ON NEURONAL FIRING MODELING VIA SPECIALLY CONFINED DIFFUSION PROCESSES

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Abstract. First passage time problems for diffusion processes have been extensively investigated to model neuronal firing activity or extinction times in population dynamics (see, for instance, [10]). In this paper we study the asymptotic behavior of first passage times densities for a class of specially confined temporally homogeneous diffusion processes in the presence of an entrance or a reflecting boundary. The emphasis is on problems of a rather mathematical nature, concerning the behavior of the first passage time density and of its moments when the neuronal firing threshold is in the neighborhood of the reflecting boundary, and when it moves indefinitely away from it. Our asymptotic results are obtained without need to determine beforehand the transition probability density in the presence of entrance or reflecting boundaries; they depend, instead, only on drift, infinitesimal variance, threshold and on the entrance or the reflecting boundary of the process. Some evaluations of moments of first passage time, in particular, mean and variance, are performed by solving numerically, or analytically whenever possible, Siegert's recursion equations [12], and by comparing the results with those obtained through our approximate formulas. In the case where the transition probability density is known, the goodness of the obtained approximations can be verified. Such results appear to be useful for neuronal modeling in the presence of reversal potential especially to pinpoint the role of the involved parameters in various models, some of which are the object of a somewhat detailed analysis.

1 Introduction

The purpose of this paper is to provide the necessary mathematical framework to approach the single neuron's firing description by means of models based on the theory of stochastic diffusion processes. Although this is undoubtedly a very much trotted ground (see, for instance, [11] and references therein), our present approach differs substantially in that it makes use of our notion of BF-processes, that will be introduced in the sequel after provided the necessary mathematical background and proving several basic analytic results. The last part of this paper will finally be specifically centred on the discussion of neuronal models and on the outline of several computational results.

Let $\{X(t),\, t \geq 0\}$ be a time–homogeneous diffusion process defined over the interval $I \equiv [r, r_2)$, where r is a *regular* or *entrance* boundary and r_2 is a natural boundary. Further, let $A_1(x)$ and $A_2(x)$ be the drift and infinitesimal variance of $X(t)$, respectively, and let $S \in (r, r_2)$. For all $x \in (r, r_2)$, let

$$h(x) := \exp\Big\{-2 \int^x \frac{A_1(\xi)}{A_2(\xi)}\, d\xi\Big\} \qquad \text{(scale function)}$$

(1)

$$s(x) := \frac{2}{A_2(x)\, h(x)} \qquad \text{(speed density)}.$$

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Furthermore, for all $t > 0$ and $x, x_0 \in I$, let

$$f(x,t|x_0) = \frac{\partial}{\partial x} P\{X(t) < x | X(0) = x_0\}$$

denote the transition probability density function (pdf) with a reflecting condition (regular boundary) or with a zero-flux condition (entrance boundary) set at r.
Hereafter, we shall focus our attention on the properties of the first passage time (FPT) random variable

$$T = \inf_{t \geq 0}\{t : \ X(t) > S\}, \qquad X(0) = x_0 < S.$$

Let

$$g(S,t|x_0) = \frac{\partial}{\partial t} P\{T < t\}$$

be the pdf of T. An analytic approach to the evaluation of $g(S,t|x_0)$ is based on its Laplace transform (LT) with respect to t:

$$g_\lambda(S|x_0) := \int_0^{+\infty} e^{-\lambda t} g(S,t|x_0)\, dt. \tag{2}$$

Then, $g(S,t|y)$ can be obtained as the inverse LT. It is worth to point out that even though the inverse transform cannot be calculated, it can nevertheless provide useful information. Indeed, for all $n = 1, 2, \dots$, the moments of T

$$t_n(S|x_0) := E(T^n) = \int_0^{+\infty} t^n \ g(S,t|x_0) \ dt \tag{3}$$

customarily obtained via $g_\lambda(S|x_0)$ as follows:

$$t_n(S|x_0) = (-1)^n \, \frac{d^n g_\lambda(S|x_0)}{d\lambda^n}\Big|_{\lambda=0}.$$

When the end point r of the diffusion interval is a reflecting or entrance boundary, alternative approaches to calculate the FPT moments are provided by the following propositions.

Proposition 1.1 *Let $r \leq x_0 < S$, where r is a reflecting or an entrance boundary. Then, for the FPT probability one has:*

$$P(S|x_0) := \int_0^{+\infty} g(S,t|x_0) \ dt = 1.$$

Furthermore, all moments $t_n(S|x_0)$ are finite and can be iteratively calculated as

$$t_n(S|x_0) = n \int_{x_0}^{S} h(z) \ dz \int_r^z s(u)\, t_{n-1}(S|u) \ du \qquad (n = 1, 2, \dots) \tag{4}$$

where $t_0(S|x_0) = 1$.

Proof: It goes along the lines indicated in [12].

■

Proposition 1.2 *Under the assumptions of Proposition 1.1, one has*

(5)
$$t_1(S|x_0) = \chi_1(S|r) - \chi_1(x_0|r)$$
$$t_n(S|x_0) = n!\left\{\sum_{j=1}^{n-1} \frac{(-1)^{n-j-1}}{j!}\, t_j(S|x_0)\, \chi_{n-j}(S|r) + (-1)^{n-1}\Big[\chi_n(S|r) - \chi_n(x_0|r)\Big]\right\} \qquad (n = 2, 3, \dots)$$

where for all $x \in I$ we have set

(6)
$$\chi_1(x|r) = \int_r^x h(z)\, dz \int_r^z s(u)\, du$$
$$\chi_n(x|r) = \int_r^x h(z)\, dz \int_r^z s(u)\, \chi_{n-1}(u|r)\, du \qquad (n = 2, 3, \dots).$$

Proof: By making use of (4) and (6) it is immediately seen that relations (5) hold for $n = 1, 2$. We now proceed by induction and prove that if (5) holds for an arbitrarily fixed n, it also holds for $n + 1$. Indeed, from (4) and (6) it follows

$$t_{n+1}(S|x_0) = (n+1)!\left\{\sum_{j=1}^{n-1} \frac{(-1)^{n-j-1}}{(j+1)!}\, \chi_{n-j}(S|r)\, t_{j+1}(S|x_0) + (-1)^{n-1}\chi_n(S|r)\, t_1(S|x_0) + (-1)^n\Big[\chi_{n+1}(S|r) - \chi_{n+1}(x_0|r)\Big]\right\},$$

whose right-hand side is seen to yield $t_{n+1}(S|x_0)$ as defined by (5).

■

Remark 1.1 *Under the assumptions of Proposition 1.1, one has*

(7)
$$\chi_n(x|r) \le [t_1(x|r)]^n \qquad (n = 1, 2, \dots)$$

Proof: By virtue of (5) and (6), it is immediately seen that relations (7) hold for $n = 1, 2$. We now proceed by induction and prove that if (7) holds for an arbitrarily fixed n, it also holds for $n + 1$. Indeed, from (5) and (6) it follows:

(8)
$$\chi_{n+1}(x|r) = \int_r^x h(z)\, dz \int_r^z s(u)\, \chi_n(u|r)\, du \le \int_r^x h(z)\, dz \int_r^z s(u)\, [t_1(u|r)]^n\, du$$
$$\le [t_1(x|r)]^n \int_r^x h(z)\, dz \int_r^z s(u)\, du = [t_1(x|r)]^{n+1}.$$

■

In Section 2 we analyze the behavior of the of the FPT pdf and of its moments when the threshold S is in the neighborhood of the reflecting or entrance boundary r. Instead, in Section 3 we analyze the behavior of the FPT pdf and of its moments when the threshold S is moving indefinitely away from it. Our asymptotic results are obtained without need to determine beforehand the transition pdf in the presence of entrance or reflecting boundaries; they depend, instead, only on drift, infinitesimal variance, threshold and on the entrance or the reflecting boundary of the process. In Section 4 certain closed-form solutions for the FPT moments and densities are obtained. Finally, in Section 5 new asymptotic results for Wiener, Ornstein Uhlenbeck and Feller models are presented.

2 The neuronal threshold is in the neighborhood of the boundary

In Section 2.1 we analyze the general behavior of the of the FPT pdf and of its moments when the threshold S is in the neighborhood of the reflecting or entrance boundary r, whereas in Section 2.2 we consider some special cases.

2.1 General Considerations

Lemma 2.1 *If*

$$\lim_{x\downarrow r} \sqrt{A_2(x)}\, h(x) \int_r^x s(u)\, du = 0, \tag{9}$$

$$\lim_{x\downarrow r} \Big[A_1(x) - \frac{A_2'(x)}{4}\Big]\, h(x) \int_r^x s(u)\, du = \nu, \tag{10}$$

where $A_2'(x) = \dfrac{dA_2(x)}{dx}$ *and where* $-\infty < \nu < 1$ *is a real number, then*

$$\lim_{S\downarrow r} \frac{\chi_n(S|r)}{[t_1(S|r)]^n} = \frac{1}{n! \displaystyle\prod_{i=0}^{n-1} \big[1 + 2\, i\, (1-\nu)\big]} \qquad (n = 1, 2, \dots). \tag{11}$$

Proof: We proceed by induction. Since $\chi_1(S|r) = t_1(S|r)$, (11) is trivial for $n = 1$. Setting $n = 2$ in the left-hand side of (11) and making use of (4) and (6), we obtain

$$\lim_{S\downarrow r} \frac{\chi_2(S|r)}{[t_1(S|r)]^2} = \lim_{S\downarrow r} \frac{\displaystyle\int_r^S h(z)\, dz \int_r^z s(u)\, \chi_1(u|r)\, du}{\Big[\displaystyle\int_r^S h(z)\, dz \int_r^z s(u)\, du\Big]^2}$$

$$= \frac{1}{2} \lim_{S\downarrow r} \left[1 + \frac{A_2(S)\, h^2(S)\, \Big(\int_r^S s(u)\, du\Big)^2}{2\, t_1(S|r)}\right]^{-1}, \tag{12}$$

where l'Hospital's rule has been used repeatedly. Since (9) holds, by applying again of l'Hospital's rule, one has

$$\lim_{S\downarrow r} \frac{A_2(S)\, h^2(S)\, \Big[\int_r^S du\, s(u)\Big]^2}{t_1(S|r)} = 4 \lim_{S\downarrow r} \left\{1 - \Big(A_1(S) - \frac{A_2'(S)}{4}\Big)\, h(S) \int_r^S s(u)\, du\right\}$$

$$= 4\,(1-\nu), \tag{13}$$

where the last equality follows from (10). From (13) we note that $-\infty < \nu < 1$. Making use of (13) in (12), one obtains

$$\lim_{S\downarrow r} \frac{\chi_2(S|r)}{[t_1(S|r)]^2} = \frac{1}{2\,[1 + 2\,(1-\nu)]},$$

showing that (11) is satisfied for $n = 2$. Let us now assume that (11) holds for n and prove that it also holds for $n+1$. Indeed, we have

$$\lim_{S\downarrow r} \frac{\chi_{n+1}(S|r)}{[t_1(S|r)]^{n+1}} = \lim_{S\downarrow r} \frac{\displaystyle\int_r^S h(z)\, dz \int_r^z s(u)\, \chi_n(u|r)\, du}{\left[\displaystyle\int_r^S h(z)\, dz \int_r^z s(u)\, du\right]^{n+1}}$$

$$= \frac{1}{n+1} \lim_{S\downarrow r} \left\{ \frac{\chi_n(S|r)}{[t_1(S|r)]^n} \left[1 + \frac{n}{2}\, \frac{A_2(S)\, h^2(S) \left(\int_r^S s(u)\, du\right)^2}{t_1(S|r)} \right]^{-1} \right\}. \tag{14}$$

where again (4), (6), (9) and l'Hospital's rule have been used. Due to (10), and making use of the assumption that (11) holds for n, from (14) one has:

$$\lim_{S\downarrow r} \frac{\chi_{n+1}(S|r)}{[t_1(S|r)]^{n+1}} = \frac{1}{(n+1)! \displaystyle\prod_{i=0}^{n} [1 + 2\, i\, (1-\nu)]}.$$

Hence, if (11) holds for an arbitrarily fixed n, it also holds for $n+1$. This completes the proof.

