$	$V(S \rho)$	$V(T_r)$	$V(T_r)$	$V(T_r)$	$V(T_r)$
		$p_R = 0.1$	$p_R = 0.5$	$p_R = 0.9$	$p_R = 0.99$
10.	9.254218 E+4	7.681238 E+5	3.544044 E+7	2.629677 E+9	3.148772 E+11
20.	4.970295 E+3	1.310444 E+3	1.819075 E+4	6.818541 E+5	7.161989 E+7
30.	1.390880 E+3	1.385660 E+2	1.541363 E+3	3.770806 E+4	3.364468 E+6
40.	6.265060 E+2	3.874901 E+1	4.027854 E+2	8.002658 E+3	6.297560 E+5
50.	3.519348 E+2	1.638408 E+1	1.652135 E+2	2.925226 E+3	2.101676 E+5
100.	6.821593 E+1	1.804893 E+0	1.742704 E+1	2.526670 E+2	1.463751 E+4
200.	1.506377 E+1	3.008332 E-1	2.861029 E+0	3.818526 E+1	1.958992 E+3
300.	6.426684 E+0	1.168741 E-1	1.106754 E+0	1.440657 E+1	7.086384 E+2
400.	3.542131 E+0	6.147195 E-2	5.809411 E-1	7.471651 E+0	3.597818 E+2
500.	2.239493 E+0	3.778974 E-2	3.567134 E-1	4.555481 E+0	2.165615 E+2

Table 2: For the Wiener model and for the same choices of parameters of Table 1, in the second column we have listed the FPT variance  $V(S|\rho)$  with  $S = -50$  and  $\rho = -70$ , whereas in columns three, four, five and six we have respectively listed the variance of refractoriness period for  $p_R = 0.1, 0.5, 0.9, 0.99$ .

Furthermore, the mean of first passage time is

$$(31) \quad t_1(S|x) = \begin{cases} \frac{(S-x)(S+x-2\nu)}{\sigma^2}, & \mu = 0 \\ \frac{S-x}{\mu} + \frac{\sigma^2}{2\mu^2} \left[ \exp\left\{-\frac{2\mu(S-\nu)}{\sigma^2}\right\} - \exp\left\{-\frac{2\mu(x-\nu)}{\sigma^2}\right\} \right], & \mu \neq 0. \end{cases}$$

For the Wiener model (30) with  $\mu = -0.5, \sigma^2 = 10 \cdot i, 100 \cdot i$  ( $i = 1, 2, \dots, 5$ ), restricted to  $I = [\nu, +\infty)$  with  $r_1 \equiv \nu = -80$ , in the second column of Table 1 and of Table 2 we have respectively listed the mean  $t_1(S|\rho)$  and variance  $V(S|\rho)$ , numerically obtained via (7) with  $S = -50$  and  $\rho = -70$ . Note that the FPT mean and variance decrease with  $\sigma^2$ . Being  $\beta/\alpha = p_R/(1-p_R)$ , in Table 1 and in Table 2 we have respectively listed the values of mean and variance of refractoriness period, numerically obtained via (28) for  $p_R = 0.1, 0.5, 0.9, 0.99$ . We observe that  $E(T_r)$  and  $V(T_r)$  increase with  $p_R$  for any fixed  $\sigma^2$ .

$\sigma^2$	$t_1(S \rho)$	$E(T_r)$	$E(T_r)$	$E(T_r)$	$E(T_r)$
		$p_R = 0.1$	$p_R = 0.5$	$p_R = 0.9$	$p_R = 0.99$
10.	9.862135 E+3	9.901436 E+41	8.911293 E+42	8.020163 E+43	8.822180 E+44
20.	2.600359 E+2	3.452097 E+20	3.106887 E+21	2.796199 E+22	3.075818 E+23
30.	7.956655 E+1	2.140293 E+13	1.926264 E+14	1.733637 E+15	1.907001 E+16
40.	4.273886 E+1	4.978530 E+9	4.480677 E+10	4.032609 E+11	4.435870 E+12
50.	2.853092 E+1	3.149993 E+7	2.834994 E+8	2.551494 E+9	2.806644 E+10
100.	1.038152 E+1	1.006196 E+3	9.055763 E+3	8.150187 E+4	8.965206 E+5
200.	4.525217 E+0	4.073683 E+0	3.666314 E+1	3.299683 E+2	3.629651 E+3
300.	2.890905 E+0	5.465816 E-1	4.919234 E+0	4.427311 E+1	4.870042 E+2
400.	2.123662 E+0	1.839895 E-1	1.655905 E+0	1.490315 E+1	1.639346 E+2
500.	1.678216 E+0	9.103197 E-2	8.192877 E-1	7.373589 E+0	8.110948 E+1