■

Theorem 2.1 *Under the assumptions of Lemma 2.1 there results:*

$$\lim_{S\downarrow r} \frac{t_n(S|r)}{[t_1(S|r)]^n} = u_n \qquad (n = 0, 1, \dots), \tag{15}$$

where

$$u_0 = 1, \quad u_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{\displaystyle\prod_{i=0}^{k-1} [1 + 2\, i\, (1-\nu)]}\, u_{n-k} \qquad (n = 1, 2, \dots). \tag{16}$$

Proof: For $n = 0, 1$, Equation (15) holds with $u_0 = u_1 = 1$. For $n = 2, 3, \dots$, from (5) with $x_0 = r$ we have:

$$\frac{t_n(S|r)}{[t_1(S|r)]^n} = n! \left\{ \sum_{j=1}^{n-1} \frac{(-1)^{n-j-1}}{j!} \frac{t_j(S|r)}{[t_1(S|r)]^j} \frac{\chi_{n-j}(S|r)}{[t_1(S|r)]^{n-j}} + (-1)^{n-1} \frac{\chi_n(S|r)}{[t_1(S|r)]^n} \right\}. \tag{17}$$

Recalling (11), in the limit as $S \downarrow r$, from (17) it follows:

$$u_n = n! \sum_{j=1}^{n-1} \frac{(-1)^{n-j-1}\, u_j}{j!\, (n-j)! \displaystyle\prod_{i=0}^{n-j-1} [1 + 2\, i\, (1-\nu)]} + \frac{(-1)^{n-1}}{\displaystyle\prod_{i=0}^{n-1} [1 + 2\, i\, (1-\nu)]}$$

$$= \sum_{j=0}^{n-1} \binom{n}{j} \frac{(-1)^{n-j-1}}{\displaystyle\prod_{i=0}^{n-j-1} [1 + 2\, i\, (1-\nu)]}\, u_j \qquad (n = 2, 3, \dots). \tag{18}$$

Setting $k = n - j$, the right-hand side of (18) is finally seen to identify with the right-hand side of (16) for $n = 2, 3, \dots$ This completes the proof.

■

Corollary 2.1 *Under the assumptions of Lemma 2.1, as S approaches the boundary r, the following asymptotic expressions hold:*

$$t_n(S|r) \simeq [t_1(S|r)]^n\, u_n \qquad (n = 0, 1, \dots), \tag{19}$$

where u_n are defined in (16).

Proof: It follows immediately from (15).

■

Corollary 2.2 *Let u_n $(n = 0, 1, \dots)$ be defined as in (16). Then, $u_0 = 1$ and*

$$u_n = (-1)^n\, n!\, \det \begin{pmatrix} c_1 & 1 & 0 & 0 & \dots & 0 \\ c_2 & c_1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & c_{n-4} & \dots & 1 \\ c_n & c_{n-1} & c_{n-2} & c_{n-3} & \dots & c_1 \end{pmatrix} \qquad (n = 1, 2, \dots) \tag{20}$$

where

$$c_k = \frac{(-1)^k}{k! \displaystyle\prod_{i=0}^{k-1} [1 + 2\,i\,(1-\nu)]} \qquad (k = 1, 2, \dots). \tag{21}$$

Furthermore, the generating function of $u_n/n!$ is given by

$$U(z) := 1 + \sum_{n=1}^{+\infty} \frac{u_n}{n!}\, z^n = \left[{}_1F_2\Big(1; 1, \frac{1}{2\,(1-\nu)}; -\frac{z}{2\,(1-\nu)}\Big) \right]^{-1}, \tag{22}$$

where

$${}_1F_2(a; b, c; x) = 1 + \sum_{k=1}^{+\infty} \frac{(a)_k}{(b)_k\,(c)_k}\, \frac{x^k}{k!}, \tag{23}$$

denotes the generalized hypergeometric series.

Proof: Making use of (16), we obtain:

$$\begin{aligned} U(z) &= 1 + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{\displaystyle\prod_{i=0}^{k-1} [1 + 2\,i\,(1-\nu)]}\, u_{n-k} \\ &= 1 + \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{\displaystyle\prod_{i=0}^{k-1} [1 + 2\,i\,(1-\nu)]} \sum_{n=k}^{+\infty} \frac{z^n}{n!} \binom{n}{k} u_{n-k} \\ &= 1 - U(z) \sum_{k=1}^{+\infty} \frac{(-1)^k}{k! \displaystyle\prod_{i=0}^{k-1} [1 + 2\,i\,(1-\nu)]}\, z^k, \end{aligned}$$

that implies

$$U(z)=\left\{1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{k!\prod_{i=0}^{k-1}[1+2\,i\,(1-\nu)]}\,z^k\right\}^{-1}. \tag{24}$$

Hence,

$$U(z)=\Big[1+\sum_{n=1}^{+\infty}c_n\,z^n\Big]^{-1}, \tag{25}$$

where c_n's are defined in (21). Recalling that $U(z)$ is the generating function of $u_n/n!$, (20) immediately follows from (25), (cf., for instance, [8], pag. 14, n. 0.313). Furthermore, since

$$\prod_{i=0}^{k-1}[1+2\,i\,(1-\nu)]=[2\,(1-\nu)]^k\,\left(\frac{1}{2\,(1-\nu)}\right)_k,$$

from (24) one obtains:

$$U(z)=\left\{1+\sum_{k=1}^{+\infty}\frac{1}{k!\left(\dfrac{1}{2\,(1-\nu)}\right)_k}\,\Big[-\frac{z}{2\,(1-\nu)}\Big]^k\right\}^{-1}. \tag{26}$$

Setting $a=1$, $b=1$, $c=[2\,(1-\nu)]^{-1}$, $x=-\,z/[2\,(1-\nu)]$ in (23), the right-hand side of (26) is finally seen to yield to the right-hand side of (22).

■

In particular, from (20) and (21) it follows:

$$\begin{aligned} &u_0=u_1=1, \qquad u_2=\frac{1+4\,(1-\nu)}{1+2\,(1-\nu)}, \qquad u_3=\frac{1+12\,(1-\nu)+48\,(1-\nu)^2}{[1+2\,(1-\nu)]\,[1+4\,(1-\nu)]},\\ &u_4=\frac{1+26\,(1-\nu)+288\,(1-\nu)^2+1536\,(1-\nu)^3+2304\,(1-\nu)^4}{[1+2\,(1-\nu)]^2\,[1+4\,(1-\nu)]\,[1+6\,(1-\nu)]}. \end{aligned} \tag{27}$$

Lemma 2.2 *Let $\{Z(t)\,t\ge 0\}$ be a Feller diffusion process defined in $[0,+\infty)$ and characterized by drift and infinitesimal variance*

$$C_1(z)=\frac{1}{2\,(1-\nu)}, \qquad C_2(z)=2\,z \qquad (-\infty<\nu<1) \tag{28}$$

and let $\widehat{g}_\lambda(\widehat{S}|0)$ be the Laplace transform of the FPT pdf from the state 0 to the state $\widehat{S}$ ($\widehat{S}>0$). Under the assumptions of Lemma 2.1, one has:

$$\tilde{g}_\lambda(S|r):=U\big[-\lambda t_1(S|r)\big]\equiv\widehat{g}_\lambda\Big(\frac{t_1(S|r)}{2\,(1-\nu)}\Big|0\Big), \tag{29}$$

where $t_1(S|r)$ denotes the FPT mean from r through S for the process $X(t)$.

Proof: As is well known (cf., for instance, [9]) for the diffusion process (28) boundary 0 is regular for $\nu < 1/2$ and entrance for $1/2 \le \nu < 1$, whereas boundary $+\infty$ is natural. From Proposition 1.1 it follows that $P(\widehat{S}|0) = 1$ for all $\widehat{S}$ inside of $I = [0, +\infty)$ and the moments $t_n(\widehat{S}|0)$ $(n = 1, 2, \dots)$ are finite. The Laplace transform $\widehat{g}_\lambda(\widehat{S}|0)$ of the FPT pdf when a reflecting condition is set at the regular boundary $x = 0$ $(\nu < 1/2)$ or when a zero-flux condition is set at the entrance boundary $x = 0$ $(1/2 \le \nu < 1)$ is (see, for instance, [5]):

$$\widehat{g}_\lambda(\widehat{S}|0) = \left\{ \Gamma\Big(\frac{1}{2\,(1-\nu)}\Big) \left(\sqrt{\lambda\,\widehat{S}}\right)^{1-\frac{1}{2\,(1-\nu)}} I_{\frac{1}{2\,(1-\nu)}-1}\left(2\,\sqrt{\lambda\,\widehat{S}}\right) \right\}^{-1}, \qquad \widehat{S} > 0, \tag{30}$$

where

$$I_\alpha(z) = \sum_{k=0}^{+\infty} \frac{1}{k!\;\Gamma(\alpha+k+1)} \left(\frac{z}{2}\right)^{\alpha+2\,k} \tag{31}$$

denotes the modified Bessel function of first kind. Hence, setting

$$\widehat{S} = \frac{t_1(S|r)}{2\,(1-\nu)},$$

in (30), one obtains:

$$\begin{aligned} &\widehat{g}_\lambda\Big(\frac{t_1(S|r)}{2\,(1-\nu)}\Big|0\Big) \\ &\quad = \left\{ \Gamma\Big(\frac{1}{2\,(1-\nu)}\Big) \left(\sqrt{\frac{\lambda\,t_1(S|r)}{2\,(1-\nu)}}\right)^{1-\frac{1}{2\,(1-\nu)}} I_{\frac{1}{2\,(1-\nu)}-1}\left(\sqrt{\frac{2\,\lambda\,t_1(S|r)}{1-\nu}}\right) \right\}^{-1}. \end{aligned} \tag{32}$$

Recalling (22) and (26), we now note that

$$\begin{aligned} \tilde{g}_\lambda(S|r) &:= U\big[-\lambda t_1(S|r)\big] \\ &= \left[{}_1F_2\Big(1; 1, \frac{1}{2\,(1-\nu)}; \frac{\lambda t_1(S|r)}{2\,(1-\nu)}\Big)\right]^{-1} \\ &= \left\{ 1 + \sum_{k=1}^{+\infty} \frac{1}{k!\left(\dfrac{1}{2\,(1-\nu)}\right)_k} \left(\sqrt{\frac{\lambda t_1(S|r)}{2\,(1-\nu)}}\right)^{2k} \right\}^{-1}. \end{aligned} \tag{33}$$

Since, from (31) one has:

$$1 + \sum_{k=1}^{+\infty} \frac{1}{k!\,(\alpha)_k}\, x^{2\,k} = \Gamma(\alpha)\, x^{1-\alpha}\, I_{\alpha-1}(2\,x),$$

the right-hand side of (33) is finally seen to identify with the right-hand side of (32). ■

Theorem 2.2 *Let $g_\lambda(S|r)$ be the Laplace transform of the FPT pdf from the state r to the state S $(S > r)$ for the process $X(t)$. Under the assumptions of Lemma 2.1 one has:*

$$\lim_{S\downarrow r} \frac{1 - g_\lambda(S|r)}{1 - \tilde{g}_\lambda(S|r)} = 1, \tag{34}$$

where $\tilde{g}_\lambda(S|r)$ is given in (29).

Proof: From (22) and (29) one has:

$$\tilde{g}_\lambda(S|r) = U\big[-\lambda t_1(S|r)\big] = 1 + \sum_{n=1}^{+\infty} \frac{(-\lambda)^n}{n!} \big[t_1(S|r)\big]^n u_n, \tag{35}$$

whereas from (2) and (3) one obtains:

$$g_\lambda(S|r) = 1 + \sum_{n=1}^{+\infty} \frac{(-\lambda)^n}{n!} t_n(S|r) \tag{36}$$

Hence, by making use of (35) and (36), one is led to the following equality:

$$\frac{1 - g_\lambda(S|r)}{1 - \tilde{g}_\lambda(S|r)} = \frac{\displaystyle\sum_{n=1}^{+\infty} \frac{(-\lambda)^n}{n!} \frac{t_n(S|r)}{[t_1(S|r)]^n} [t_1(S|r)]^n}{\displaystyle\sum_{n=1}^{+\infty} \frac{(-\lambda)^n}{n!} \big[t_1(S|r)\big]^n u_n} \;. \tag{37}$$

Taking the limit in (37) as $S \downarrow r$ and recalling (15), we conclude that (34) holds.

■

Corollary 2.3 *Under the assumptions of Lemma 2.1, as S approaches the boundary r the following asymptotic expression holds:*

$$g_\lambda(S|r) \simeq \left\{ \Gamma\left(\frac{1}{2(1-\nu)}\right) \left(\sqrt{\frac{\lambda t_1(S|r)}{2(1-\nu)}}\right)^{1-\frac{1}{2(1-\nu)}} I_{\frac{1}{2(1-\nu)}-1}\left(\sqrt{\frac{2\lambda t_1(S|r)}{1-\nu}}\right)\right\}^{-1} \tag{38}$$

Proof: Since (34) holds, the Laplace transform of $g(S,t|r)$ admits the following asymptotic representation

$$g_\lambda(S|r) \simeq \tilde{g}_\lambda(S|r)$$

when the threshold is in the neighborhood of the reflecting or entrance boundary. Hence, recalling (29) and (32), one immediately obtains (38).

■

2.2 Special cases

Proposition 2.1 *Under the assumptions of Lemma 2.1, if $\nu = 0$ one has:*

$$U(z) = \sec\sqrt{2z} \tag{39}$$

and

$$u_0 = 1, \qquad u_n = \frac{(-1)^n E_{2n}}{(2n-1)!!} \qquad (n = 1, 2, \dots), \tag{40}$$

where $E_0, E_2, \dots$ denote Euler numbers:

$$E_0 = 1, \qquad E_{2n} = -\sum_{j=0}^{n-1} \binom{2n}{2j} E_{2j} \qquad (n = 1, 2 \dots).$$

Furthermore,

$$\tilde{g}_\lambda(S|r) = \operatorname{sech}\sqrt{2\,\lambda\, t_1(S|r)}$$

and

$$\tilde{g}(S,t|r) = \sqrt{\frac{2\,t_1(S|r)}{\pi\, t^3}}\sum_{k=0}^{+\infty}(-1)^k(2\,k+1)\exp\Big\{-\frac{(2\,k+1)^2 t_1(S|r)}{2\,t}\Big\}. \tag{41}$$

Proof: Setting $\nu = 0$ in (26) and making use of identities

$$\Big(\frac{1}{2}\Big)_k = \frac{(2k-1)!!}{2^k}, \qquad (2\,k)! = 2^k\, k!\,(2k-1)!!\,,$$

one obtains:

$$U(z) = \left\{1+\sum_{k=1}^{+\infty}\frac{1}{k!\Big(\dfrac{1}{2}\Big)_k}\Big(-\frac{z}{2}\Big)^k\right\}^{-1} = \left\{1+\sum_{k=1}^{+\infty}\frac{(-z)^k}{k!\,(2k-1)!!}\right\}^{-1} = \left\{\sum_{k=0}^{+\infty}\frac{(-1)^k\,(2z)^k}{(2k)!}\right\}^{-1}. \tag{42}$$

Recalling that (cf., for instance, [8], pag. 34, n. 1.411.3)

$$\cos x = \sum_{k=0}^{+\infty}\frac{(-1)^k\, x^{2k}}{(2k)!},$$

(42) takes the following form:

$$U(z) = \Big\{\cos\sqrt{2z}\Big\}^{-1} \equiv \sec\sqrt{2z}.$$

Furthermore, since (cf., for instance, [1], pag. 75, n. 4.3.69)

$$\sec x = \sum_{n=0}^{+\infty}\frac{(-1)^n\, E_{2n}\, x^{2n}}{(2n)!} \qquad \Big(|x| < \frac{\pi}{2}\Big),$$

where E_{2n} denote Euler numbers, one also obtains:

$$U(z) = \sum_{n=0}^{+\infty}\frac{(-1)^n\, E_{2n}\,(2z)^n}{(2n)!} \qquad \Big(\big|\sqrt{2z}\big| < \frac{\pi}{2}\Big). \tag{43}$$

A comparison of (43) and the first of (22) shows that relations (40) hold. Furthermore, from (29) and (39) it follows

$$\tilde{g}_\lambda(S|r) = U\big[-\lambda t_1(S|r)\big] = \sec\sqrt{-2\lambda t_1(S|r)} \equiv \operatorname{sech}\sqrt{2\,\lambda\, t_1(S|r)}. \tag{44}$$

Since (see, for instance, [8], pag. 23 n. 1.232.2)

$$\operatorname{sech} x = 2\sum_{k=0}^{+\infty}(-1)^k\,\exp\{-(2\,k+1)\,x\} \qquad (x>0),$$

from (44) one has

$$\tilde{g}_\lambda(S|r) = 2\sum_{k=0}^{+\infty}(-1)^k\exp\Big\{-(2\,k+1)\,\sqrt{2\,\lambda\, t_1(S|r)}\Big\}. \tag{45}$$

Making use of the known formula

$$\int_0^{+\infty} e^{-\lambda\, t} \Big[\frac{\sqrt{\alpha}}{2\sqrt{\pi}}\, \frac{1}{t\sqrt{t}}\, e^{-\alpha/(4\,t)} \Big]\, dt = e^{-\sqrt{\alpha\,\lambda}} \qquad (\mathbf{Re}\ \alpha > 0),$$

(cf., for instance, [3], pag. 245, n. (1)), (41) follows from (45).

∎

Note that, since

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385,$$

from (40) in particular one has:

$$u_0 = u_1 = 1, \qquad u_2 = \frac{5}{3}\,, \qquad u_3 = \frac{61}{15}\,, \qquad u_4 = \frac{277}{21}\,,$$

that identifies with (27) for $\nu = 0$. Furthermore, making use of Proposition 2.1, if $\nu = 0$ from (19) and (34) one obtains as $S \downarrow r$ the following asymptotic expressions:

$$\begin{aligned} &t_n(S|r) \simeq \frac{(-1)^n\, E_{2\,n}}{(2\,n-1)!!}\, [t_1(S|r)]^n \qquad (n = 0, 1, \dots),\\ &g_\lambda(S|r) \simeq \operatorname{sech}\sqrt{2\,\lambda\, t_1(S|r)},\\ &g(S,t|r) \simeq \sqrt{\frac{2\,t_1(S|r)}{\pi\, t^3}} \sum_{k=0}^{+\infty} (-1)^k (2\,k+1) \exp\Big\{ -\frac{(2\,k+1)^2\, t_1(S|r)}{2\,t} \Big\}. \end{aligned}$$

Proposition 2.2 *Under the assumptions of Lemma 2.1, if* $\nu = 2/3$ *one has:*

(46) $$U(z) = \sqrt{6z}\, \csc\sqrt{6z}$$

and

(47) $$u_0 = 1, \qquad u_n = \frac{(-1)^{n+1}\, 3^n (2^{2n} - 2)\, B_{2\,n}}{(2\,n-1)!!} \qquad (n = 1, 2, \dots),$$

where $B_0, B_2, \dots$ *denote Bernoulli numbers:*

$$B_0 = 1, \qquad B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k \qquad (n = 1, 2, \dots)$$

Furthermore,

$$\tilde{g}_\lambda(S|r) = \sqrt{6\,\lambda\, t_1(S|r)}\, \operatorname{csch}\sqrt{6\,\lambda\, t_1(S|r)}$$

and

(48) $$\tilde{g}(S,t|r) = \sqrt{\frac{6\,t_1(S|r)}{\pi\, t^5}} \sum_{k=0}^{+\infty} \Big[3\,(2\,k+1)^2\, t_1(S|r) - t \Big]\, \exp\Big\{ -\frac{3\,(2\,k+1)^2\, t_1(S|r)}{2\,t} \Big\}.$$

Proof: Setting $\nu = 2/3$ in (26) and making use of identities

$$\Big(\frac{3}{2}\Big)_k = \frac{(2k+1)!!}{2^k}, \qquad (2\,k+1)! = 2^k\, k!\, (2k+1)!!\,,$$

one obtains:

$$U(z) = \left\{ 1 + \sum_{k=1}^{+\infty} \frac{1}{k! \left(\frac{3}{2}\right)_k} \left(-\frac{3\,z}{2} \right)^k \right\}^{-1} = \left\{ \sum_{k=0}^{+\infty} \frac{(-3z)^k}{k!\,(2k+1)!!} \right\}^{-1} = \left\{ \sum_{k=0}^{+\infty} \frac{(-1)^k\,(6z)^k}{(2k+1)!} \right\}^{-1}. \tag{49}$$

Recalling that (cf., for instance, [8], pag. 34, n. 1.411.1)

$$\sin x = \sum_{k=0}^{+\infty} \frac{(-1)^k\, x^{2k+1}}{(2k+1)!},$$

from (49) one has:

$$U(z) = \frac{\sqrt{6z}}{\sin\sqrt{6z}} = \sqrt{6z}\,\csc\sqrt{6z}.$$

Furthermore, since (cf., for instance, [1], pag. 75, n. 4.3.68)

$$\csc x = \frac{1}{x} - \frac{2}{x} \sum_{n=1}^{+\infty} \frac{(-1)^n\,\left(2^{2n-1}-1\right) B_{2n}\, x^{2n}}{(2n)!} \qquad \big(|x| < \pi\big),$$

where B_{2n} denote Bernoulli numbers, one also obtains:

$$U(z) = 1 + 2 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} 3^n \left(2^{2n-1}-1\right) B_{2n}\,(2z)^n}{(2n)!} \qquad \Big(6z < \pi^2\Big). \tag{50}$$