Table 3: OU model with  $\vartheta = 5, \rho = -70$  and  $\sigma^2 = 10 \cdot i, 100 \cdot i$  ( $i = 1, 2, \dots, 5$ ), restricted to  $I = [\nu, +\infty)$  with  $\nu = -80$ . In the second column we have listed the FPT mean  $t_1(S|\rho)$  with  $S = -50$ , whereas in columns three, four, five and six we have respectively listed the mean of refractoriness period for  $p_R = 0.1, 0.5, 0.9, 0.99$ .

$\sigma^2$	$V(S \rho)$	$V(T_r)$	$V(T_r)$	$V(T_r)$	$V(T_r)$
		$p_R = 0.1$	$p_R = 0.5$	$p_R = 0.9$	$p_R = 0.99$
10.	9.713857 E+7	9.803844 E+83	7.941114 E+85	6.432302 E+87	7.783086 E+89
20.	6.554937 E+4	1.191697 E+41	9.652749 E+42	7.818727 E+44	9.460659 E+46
30.	5.898427 E+3	4.580854 E+26	3.710492 E+28	3.005498 E+30	3.636653 E+32
40.	1.654790 E+3	2.478576 E+19	2.007647 E+21	1.626194 E+23	1.967694 E+25
50.	7.239524 E+2	9.922473 E+14	8.037191 E+16	6.510124 E+18	7.877250 E+20
100.	9.240940 E+1	1.029849 E+6	8.216362 E+7	6.643966 E+9	8.037646 E+11
200.	1.728177 E+1	4.589508 E+1	1.607888 E+3	1.112524 E+5	1.320047 E+7
300.	7.020488 E+0	2.765114 E+0	4.639613 E+1	2.159884 E+3	2.393706 E+5
400.	3.780330 E+0	6.379705 E-1	8.179088 E+0	2.710374 E+2	2.741284 E+4
500.	2.357829 E+0	2.431146 E-1	2.784683 E+0	7.339088 E+1	6.787980 E+3

Table 4: For the OU model and for the same choices of parameters of Table 3 in the second column we have listed the FPT variance  $V(S|\rho)$  with  $S = -50$  and  $\rho = -70$ , whereas in columns three, four, five and six we have respectively listed the variance of refractoriness period for  $p_R = 0.1, 0.5, 0.9, 0.99$ .

**3.2 OU model** The OU neuronal model is defined as the diffusion process  $X(t)$  characterized by the following drift and infinitesimal variance:

$$(32) \quad A_1(x) = -\frac{1}{\vartheta}(x - \rho), \quad A_2 = \sigma^2 \quad (\rho \in \mathbf{R}, \sigma > 0, \vartheta > 0),$$

restricted to  $I = [\nu, +\infty)$ , where on the regular boundary  $x = \nu$  a reflecting condition is imposed. For such process the scale and speed functions are

$$h(x) = \exp\left\{\frac{x^2}{\vartheta\sigma^2} - \frac{2\rho x}{\vartheta\sigma^2}\right\}, \quad k(x) = \frac{2}{\sigma^2} \exp\left\{-\frac{x^2}{\vartheta\sigma^2} + \frac{2\rho x}{\vartheta\sigma^2}\right\}.$$