A comparison of (50) and the first of (22) shows that relations (47) hold. Furthermore, from (29) and (46) it follows

$$\begin{aligned} \tilde{g}_\lambda(S|r) &= U\big[-\lambda t_1(S|r)\big] = \sqrt{-6\,\lambda\, t_1(S|r)}\,\csc\sqrt{-6\,\lambda t_1(S|r)} \\ &\equiv \sqrt{6\,\lambda\, t_1(S|r)}\,\operatorname{csch}\sqrt{6\,\lambda\, t_1(S|r)}. \end{aligned} \tag{51}$$

Since (see, for instance, [8], pag. 23 n. 1.232.3)

$$\operatorname{csch} x = 2 \sum_{k=0}^{+\infty} \exp\{-(2\,k+1)\,x\} \qquad (x > 0),$$

from (51) one has

$$\tilde{g}_\lambda(S|r) = 2\,\sqrt{6\,\lambda\, t_1(S|r)} \sum_{k=0}^{+\infty} \exp\Big\{-(2\,k+1)\,\sqrt{6\,\lambda\, t_1(S|r)}\Big\}. \tag{52}$$

Making use of

$$\int_0^{+\infty} e^{-\lambda\, t} \left[\frac{1}{\sqrt{\pi\, t^5}} \left(\frac{\alpha}{4} - \frac{t}{2} \right) e^{-\alpha/(4\,t)} \right] dt = \sqrt{\lambda}\, e^{-\sqrt{\alpha\,\lambda}} \qquad (\mathbf{Re}\ \alpha > 0),$$

(cf., for instance, [3], pag. 246, n. (5)), from (52) one obtains (48).

■

Note that

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30},$$

while from (47) in particular one has:

$$u_0 = u_1 = 1, \qquad u_2 = \frac{7}{5}\,, \qquad u_3 = \frac{93}{35}\,, \qquad u_4 = \frac{1143}{175}\,,$$

that identifies with (27) for $\nu = 2/3$. Furthermore, making use of Proposition 2.2, if $\nu = 2/3$ from (19) and (34) one obtains as $S \downarrow r$ the following asymptotic formulas:

$$\begin{aligned}
&t_n(S|r) \simeq \frac{(-1)^{n+1}\, 3^n (2^{2\,n} - 2)\, B_{2\,n}}{(2\,n-1)!!}\, [t_1(S|r)]^n \qquad (n = 0, 1, \dots),\\
&g_\lambda(S|r) \simeq \sqrt{6\,\lambda\, t_1(S|r)}\ \mathrm{csch}\sqrt{6\,\lambda\, t_1(S|r)},\\
&g(S,t|r) \simeq \sqrt{\frac{6\, t_1(S|r)}{\pi\, t^5}} \sum_{k=0}^{+\infty} \Big[3\,(2\,k+1)^2\, t_1(S|r) - t\Big]\ \exp\Big\{-\frac{3\,(2\,k+1)^2\, t_1(S|r)}{2\,t}\Big\}.
\end{aligned}$$

3 The threshold moves indefinitely away from the boundary

In Sections 3.1 and 3.2 we analyze in two different cases the general behavior of the FPT pdf and of its moments when the threshold S moves indefinitely away from the reflecting or entrance boundary r, whereas in Section 3.3 we consider some special cases.

3.1 Case (a): General Considerations

Lemma 3.1 *If*

$$\lim_{x \uparrow r_2} \sqrt{A_2(x)}\, h(x) \int_r^x s(u)\, du = +\infty, \tag{53}$$

$$\lim_{x \uparrow r_2} \left[A_1(x) - \frac{A_2'(x)}{4}\right]\, h(x) \int_r^x s(u)\, du = \gamma, \tag{54}$$

where $-\infty < \gamma < 1$ is a real number, then

$$\lim_{S \uparrow r_2} \frac{\chi_n(S|r)}{[t_1(S|r)]^n} = \frac{1}{n! \displaystyle\prod_{i=0}^{n-1} [1 + 2\,i\,(1-\gamma)]} \qquad (n = 1, 2, \dots). \tag{55}$$

Proof: Since $\chi_1(S|r) = t_1(S|r)$, (55) is trivial for $n = 1$. Recalling that r_2 is a natural boundary, making use of (4) and (6), we obtain

$$\begin{aligned}
\lim_{S \uparrow r_2} \frac{\chi_k(S|r)}{[t_1(S|r)]^k} &= \lim_{S \uparrow r_2} \frac{\displaystyle\int_r^S h(z)\, dz \int_r^z s(u)\, \chi_{k-1}(u|r)\, du}{\left[\displaystyle\int_r^S h(z)\, dz \int_r^z s(u)\, du\right]^k}\\
&= \frac{1}{k} \lim_{S \uparrow r_2} \left\{ \frac{\chi_{k-1}(S|r)}{[t_1(S|r)]^{k-1}} \left[1 + \frac{k-1}{2}\, \frac{A_2(S)\, h^2(S) \left(\int_r^S s(u)\, du\right)^2}{t_1(S|r)}\right]^{-1} \right\} \qquad (56)\\
&\qquad\qquad (k = 2, 3, \dots).
\end{aligned}$$

where l'Hospital's rule has been used repeatedly. We note that setting $k = 2$ in (56) one has:

$$\lim_{S\uparrow r_2} \frac{\chi_2(S|r)}{[t_1(S|r)]^2} = \frac{1}{2} \lim_{S\uparrow r_2} \left[1 + \frac{1}{2}\, \frac{A_2(S)\, h^2(S) \left(\int_r^S s(u)\; du\right)^2}{t_1(S|r)}\right]^{-1}. \tag{57}$$

We now proceed by induction. Since (53) holds, by applying l'Hospital's rule one obtains:

$$\begin{aligned} \lim_{S\uparrow r_2} \frac{A_2(S)\, h^2(S) \left(\int_r^S s(u)\; du\right)^2}{t_1(S|r)} &= 4 \lim_{S\uparrow r_2} \left\{1 - \left(A_1(S) - \frac{A_2'(S)}{4}\right) h(S) \int_r^S s(u)\; du\right\} \\ &= 4\,(1-\gamma), \end{aligned} \tag{58}$$

where the last equality follows from (54). Hence, making use of (58) in (57), one has

$$\lim_{S\uparrow r_2} \frac{\chi_2(S|r)}{[t_1(S|r)]^2} = \frac{1}{2\,[1 + 2\,(1-\gamma)]}\,,$$

that shows that (55) is satisfied for $n = 2$. Let us now assume that (55) holds for n and prove that it also holds for $n + 1$. Indeed, from (56) and (58) we have

$$\lim_{S\uparrow r_2} \frac{\chi_{n+1}(S|r)}{[t_1(S|r)]^{n+1}} = \frac{1}{(n+1)\,[1 + 2\,n\,(1-\gamma)]} \lim_{S\uparrow r_2} \frac{\chi_n(S|r)}{[t_1(S|r)]^n} = \frac{1}{(n+1)! \displaystyle\prod_{i=0}^{n} [1 + 2\,i\,(1-\gamma)]}\,.$$

This completes the proof.

∎

Theorem 3.1 *Under the assumptions of Lemma 3.1 one has:*

$$\lim_{S\uparrow r_2} \frac{t_n(S|r)}{[t_1(S|r)]^n} = v_n \qquad (n = 0, 1, \dots), \tag{59}$$

where

$$v_0 = 1, \quad v_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{\displaystyle\prod_{i=0}^{k-1} [1 + 2\,i\,(1-\gamma)]}\, v_{n-k} \qquad (n = 1, 2, \dots). \tag{60}$$

Proof: Equation (59) holds with $v_0 = v_1 = 1$ for $n = 0, 1$. For $n = 2, 3, \dots$, taking the limit as $S \uparrow r_2$ in (17) and making use of (55), one obtains:

$$\begin{aligned} v_n &= n! \sum_{j=1}^{n-1} \frac{(-1)^{n-j-1}\, v_j}{j!\,(n-j)! \displaystyle\prod_{i=0}^{n-j-1} [1 + 2\,i\,(1-\gamma)]} + \frac{(-1)^{n-1}}{\displaystyle\prod_{i=0}^{n-1} [1 + 2\,i\,(1-\gamma)]} \\ &= \sum_{j=0}^{n-1} \binom{n}{j} \frac{(-1)^{n-j-1}}{\displaystyle\prod_{i=0}^{n-j-1} [1 + 2\,i\,(1-\gamma)]}\, v_j \qquad (n = 2, 3, \dots). \end{aligned} \tag{61}$$

Setting $k = n - j$, the right-hand side of (61) is finally seen to identify with the right-hand side of (60) for $n = 2, 3, \dots$ This completes the proof.

∎

Corollary 3.1 *Under the assumptions of Lemma 3.1, as S approaches the boundary r_2, the following asymptotic expressions hold:*

$$t_n(S|r) \simeq [t_1(S|r)]^n \, v_n \qquad (n = 0, 1, \dots),$$

where v_n are defined in (60).

Proof: It follows immediately from (59).

■

Corollary 3.2 *Let v_n $(n = 0, 1, \dots)$ be defined as in (60). Then, $v_0 = 1$ and*

$$v_n = (-1)^n \, n! \, \det \begin{pmatrix} d_1 & 1 & 0 & 0 & \dots & 0 \\ d_2 & d_1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n-1} & d_{n-2} & d_{n-3} & d_{n-4} & \dots & 1 \\ d_n & d_{n-1} & d_{n-2} & d_{n-3} & \dots & d_1 \end{pmatrix} \qquad (n = 1, 2, \dots) \tag{62}$$

where

$$d_k = \frac{(-1)^k}{k! \displaystyle\prod_{i=0}^{k-1} [1 + 2\, i\, (1-\gamma)]} \qquad (k = 1, 2, \dots).$$

Furthermore, the generating function of $v_n/n!$ is given by

$$V(z) := 1 + \sum_{n=1}^{+\infty} \frac{v_n}{n!} \, z^n = \left\{ 1 + \sum_{k=1}^{+\infty} \frac{1}{k! \left(\dfrac{1}{2\,(1-\gamma)} \right)_k} \left[-\frac{z}{2\,(1-\gamma)} \right]^k \right\}^{-1}$$

$$= \left[{}_1F_2\left(1; 1, \frac{1}{2\,(1-\gamma)}; -\frac{z}{2\,(1-\gamma)}\right) \right]^{-1}. \tag{63}$$

where ${}_1F_2(a; b, c; x)$ denotes the generalized hypergeometric series.

Proof: It follows by arguments similar to those of Corollary 2.2.

■

In particular, from (62) one obtains:

$$v_0 = v_1 = 1, \qquad v_2 = \frac{1 + 4\,(1-\gamma)}{1 + 2\,(1-\gamma)}, \qquad v_3 = \frac{1 + 12\,(1-\gamma) + 48\,(1-\gamma)^2}{[1 + 2\,(1-\gamma)]\,[1 + 4\,(1-\gamma)]},$$

$$v_4 = \frac{1 + 26\,(1-\gamma) + 288\,(1-\gamma)^2 + 1536\,(1-\gamma)^3 + 2304\,(1-\gamma)^4}{[1 + 2\,(1-\gamma)]^2\,[1 + 4\,(1-\gamma)]\,[1 + 6\,(1-\gamma)]}.$$

Theorem 3.2 *Under the assumptions of Lemma 3.1 one has:*

$$\lim_{S \uparrow r_2} \frac{1 - g_\lambda(S|r)}{1 - \tilde{\tilde{g}}_\lambda(S|r)} = 1, \tag{64}$$

where

$$\tilde{\tilde{g}}_\lambda(S|r) := V\left[-\lambda t_1(S|r)\right]$$

$$= \left\{ \Gamma\left(\frac{1}{2\,(1-\gamma)}\right) \left(\sqrt{\frac{\lambda\, t_1(S|r)}{2\,(1-\gamma)}} \right)^{1 - \frac{1}{2\,(1-\gamma)}} I_{\frac{1}{2\,(1-\gamma)} - 1} \left(\sqrt{\frac{2\,\lambda\, t_1(S|r)}{1-\gamma}} \right) \right\}^{-1}. \tag{65}$$

Proof: Proceeding along the lines indicated in Lemma 2.2, the function $V\left[-\lambda t_1(S|r)\right]$ is seen to identify with the Laplace transform of the FPT pdf from state 0 to state $\widehat{S} = t_1(S|r)/[2(1-\gamma)]$ for a time–homogeneous diffusion process defined in $[0,+\infty)$ and characterized by drift $C_1(x) = [2\,(1-\gamma)]^{-1}$ and infinitesimal variance $C_2(x) = 2\,x$, with $-\infty < \gamma < 1$. Furthermore, from (63) and (65) one obtains:

$$\tilde{\tilde{g}}_\lambda(S|r) = V\left[-\lambda t_1(S|r)\right] = 1 + \sum_{n=1}^{+\infty} \frac{(-\lambda)^n}{n!}\left[t_1(S|r)\right]^n v_n. \tag{66}$$

Hence, making use of (36) and (66) it follows:

$$\frac{1-g_\lambda(S|r)}{1-\tilde{\tilde{g}}_\lambda(S|r)} = \frac{\displaystyle\sum_{n=1}^{+\infty} \frac{(-\lambda)^n}{n!}\,\frac{t_n(S|r)}{[t_1(S|r)]^n}\,[t_1(S|r)]^n}{\displaystyle\sum_{n=1}^{+\infty} \frac{(-\lambda)^n}{n!}\left[t_1(S|r)\right]^n v_n}\,. \tag{67}$$

Taking the limit in (67) as $S \uparrow r_2$ and recalling (59), we conclude that (64) holds.

■

Corollary 3.3 *Under the assumptions of Lemma 3.1, as* S *approaches the boundary* r_2*, the following asymptotic expression holds:*

$$g_\lambda(S|r) \simeq \left\{\Gamma\left(\frac{1}{2(1-\gamma)}\right)\left(\sqrt{\frac{\lambda t_1(S|r)}{2(1-\gamma)}}\right)^{1-\frac{1}{2(1-\gamma)}} I_{\frac{1}{2(1-\gamma)}-1}\left(\sqrt{\frac{2\lambda t_1(S|r)}{1-\gamma}}\right)\right\}^{-1}$$

Proof: It is a consequence of (64).

■

3.2 Case (b): General Considerations

Lemma 3.2 *If*

$$\lim_{x\uparrow r_2} \sqrt{A_2(x)}\,h(x)\int_r^x s(u)\,du = +\infty \tag{68}$$

$$\lim_{x\uparrow r_2} \left[A_1(x) - \frac{A_2'(x)}{4}\right] h(x)\int_r^x s(u)\,du = -\infty, \tag{69}$$

then

$$\lim_{S\uparrow r_2} \frac{\chi_n(S|r)}{[t_1(S|r)]^n} = \begin{cases} 1, & n=1 \\ 0, & n=2,3,\ldots\,. \end{cases} \tag{70}$$

Proof: Since $\chi_1(S|r) = t_1(S|r)$, (70) is trivial for $n=1$. Moreover, relations (56) and (57) hold. By virtue of (68), and by applying l'Hospital's rule, one obtains:

$$\lim_{S\uparrow r_2} \frac{A_2(S)\,h^2(S)\left(\int_r^S s(u)\,du\right)^2}{t_1(S|r)} = 4\lim_{S\uparrow r_2}\left\{1-\left(A_1(S)-\frac{A_2'(S)}{4}\right)h(S)\int_r^S s(u)\,du\right\} = +\infty, \tag{71}$$

where the last equality follows from (69). Hence, making use of (71) in (57), one has

$$\lim_{S\uparrow r_2} \frac{\chi_2(S|r)}{[t_1(S|r)]^2} = 0\,,$$

that shows that (70) is satisfied for $n = 2$. Let us now assume that (70) holds for n and prove that it also holds for $n+1$. Indeed, this follows from (56) and (71) since

$$\lim_{S\uparrow r_2} \frac{\chi_{n+1}(S|r)}{[t_1(S|r)]^{n+1}} = 0.$$

This completes the proof.

■

Theorem 3.3 *Under the assumptions of Lemma 3.2 one has:*

$$\lim_{S\uparrow r_2} \frac{t_n(S|r)}{[t_1(S|r)]^n} = n! \qquad (n = 0, 1, \dots). \tag{72}$$

Proof: Equation (72) holds for $n = 0, 1$. We now set:

$$b_0 = b_1 = 1, \qquad b_n := \lim_{S\uparrow r_2} \frac{t_n(S|r)}{[t_1(S|r)]^n} \qquad (n = 2, 3, \dots). \tag{73}$$

Taking the limit as $S \uparrow r_2$ in (17) and making use of (70), one obtains:

$$b_n = \frac{n!}{(n-1)!}\, b_{n-1} \qquad (n = 1, 2, \dots),$$

that implies $b_n = n!$ for $n = 0, 1, \dots$ Hence, (72)'s follow from (73).

■

Corollary 3.4 *Under the assumptions of Lemma 3.2, as S approaches the boundary r_2 one has:*

$$t_n(S|r) \simeq n!\, [t_1(S|r)]^n \qquad (n = 0, 1, \dots).$$

Proof: It follows from (72).

■

Theorem 3.4 *Under the assumptions of Lemma 3.2 one has:*

$$\lim_{S\uparrow r_2} \frac{1 - g_\lambda(S|r)}{1 - [1 + \lambda\, t_1(S|r)]^{-1}} = 1\,. \tag{74}$$

Proof: Making use of (36), one obtains:

$$\frac{1 - g_\lambda(S|r)}{1 - [1 + \lambda\, t_1(S|r)]^{-1}} = \frac{\displaystyle\sum_{n=1}^{+\infty} \frac{(-\lambda)^n}{n!}\, \frac{t_n(S|r)}{[t_1(S|r)]^n}\, [t_1(S|r)]^n}{\displaystyle\sum_{n=1}^{+\infty} \frac{(-\lambda)^n}{n!}\, n!\, \big[t_1(S|r)\big]^n}\,. \tag{75}$$

Taking the limit in (75) as $S \uparrow r_2$ and recalling (72), we conclude that (74) holds.

■

Corollary 3.5 *Under the assumptions of Lemma 3.2, as S approaches the boundary r_2 the following asymptotic expressions hold:*

$$g_\lambda(S|r) \simeq [1+\lambda\, t_1(S|r)]^{-1}$$
$$g(S,t|r) \simeq \frac{1}{t_1(S|r)}\, \exp\Big\{-\frac{t}{t_1(S|r)}\Big\}.$$

Proof: They follow from (74).

■

3.3 Special cases

By arguments similar to those of Proposition 2.1, under the assumptions of Lemma 3.1, if $\gamma = 0$ one obtains the following asymptotic expressions as $S \uparrow r_2$:

$$t_n(S|r) \simeq \frac{(-1)^n\, E_{2\,n}}{(2\,n-1)!!}\,[t_1(S|r)]^n \qquad (n=0,1,\dots),$$
$$g_\lambda(S|r) \simeq \operatorname{sech}\sqrt{2\,\lambda\, t_1(S|r)},$$
$$g(S,t|r) \simeq \sqrt{\frac{2\,t_1(S|r)}{\pi\, t^3}}\,\sum_{k=0}^{+\infty}(-1)^k(2\,k+1)\exp\Big\{-\frac{(2\,k+1)^2\, t_1(S|r)}{2\,t}\Big\},$$

where $E_0, E_2, \dots$ denote Euler numbers. Furthermore, by arguments similar to those of Proposition 2.2, under the assumptions of Lemma 3.1, if $\gamma = 2/3$ one obtains the following asymptotic expressions as $S \uparrow r_2$:

$$t_n(S|r) \simeq \frac{(-1)^{n+1}\, 3^n (2^{2\,n}-2)\, B_{2\,n}}{(2\,n-1)!!}\,[t_1(S|r)]^n \qquad (n=0,1,\dots),$$
$$g_\lambda(S|r) \simeq \sqrt{6\,\lambda\, t_1(S|r)}\,\operatorname{csch}\sqrt{6\,\lambda\, t_1(S|r)},$$
$$g(S,t|r) \simeq \sqrt{\frac{6\,t_1(S|r)}{\pi\, t^5}}\,\sum_{k=0}^{+\infty}\Big[3\,(2\,k+1)^2\, t_1(S|r) - t\Big]\,\exp\Big\{-\frac{3\,(2\,k+1)^2\, t_1(S|r)}{2\,t}\Big\}.$$

4 Some closed form results

In this Section we restrict our attention to a particular class of time-homogeneous diffusion processes defined over the interval $I \equiv [r, r_2)$, where r is a regular or an entrance boundary and r_2 is a natural boundary. For these processes we shall prove that the transition pdf, the Laplace transform of the FPT pdf and its moments can be explicitly obtained in terms of the mean first passage time. Some of the considerations to follow are heavily based on arguments in [7].

Definition 4.1 *A diffusion process $\{X(t),\, t \ge 0\}$ defined over the interval $I \equiv [r, r_2)$ will be said to be BF*[1] *if for all x inside I its infinitesimal moments $A_1(x)$ and $A_2(x)$ satisfy the following conditions:*

$$\lim_{a\downarrow r}\int_a^x \frac{dz}{\sqrt{A_2(z)}} < +\infty, \qquad \lim_{b\uparrow r_2}\int_x^b \frac{dz}{\sqrt{A_2(z)}} = +\infty, \tag{76}$$

$$A_1(x) = \frac{A_2'(x)}{4} + \frac{\nu}{2\,(1-\nu)}\;\frac{\sqrt{A_2(x)}}{\displaystyle\int_r^x \frac{dz}{\sqrt{A_2(z)}}}\,, \tag{77}$$

where $-\infty < \nu < 1$ is a real number.

[1] This stands for Bessel-Feller, as made clear in the sequel.