Furthermore, the mean of first passage time is:

$$(33) \quad \begin{aligned} t_1(S|x) = & \vartheta \sum_{k=0}^{+\infty} \frac{2^k}{(k+1)(2k+1)!!} \left[ \left(\frac{S-\rho}{\sigma\sqrt{\vartheta}}\right)^{2k+2} - \left(\frac{x-\rho}{\sigma\sqrt{\vartheta}}\right)^{2k+2} \right] \\ & - 2\vartheta \exp\left\{-\frac{(\nu-\rho)^2}{\sigma^2\vartheta}\right\} \sum_{k=0}^{+\infty} \frac{2^k}{(2k+1)!!} \left(\frac{\nu-\rho}{\sigma\sqrt{\vartheta}}\right)^{2k+1} \\ & \times \sum_{k=0}^{+\infty} \frac{1}{(2k+1)k!} \left[ \left(\frac{S-\rho}{\sigma\sqrt{\vartheta}}\right)^{2k+1} - \left(\frac{x-\rho}{\sigma\sqrt{\vartheta}}\right)^{2k+1} \right]. \end{aligned}$$

For the OU model (32) with  $\vartheta = 5, \rho = -70, \sigma^2 = 10 \cdot i, 100 \cdot i$  ( $i = 1, 2, \dots, 5$ ), restricted to  $I = [\nu, +\infty)$  with  $\nu = -80$ , in the second column of Table 3 and of Table 4 we have respectively listed the mean  $t_1(S|\rho)$  and variance  $V(S|\rho)$ , numerically obtained via (7) with  $S = -50$ . Furthermore, in Table 3 and in Table 4 we have also listed the values of mean and variance of refractoriness period for  $p_R = 0.1, 0.5, 0.9, 0.99$ . Similarly to the case of the Wiener model, for the OU model the FPT mean and variance decrease with  $\sigma^2$ ; furthermore,  $E(T_r)$  and  $V(T_r)$  increase with  $p_R$  for any fixed  $\sigma^2$ .

**3.3 Feller model** The Feller neuronal model is defined as the diffusion process  $X(t)$  characterized by the following drift and infinitesimal variance:

$$(34) \quad A_1(x) = -\frac{1}{\vartheta}(x - \rho), \quad A_2(x) = 2\xi(x - \nu) \quad (\rho, \nu \in \mathbf{R}, \rho > \nu, \vartheta > 0, \xi > 0).$$

$\xi$	$t_1(S \rho)$	$E(T_r)$	$E(T_r)$	$E(T_r)$	$E(T_r)$
		$p_R = 0.1$	$p_R = 0.5$	$p_R = 0.9$	$p_R = 0.99$
0.5	3.768002 E+2	4.103229 E+15	3.692906 E+16	3.323615 E+17	3.655977 E+18
1.0	8.029989 E+1	2.425535 E+7	2.182981 E+8	1.964683 E+9	2.161152 E+10
1.5	4.661249 E+1	4.020549 E+4	3.618494 E+5	3.256645 E+6	3.582309 E+7
2.0	3.470051 E+1	1.573636 E+3	1.416272 E+4	1.274645 E+5	1.402110 E+6
2.5	2.866867 E+1	2.204449 E+2	1.984004 E+3	1.785603 E+4	1.964164 E+5
3.0	2.502681 E+1	5.871921 E+1	5.284729 E+2	4.756256 E+3	5.231882 E+4
3.5	2.258692 E+1	2.264293 E+1	2.037863 E+2	1.834077 E+3	2.017485 E+4
4.0	2.083633 E+1	1.102071 E+1	9.918635 E+1	8.926772 E+2	9.819449 E+3
4.5	1.951789 E+1	6.271095 E+0	5.643986 E+1	5.079587 E+2	5.587546 E+3
5.0	1.848842 E+1	3.983514 E+0	3.585162 E+1	3.226646 E+2	3.549311 E+3

Table 5: Feller model with  $\vartheta = 5$ ,  $\rho = -70$ ,  $\nu = -80$  and  $\xi = 0.5 \cdot i$  ( $i = 1, 2, \dots, 10$ ). In second column we have listed the FPT mean  $t_1(S|\rho)$  with  $S = -50$ , whereas in columns three, four, five and six we have respectively listed the mean of refractoriness period for  $p_R = 0.1, 0.5, 0.9, 0.99$ .