Remark 4.1 *Let $-\infty < \nu < 1$ be a real number. Conditions (76) and (77) are equivalent to*

$$\lim_{x\downarrow r} \sqrt{A_2(x)}\, h(x) \int_r^x s(u)\, du = 0, \qquad \lim_{x\uparrow r_2} \sqrt{A_2(x)}\, h(x) \int_r^x s(u)\, du = +\infty, \tag{78}$$

$$\Big[A_1(x) - \frac{A_2'(x)}{4}\Big]\, h(x) \int_r^x s(u)\, du = \nu, \tag{79}$$

Proof: First of all, we shall prove that (78) and (79) hold if the assumptions (76) and (77) are satisfied. Indeed, from (77) one has:

$$\frac{2\, A_1(x)}{A_2(x)} = \frac{A_2'(x)}{2\, A_2(x)} + \frac{\nu}{1-\nu}\, \frac{1}{\sqrt{A_2(x)} \displaystyle\int_r^x \frac{dz}{\sqrt{A_2(z)}}}$$

that, recalling (1), implies

$$h(x) = \frac{c}{\sqrt{A_2(x)}} \left[\int_r^x \frac{dz}{\sqrt{A_2(z)}}\right]^{-\nu/(1-\nu)}$$

$$s(x) = \frac{2}{c\, \sqrt{A_2(x)}} \left[\int_r^x \frac{dz}{\sqrt{A_2(z)}}\right]^{\nu/(1-\nu)}.$$

where $c > 0$ is an arbitrary constant. Since $-\infty < \nu < 1$ and (76) hold, one has:

$$h(x) \int_r^x s(u)\, du = \frac{2\,(1-\nu)}{\sqrt{A_2(x)}} \int_r^x \frac{du}{\sqrt{A_2(u)}}\ . \tag{80}$$

Hence, making use of (76), (77) and (80), one is immediately led to (78) and (79).

We now prove that (78) and (79) imply (76) and (77). We note that from (1) it follows:

$$A_1(x)\, h(x) \int_r^x du\, s(u) = -\frac{A_2(x)}{2}\, \frac{d}{dx}\Big[h(x) \int_r^x s(u)\, du\Big] + 1. \tag{81}$$

Making use of (81) in (79) one obtains:

$$\frac{d}{dx}\left\{\sqrt{A_2(x)}\, h(x) \int_r^x s(u)\, du\right\} = \frac{2\,(1-\nu)}{\sqrt{A_2(x)}}\,,$$

or, equivalently, since (78) holds:

$$h(x) \int_r^x s(u)\, du = \frac{2\,(1-\nu)}{\sqrt{A_2(x)}} \int_r^x \frac{dz}{\sqrt{A_2(z)}}\ . \tag{82}$$

Hence, making use of (78), (79) and (82), one immediately obtains (76) and (77).

■

Note that the first of (78) correspond to (9) of Lemma 2.1, whereas the second of (78) correspond to (53) of Lemma 3.1. Furthermore, (79) implies that (10) of Lemma 2.1 holds and also that (54) of Lemma 3.1 is satisfied with $\gamma = \nu$.

Under the assumptions (76) and (77), it is possible to show that a BF process can be transformed into a Bessel process and also into a Feller process.

Proposition 4.1 *Let $\{X(t),\, t \geq 0\}$ be BF and let*

$$Y(t) = \int_r^{X(t)} \frac{dz}{\sqrt{A_2(z)}}\ . \tag{83}$$

Then, $\{Y(t),\, t \geq 0\}$ is a Bessel process defined in $[0, +\infty)$ and characterized by drift and infinitesimal variance

$$B_1(y) = \frac{\nu}{2(1-\nu)\, y}\,, \qquad B_2(y) = 1. \tag{84}$$

Proof: We first note that a BF process is defined by the Stratonovich stochastic equation (see, for instance, [2]):

$$\frac{dx(t)}{dt} = \Big[A_1(x) - \frac{A_2'(x)}{4}\Big] + \sqrt{A_2(x)}\;\Lambda(t) = \frac{\nu}{2\,(1-\nu)}\;\frac{\sqrt{A_2(x)}}{\displaystyle\int_r^x \frac{dz}{\sqrt{A_2(z)}}} + \sqrt{A_2(x)}\;\Lambda(t), \tag{85}$$

where $\Lambda(t)$ is a zero-mean, delta-correlated stationary normal process having unit intensity (white noise). From (83) and (85) one obtains

$$\frac{dy(t)}{dt} = \frac{1}{\sqrt{A_2(x)}}\;\frac{dx(t)}{dt} = \frac{\nu}{2\,(1-\nu)\,y} + \Lambda(t),$$

which is the Stratonovich stochastic equation of the Bessel process defined by (84).

■

Proposition 4.2 *Let $\{X(t),\, t \geq 0\}$ be BF and let*

$$Z(t) = \frac{1}{2}\left(\int_r^{X(t)} \frac{dz}{\sqrt{A_2(z)}}\right)^2. \tag{86}$$

Then, $\{Z(t),\, (t \geq 0\}$ is a Feller process defined in $[0, +\infty)$ and characterized by drift and infinitesimal variance (28).

Proof: By making use of (85) and (86) one obtains

$$\frac{dz(t)}{dt} = \frac{1}{\sqrt{A_2(x)}}\;\int_r^x \frac{dz}{\sqrt{A_2(z)}}\;\frac{dx(t)}{dt} = \frac{\nu}{2\,(1-\nu)} + \sqrt{2\,z}\;\Lambda(t),$$

which is the Stratonovich stochastic equation of the Feller process defined in (28).

■

As is well known (see, for instance, [9]) for both diffusion processes $Y(t)$ and $Z(t)$ characterized by (84) and (28), respectively, boundary 0 is regular for $\nu < 1/2$ and entrance for $1/2 \leq \nu < 1$, whereas boundary $+\infty$ is natural. Hence, for the BF process $X(t)$ boundary r is regular for $\nu < 1/2$ and entrance for $1/2 \leq \nu < 1$, whereas boundary r_2 is natural. From Proposition 1.1 it follows that for all S inside of I and for all x_0 such that $r \leq x_0 < S$, $P(S|x_0) = 1$ and the moments $t_n(S|x_0)$ $(n = 1, 2, \dots)$ are finite.

Theorem 4.1 *Let $\{X(t),\, t \geq 0\}$ be BF. For all S in I and for all x_0 such that $r \leq x_0 < S$ one has*

$$t_1(S|x_0) = (1-\nu)\left[\left(\int_r^S \frac{dz}{\sqrt{A_2(z)}}\right)^2 - \left(\int_r^{x_0} \frac{dz}{\sqrt{A_2(z)}}\right)^2\right]. \tag{87}$$

Proof: Substituting (80) in (4) with $n = 1$ one obtains:

$$t_1(S|x_0) = \int_{x_0}^S dz\; h(z) \int_r^z s(u)\; du = 2(1-\nu) \int_{x_0}^S \frac{1}{\sqrt{A_2(z)}} \left(\int_r^z \frac{d\xi}{\sqrt{A_2(\xi)}}\right) dz$$

that immediately leads to (87).

■

Proposition 4.3 *For a BF process, the transition pdf in the presence of a reflecting (regular boundary) or a zero-flux condition (entrance boundary) at r, is given by*

$$\begin{aligned} f(x,t|x_0) &= \frac{1}{t}\sqrt{\frac{t_1(x|r)}{(1-\nu)\,A_2(x)}}\left[\frac{t_1(x|r)}{t_1(x_0|r)}\right]^{\frac{2\nu-1}{4(1-\nu)}} \exp\left\{-\frac{t_1(x|r)+t_1(x_0|r)}{2\,(1-\nu)\,t}\right\} \\ &\times I_{\frac{2\nu-1}{2(1-\nu)}}\left(\frac{\sqrt{t_1(x|r)t_1(x_0|r)}}{(1-\nu)\,t}\right), \end{aligned} \tag{88}$$

where $I_\alpha(z)$ denotes the modified Bessel function of first kind. Furthermore, the LT of the FPT pdf is given by

$$g_\lambda(S|x_0) = \left[\frac{t_1(S|r)}{t_1(x_0|r)}\right]^{\frac{2\nu-1}{4(1-\nu)}} \frac{I_{\frac{2\nu-1}{2(1-\nu)}}\left(\sqrt{\dfrac{2\,\lambda\, t_1(x_0|r)}{1-\nu}}\right)}{I_{\frac{2\nu-1}{2(1-\nu)}}\left(\sqrt{\dfrac{2\,\lambda\, t_1(S|r)}{1-\nu}}\right)} \qquad (r \leq x_0 < S < r_2). \tag{89}$$

Proof: Let $Z(t)$ be the Feller diffusion process defined in (28). Its transition pdf $f_F(\widehat{x},t|\widehat{x}_0)$ in the presence of a reflecting condition ($\nu < 1/2$) or of a zero-flux condition ($1/2 \leq \nu < 1$) at $\widehat{x} = 0$ is given by (see, for instance, [5]):

$$f_F(\widehat{x},t|\widehat{x}_0) = \frac{1}{t}\left(\frac{\widehat{x}}{\widehat{x}_0}\right)^{\frac{2\nu-1}{4(1-\nu)}} \exp\left\{-\frac{\widehat{x}+\widehat{x}_0}{t}\right\} I_{\frac{2\nu-1}{2(1-\nu)}}\left(2\frac{\sqrt{\widehat{x}\,\widehat{x}_0}}{t}\right). \tag{90}$$

Recalling (86) and (87) and making use of transformation

$$\widehat{x} = \frac{t_1(x|r)}{2\,(1-\nu)}, \qquad \widehat{x}_0 = \frac{t_1(x_0|r)}{2\,(1-\nu)}, \qquad f(x,t|x_0) = \frac{1}{2\,(1-\nu)}\,\frac{dt_1(x|r)}{dx}\, f_F(\widehat{x},t|\widehat{x}_0),$$

from (90) relation (88) follows. Recalling that (30) holds, the Laplace transform $\widehat{g}_\lambda(\widehat{S}|\widehat{x}_0)$ of the FPT pdf of Feller process when a reflecting condition is set at the regular boundary $x = 0$ ($\nu < 1/2$) or when a zero-flux condition is set at the entrance boundary $x = 0$ ($1/2 \leq \nu < 1$), is given by:

$$\widehat{g}_\lambda(\widehat{S}|\widehat{x}_0) = \frac{\widehat{g}_\lambda(\widehat{S}|0)}{\widehat{g}_\lambda(\widehat{x}_0|0)} = \left(\frac{\widehat{x}_0}{\widehat{S}}\right)^{\frac{1}{2}-\frac{1}{4\,(1-\nu)}} \frac{I_{\frac{1}{2\,(1-\nu)}-1}\left(2\sqrt{\lambda\,\widehat{x}_0}\right)}{I_{\frac{1}{2\,(1-\nu)}-1}\left(2\sqrt{\lambda\,\widehat{S}}\right)} \tag{91}$$

where $0 \le \widehat{x}_0 < \widehat{S}$. Hence, making use of transformation

$$\widehat{S} = \frac{t_1(S|r)}{2\,(1-\nu)}\,, \qquad \widehat{x}_0 = \frac{t_1(x_0|r)}{2\,(1-\nu)}\,, \qquad g(S,t|x_0) = \widehat{g}(\widehat{S},t|\widehat{x}_0),$$

from (91) relation (89) immediately follows.

■

Theorem 4.2 *If $x_0 = r$, the FPT–moments of a BF process are given by*

$$t_n(S|r) = u_n\,\big[t_1(S|r)\big]^n \qquad (n=0,1,\dots), \tag{92}$$

where $u_0, u_1, \dots$ are recursively defined in (16).