$\xi$	$V(S \rho)$	$V(T_r)$	$V(T_r)$	$V(T_r)$	$V(T_r)$
		$p_R = 0.1$	$p_R = 0.5$	$p_R = 0.9$	$p_R = 0.99$
0.5	1.395404 E+5	1.683649 E+31	1.363755 E+33	1.104642 E+35	1.336617 E+37
1.0	6.372482 E+3	5.883257 E+14	4.765411 E+16	3.859980 E+18	4.670576 E+20
1.5	2.241795 E+3	1.620153 E+9	1.309681 E+11	1.060603 E+13	1.283297 E+15
2.0	1.304116 E+3	2.585121 E+6	2.015619 E+8	1.625601 E+10	1.966008 E+12
2.5	9.313963 E+2	6.143569 E+4	4.051826 E+6	3.198777 E+8	3.85908 E+10
3.0	7.390905 E+2	6.492913 E+3	3.066862 E+5	2.286843 E+7	2.739949 E+9
3.5	6.238662 E+2	1.592475 E+3	5.124507 E+4	3.451142 E+6	4.079672 E+8
4.0	5.478171 E+2	6.147816 E+2	1.427634 E+4	8.366949 E+5	9.684436 E+7
4.5	4.940875 E+2	3.064866 E+2	5.588538 E+3	2.795397 E+5	3.144373 E+7
5.0	4.541290 E+2	1.789263 E+2	2.751622 E+3	1.172086 E+5	1.272925 E+7

Table 6: For the Feller model and for the same choices of parameters of Table 5, in the second column we have listed the FPT variance  $V(S|\rho)$  with  $S = -50$  and  $\rho = -70$ , whereas in columns three, four, five and six we have respectively listed the variance of refractoriness period for  $p_R = 0.1, 0.5, 0.9, 0.99$ .

defined in  $I = [\nu, +\infty)$ , where  $x = \nu$  is regular if  $\rho - \nu < \xi \vartheta$  and entrance if  $\rho - \nu \geq \xi \vartheta$ , whereas the boundary  $+\infty$  is natural. For such process the scale and speed functions are:

$$h(x) = \exp\left\{\frac{x}{\vartheta \xi}\right\} (x - \nu)^{-(\rho - \nu)/(\vartheta \xi)}, \quad k(x) = \frac{1}{\xi} \exp\left\{-\frac{x}{\vartheta \xi}\right\} (x - \nu)^{(\rho - \nu)/(\vartheta \xi) - 1}.$$

The mean of the firing time can be calculated; for  $x < S$  one obtains

(35)

$$t_1(S|x) = \frac{\vartheta}{\rho - \nu} \left[ S - x + \sum_{k=1}^{\infty} \left(\frac{1}{\vartheta}\right)^k \frac{(S - \nu)^{k+1} - (x - \nu)^{k+1}}{k + 1} \left\{ \prod_{i=1}^k \left(\frac{\rho - \nu}{\vartheta} + \xi i\right) \right\}^{-1} \right].$$

For the Feller model (34) with  $\vartheta = 5$ ,  $\rho = -70$ ,  $\nu = -80$ ,  $\xi = 0.5 \cdot i$  ( $i = 1, 2, \dots, 10$ ), in the second column of Table 5 and of Table 6 we have respectively listed the mean  $t_1(S|\rho)$  and variance  $V(S|\rho)$ , numerically obtained via (7) with  $S = -50$ . Note that the FPT mean and variance decrease with  $\xi$ . Furthermore, in Table 5 and in Table 6 we have listed the values of mean and variance of refractoriness period for  $p_R = 0.1, 0.5, 0.9, 0.99$ . We note that  $E(T_r)$  and  $V(T_r)$  increase with  $p_R$  for any fixed  $\xi$ .

We conclude by pointing out that the purpose of the present note was to establish the quantitative foundations to a viable way to include refractoriness in neuronal diffusion

models. Implementation of our approach to data analysis will be the object of future endeavors.

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