Proof: If $x_0 = r$, from (22) and (29) one has:

$$g_\lambda(S|r) = \widehat{g}_\lambda(\widehat{S}|0) \equiv U\big[-\lambda t_1(S|r)\big] = 1 + \sum_{n=1}^{+\infty} \frac{u_n}{n!}\,\big[-\lambda t_1(S|r)\big]^n. \tag{93}$$

Comparing the right-hand side of (93) with the right-hand side of (36), relations (92) follow.

■

Theorem 4.3 *If $r \le x_0 < S < r_2$, the FPT–moments of a BF process are given by*

$$\begin{aligned} t_n(S|x_0) = \Bigg\{ u_n + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{u_{n-k}}{\prod_{i=0}^{k-1}[1+2\,i\,(1-\nu)]} \\ \times \left[\frac{t_1(x_0|r)}{t_1(S|r)}\right]^k \Bigg\}\, \big[t_1(S|r)\big]^n \qquad (n=2,3,\dots). \end{aligned} \tag{94}$$

Proof: The case $x_0 = r$ has already been proved in (92). Let us now consider the case $r < x_0 < S < r_2$. From (33) it follows

$$\begin{aligned} \frac{1}{g_\lambda(x_0|r)} \equiv \frac{1}{U\big[-\lambda t_1(x_0|r)\big]} &= 1 + \sum_{k=1}^{+\infty} \frac{1}{k!\,\left(\frac{1}{2(1-\nu)}\right)_k} \left[\frac{\lambda\,t_1(x_0|r)}{2\,(1-\nu)}\right]^k \\ &= 1 + \sum_{k=1}^{+\infty} \frac{[\lambda\,t_1(x_0|r)]^k}{k!\,\prod_{i=0}^{k-1}[1+2\,i\,(1-\nu)]}\,. \end{aligned} \tag{95}$$

Hence, making use of (93) and (95), one obtains:

$$\begin{aligned} g_\lambda(S|x_0) = \frac{g_\lambda(S|r)}{g_\lambda(x_0|r)} = 1 - \lambda\,t_1(S|x_0) + \sum_{n=2}^{+\infty} \frac{[-\lambda\,t_1(S|r)]^n}{n!} \\ \times \Bigg\{ u_n + \sum_{k=1}^{n} \binom{n}{k} \frac{u_{n-k}}{\prod_{i=0}^{k-1}[1+2\,i\,(1-\nu)]} \left[-\frac{t_1(x_0|r)}{t_1(S|r)}\right]^k \Bigg\}, \end{aligned} \tag{96}$$

whereas from (2) and (3) one has:

$$g_\lambda(S|x_0) = 1 + \sum_{n=1}^{+\infty} \frac{(-\lambda)^n}{n!}\, t_n(S|x_0). \tag{97}$$

Comparing the right-hand side of (96) with the right-hand side of (97), relations (94) follow. ■

A few examples of BF processes are the following:

(i) (*Wiener process*)

$$A_1 = 0 \qquad A_2 = \sigma^2, \qquad I = [r, +\infty), \qquad (\sigma > 0,\ r \in \mathbf{R})$$

The boundary $x = r$ is regular and (77) holds with $\nu = 0$.

(ii) (*Lognormal process*)

$$A_1(x) = \frac{\sigma^2\, x}{2} \qquad A_2(x) = \sigma^2\, x^2, \qquad I = [r, +\infty), \qquad (\sigma > 0,\ r > 0).$$

The boundary $x = r$ is regular and (77) holds with $\nu = 0$.

(iii) (*Feller process*)

$$A_1 = q \qquad A_2(x) = 2\,\xi\, x, \qquad I = [0, +\infty), \qquad (q > 0,\, \xi > 0).$$

The boundary $x = 0$ is regular if $0 < q < \xi$ and entrance if $q \geq \xi$, and (77) holds with $\nu = 1 - \xi/(2q)$.

(iv) (*Bessel process*)

$$A_1(x) = \frac{a}{x} \qquad A_2 = \sigma^2, \qquad I = [0, +\infty), \qquad (a > -\,\sigma^2/2,\, \sigma > 0).$$

The boundary $x = 0$ is regular if $-\sigma^2/2 < a < \sigma^2/2$ and entrance if $a \geq \sigma^2/2$, and (77) holds with $\nu = 2a/(2a + \sigma^2)$.

(v) (*Polynomial process*)

$$A_1(x) = \mu\, x^\alpha \qquad A_2(x) = \beta^2 x^{\alpha+1}, \qquad I = [0, +\infty) \qquad (\alpha < 1,\, \mu > \alpha\, \beta^2/2,\, \beta > 0).$$

The boundary $x = 0$ is regular if $\alpha\,\beta^2/2 < \mu < \beta^2/2$ and entrance if $\mu \geq \beta^2/2$, and (77) holds with

$$\nu = \frac{2}{\beta^2}\Big(\mu - \frac{\alpha+1}{4}\,\beta^2\Big)\Big(\frac{2\mu}{\beta^2} - \alpha\Big)^{-1}.$$

Finally, we note that the polynomial process *(v)* identifies with process *(i)* if $\mu = 0$, $\alpha = -1$, $\beta^2 = \sigma^2$, $r = 0$, with process *(iii)* if $\mu = q$, $\alpha = 0$, $\beta^2 = 2\,\xi$ and with process *(iv)* if $\mu = a$, $\alpha = -1$, $\beta^2 = \sigma^2$. On the contrary, it cannot include process *(ii)* since this is defined for $r > 0$.

5 Neuronal models

The purpose of this Section is to point out certain theoretical and computational results obtained by us in order to provide a quantitative description of the input-output behavior of single neurons subject to a diffusion-like dynamics. We shall assume that the neuron's membrane potential is modeled by a one-dimensional diffusion process $X(t)$ starting at a point $x_0 \in (r, r_2)$, where r is a regular or entrance boundary and $r_2 = +\infty$ is a natural boundary. The threshold potential, denoted by $S > x_0$, will be assumed to be a deterministic continuous function of time so that T, the FPT through S, is the theoretical counterpart of the interspike interval. We recall that the importance of interspike intervals is due to the generally accepted hypothesis that the information transferred within the nervous system is usually encoded by the timing of occurrence of neuronal spikes. Hence, the determination of the firing pdf for a neuron modeled by a diffusion process $X(t)$ is an FPT problem in which the unknown is the FPT pdf through a preassigned threshold S.

5.1 Wiener model

The year 1964 marks the beginning of the history of neuronal models based on diffusion processes. In a much celebrated article, Gerstein and Mandelbrot [4] at that time postulated that for a number of experimentally monitored neurons subject to spontaneous activity the firing pdf could be modeled by the FPT pdf of a Wiener process. Indeed, these authors were able to show that, by suitably choosing the parameters of the model, the experimentally recorded interspike intervals histograms of many units could be fitted to an excellent degree of approximation by means of the FPT pdf of a Wiener process, i.e. of the time homogeneous diffusion process defined in $I = (-\infty, +\infty)$ and characterized by the constant infinitesimal moments

$$A_1(x) = \mu \qquad A_2(x) = \sigma^2, \qquad (\mu \in \mathbf{R}, \sigma > 0). \tag{98}$$

S	$t_1(S\|r)$	$t_2(S\|r)$	$m_2(S\|r)$	$t_3(S\|r)$	$m_3(S\|r)$
-79.9	5.084386 E-3	4.319985 E-5	4.308496 E-5	5.377643 E-7	5.345077 E-7
-79.8	2.068367 E-2	7.168260 E-4	7.130238 E-4	3.642371 E-5	3.598500 E-5
-79.7	4.733697 E-2	3.764514 E-3	3.734648 E-3	4.392519 E-4	4.313601 E-4
-79.6	8.561103 E-2	1.234563 E-2	1.221541 E-2	2.613956 E-3	2.551689 E-3
-79.5	1.361017 E-1	3.128400 E-2	3.087277 E-2	1.056529 E-2	1.025248 E-2
-79.4	1.994352 E-1	6.734980 E-2	6.629069 E-2	3.343978 E-2	3.225850 E-2
-79.3	2.762702 E-1	1.295786 E-1	1.272087 E-1	8.941508 E-2	8.575129 E-2
-79.2	3.672988 E-1	2.296316 E-1	2.248473 E-1	2.113498 E-1	2.015102 E-1
-79.1	4.732487 E-1	3.822031 E-1	3.732740 E-1	4.547051 E-1	4.310295 E-1
-79.0	5.948851 E-1	6.054782 E-1	5.898138 E-1	9.083671 E-1	8.561263 E-1

Table 1: Wiener process with $\mu = -0.5$, $\sigma^2 = 2$, restricted to $I = [r, +\infty)$ with $r = -80$. In columns two, three and five we have listed the FPT moments $t_1(S|r)$, $t_2(S|r)$ and $t_3(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 0.1, 0.2, \dots, 1$; instead, in columns four and six the approximate values $m_2(S|r) = 5\, t_1^2(S|r)/3$ and $m_3(S|r) = 61\, t_1^3(S|r)/15$ are indicated.

As is well known, for such a process $r_1 = -\infty$ and $r_2 = +\infty$ are natural boundaries. We now consider a Wiener process characterized by (98) and restricted to $I = [r, +\infty)$, where on the regular boundary $x = r$ a reflecting condition is imposed. For such process one has:

$$t_1(S|r) = \begin{cases} \dfrac{(S-r)^2}{\sigma^2}, & \mu = 0 \\ \dfrac{S-r}{\mu} + \dfrac{\sigma^2}{2\mu^2}\Big[\exp\Big\{-\dfrac{2\mu(S-r)}{\sigma^2}\Big\} - 1\Big], & \mu \neq 0. \end{cases} \tag{99}$$

S	$t_1(S\|r)$	$t_2(S\|r)$	$m_2(S\|r)$	$t_3(S\|r)$	$m_3(S\|r)$
-79.	7.182818 E-1	9.049849 E-1	1.031858 E+0	1.689703 E+0	2.223494 E+0
-78.	4.389056 E+0	3.530574 E+1	3.852763 E+1	4.237908 E+2	5.072997 E+2
-77.	1.608554 E+1	4.904890 E+2	5.174890 E+2	2.239169 E+4	2.497227 E+4
-76.	4.959815 E+1	4.772757 E+3	4.919953 E+3	6.885009 E+5	7.320617 E+5
-75.	1.424132 E+2	3.991836 E+4	4.056302 E+4	1.678092 E+7	1.733012 E+7
-74.	3.964288 E+2	3.118250 E+5	3.143116 E+5	3.679008 E+8	3.738065 E+8
-73.	1.088633 E+3	2.361388 E+6	2.370244 E+6	7.683175 E+9	7.740980 E+9
-72.	2.971958 E+3	1.763516 E+7	1.766507 E+7	1.569663 E+11	1.574995 E+11
-71.	8.093084 E+3	1.308987 E+8	1.309960 E+8	3.175758 E+12	3.180485 E+12
-70.	2.201547 E+4	9.690530 E+8	9.693615 E+8	6.398208 E+13	6.402283 E+13
-69.	5.986214 E+4	7.165994 E+9	7.166952 E+9	1.286743 E+15	1.287087 E+15
-68.	1.627418 E+5	5.296685 E+10	5.296978 E+10	2.585833 E+16	2.586119 E+16
-67.	4.423994 E+5	3.914256 E+11	3.914344 E+11	5.194876 E+17	5.195111 E+17
-66.	1.202589 E+6	2.892416 E+12	2.892442 E+12	1.043507 E+19	1.043526 E+19
-65.	3.269001 E+6	2.137266 E+13	2.137274 E+13	2.096010 E+20	2.096025 E+20
-64.	8.886094 E+6	1.579251 E+14	1.579253 E+14	4.210005 E+21	4.210017 E+21
-63.	2.415493 E+7	1.166921 E+15	1.166922 E+15	8.456066 E+22	8.456076 E+22
-62.	6.565995 E+7	8.622456 E+15	8.622458 E+15	1.698450 E+24	1.698451 E+24
-61.	1.784823 E+8	6.371184 E+16	6.371185 E+16	3.411430 E+25	3.411431 E+25
-60.	4.851652 E+8	4.707705 E+17	4.707705 E+17	6.852043 E+26	6.852043 E+26

Table 2: Wiener process with $\mu = -1$, $\sigma^2 = 2$, restricted to $I = [r, +\infty)$ with $r = -80$. In columns two, three and five we have listed the FPT moments $t_1(S|r)$, $t_2(S|r)$ and $t_3(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 1, 2, \dots, 20$; instead, in columns four and six the approximate values $m_2(S|r) = 2\,t_1^2(S|r)$ and $m_3(S|r) = 6\,t_1^3(S|r)$ are indicated.

As proved in Section 4, the restricted Wiener process is BF if $\mu = 0$. Since (77) holds with $\nu = 0$, if $\mu = 0$ from (92) one obtains:

$$t_n(S|r) = \frac{(-1)^n E_{2n}}{(2n-1)!!} \left[t_1(S|r)\right]^n \qquad (n = 0, 1, \dots),$$

where $E_0, E_2, \dots$ denote Euler numbers. If $\mu \neq 0$, since (9) and (10) hold with $\nu = 0$, from Theorem 2.1 and from Proposition 2.1 one has:

$$\lim_{S \downarrow r} \frac{t_n(S|r)}{[t_1(S|r)]^n} = \frac{(-1)^n E_{2n}}{(2n-1)!!} \qquad (n = 0, 1, \dots). \tag{100}$$

In Table 1 for the Wiener process with $\mu = -0.5$ and $\sigma^2 = 2$, restricted to $I = [r, +\infty)$ with $r = -80$, we have listed the FPT moments $t_1(S|r)$, $t_2(S|r)$ and $t_3(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 0.1, 0.2, \dots, 1$. The mean FPT is evaluated via (99), whereas $t_2(S|r)$ and $t_3(S|r)$ are numerically evaluated via (4) with $x_0 = r = -80$. Furthermore, in columns four and six we have listed the approximate values $m_2(S|r) = 5\,t_1^2(S|r)/3$ and $m_3(S|r) = 61\,t_1^3(S|r)/15$. Note that the goodness of the approximation of $t_2(S|r)$ by $m_2(S|r)$ and $t_3(S|r)$ by $m_3(S|r)$ improves as ϵ is decreased, i.e. as S moves to the neighborhood of the reflecting boundary. Furthermore, we note that conditions (68) and (69) hold if $\mu < 0$. Hence, from Theorem 3.3 one has:

$$\lim_{S \uparrow +\infty} \frac{t_n(S|r)}{[t_1(S|r)]^n} = n! \qquad (n = 0, 1, \dots). \tag{101}$$

Furthermore, if $\mu < 0$ the FPT pdf $g(S, t|r)$ exhibits the following exponential behavior:

$$g(S, t|r) \simeq \frac{1}{t_1(S|r)} \exp\left\{-\frac{t}{t_1(S|r)}\right\} \qquad (S \uparrow +\infty). \tag{102}$$

S	$CV(S\|r)$	$\Sigma(S\|r)$	S	$CV(S\|r)$	$\Sigma(S\|r)$
-79.	8.683828 E-1	1.981136	-69.	9.998663 E-1	2.
-78.	9.125509 E-1	1.992384	-68.	9.999447 E-1	2.
-77.	9.463878 E-1	1.997357	-67.	9.999774 E-1	2.
-76.	9.696203 E-1	1.999211	-66.	9.999909 E-1	2.
-75.	9.839790 E-1	1.999795	-65.	9.999963 E-1	2.
-74.	9.920573 E-1	1.999953	-64.	9.999985 E-1	2.
-73.	9.962566 E-1	1.999990	-63.	9.999994 E-1	2.
-72.	9.983053 E-1	1.999998	-62.	9.999998 E-1	2.
-71.	9.992565 E-1	2.	-61.	9.999999 E-1	2.
-70.	9.996817 E-1	2.	-60.	1.	2.

Table 3: Coefficient of variation $CV(S|r)$ and skewness $\Sigma(S|r)$ of the FPT are listed for the same process and for the same thresholds of Table 2.

In Table 2 for the Wiener process with $\mu = -1$ and $\sigma^2 = 2$, restricted to $I = [r, +\infty)$ with $r = -80$, we have listed the FPT moments $t_1(S|r)$, $t_2(S|r)$ and $t_3(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 1, 2, \ldots, 20$. Instead, in columns four and six we have listed the approximate values $m_2(S|r) = 2\,t_1^2(S|r)$ and $m_3(S|r) = 6\,t_1^3(S|r)$. Note that the goodness of the approximation of $t_2(S|r)$ by $m_2(S|r)$ and $t_3(S|r)$ by $m_3(S|r)$ improves as ϵ increases, i.e. when the firing threshold S is moving indefinitely away from the reflecting boundary. In Table 3 for the same process and for the same thresholds of Table 2, the coefficient of variation $CV(S|r) \equiv \sqrt{Var(S|r)}/t_1(S|r)$ and the skewness $\Sigma(S|r) \equiv [t_3(S|r) + 2\,t_1^3(S|r) - 3\,t_1(S|r)\,t_2(S|r)]/[Var(S|r)]^{3/2}$ of the FPT are indicated. Note that $CV(S|r)$ increases with S and approaches 1 for large thresholds; furthermore, also $\Sigma(S|r)$ increases with S to approach 2 for large thresholds. All this is clearly suggestive of the exponential approximation (102) to the firing pdf for large thresholds.

The neuronal model based on the Wiener process does not include the well-known spontaneous exponential decay of the neuron's membrane potential that occurs between successive PSP's. Hereafter we shall thus discuss another diffusion model for neuronal activity that includes this specific feature. This is customarily denoted as the Ornstein-Uhlenbeck neuronal model because of its analogy with the well-known model used by these two authors to describe the Brownian motion.

5.2 Ornstein Uhlenbeck model

The Ornstein Uhlenbeck neuronal model is defined as the diffusion process characterized by the following drift and infinitesimal variance:

$$A_1(x) = -\frac{1}{\vartheta}(x - \varrho) \qquad A_2 = \sigma^2 \qquad (\varrho \in \mathbf{R},\ \sigma > 0,\ \vartheta > 0). \tag{103}$$

restricted to $I = [r, +\infty)$, where r is a regular boundary (with a reflecting condition) and $+\infty$ is a natural boundary. Comparing (98) with (103), we see that now the drift is state-dependent. However, in the limit as $\vartheta \to +\infty$ moments (103) tend to moments (98) with $\mu = 0$, meaning that the Ornstein Uhlenbeck model yields the Wiener model for infinitely large ϑ. In the absence of randomness ($\sigma = 0$), the membrane potential $X(t)$ exponentially decays to the resting potential ϱ with a time constant ϑ.

The FPT problem for constant thresholds is in general not solvable in closed form. However,

S	$t_1(S\|r)$	$t_2(S\|r)$	$m_2(S\|r)$	$t_3(S\|r)$	$m_3(S\|r)$
-79.9	9.933995 E-4	1.642992 E-6	1.644738 E-6	3.977012 E-9	3.986671 E-9
-79.8	3.947719 E-3	2.591928 E-5	2.597414 E-5	2.489882 E-7	2.501941 E-7
-79.7	8.825289 E-3	1.294004 E-4	1.298095 E-4	2.775179 E-6	2.795280 E-6
-79.6	1.558994 E-2	4.033832 E-4	4.050768 E-4	1.526194 E-5	1.540890 E-5
-79.5	2.420693 E-2	9.715477 E-4	9.766254 E-4	5.700033 E-5	5.768428 E-5
-79.4	3.464292 E-2	1.987805 E-3	2.000220 E-3	1.666835 E-4	1.690760 E-4
-79.3	4.686593 E-2	3.634324 E-3	3.660693 E-3	4.117376 E-4	4.186107 E-4
-79.2	6.084527 E-2	6.119718 E-3	6.170246 E-3	8.989553 E-4	9.160499 E-4
-79.1	7.655152 E-2	9.677392 E-3	9.766892 E-3	1.786229 E-3	1.824316 E-3
-79.0	9.395647 E-2	1.456403 E-2	1.471303 E-2	3.295210 E-3	3.373018 E-3

Table 4: Ornstein Uhlenbeck process with $\vartheta = 10$, $\varrho = -70$ and $\sigma^2 = 10$, restricted to $I = [r, +\infty)$ with $r = -80$. In columns two, three and five are listed the FPT moments $t_1(S|r)$, $t_2(S|r)$ and $t_3(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 0.1, 0.2, \ldots, 1$; instead, in columns four and six the approximate values $m_2(S|r) = 5\, t_1^2(S|r)/3$ and $m_3(S|r) = 61\, t_1^3(S|r)/15$ are listed.

by use of Siegert's formula (4), the mean FPT can be obtained as

$$\begin{aligned} t_1(S|r) = \vartheta \sum_{k=0}^{+\infty} \frac{2^k}{(k+1)\,(2k+1)!!} \left[\left(\frac{S-\varrho}{\sigma\sqrt{\vartheta}} \right)^{2k+2} - \left(\frac{r-\varrho}{\sigma\sqrt{\vartheta}} \right)^{2k+2} \right] \\ -2\,\vartheta \exp\Big\{ -\frac{(r-\varrho)^2}{\sigma^2\vartheta} \Big\} \sum_{k=0}^{+\infty} \frac{2^k}{(2k+1)!!} \left(\frac{r-\varrho}{\sigma\sqrt{\vartheta}} \right)^{2k+1} \\ \times \sum_{k=0}^{+\infty} \frac{1}{(2k+1)\,k!} \left[\left(\frac{S-\varrho}{\sigma\sqrt{\vartheta}} \right)^{2k+1} - \left(\frac{r-\varrho}{\sigma\sqrt{\vartheta}} \right)^{2k+1} \right]. \end{aligned}$$

Since (9) and (10) are satisfied with $\nu = 0$, from Theorem 2.1 and from Proposition 2.1 one sees that (100) holds. In Table 4 for the Ornstein Uhlenbeck process with $\vartheta = 10$, $\varrho = -70$ and $\sigma^2 = 10$, restricted to $I = [r, +\infty)$ with $r = -80$, we have listed the FPT moments $t_1(S|r)$, $t_2(S|r)$ and $t_3(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 0.1, 0.2, \ldots, 1$. The FPT moments $t_1(S|r)$, $t_2(S|r)$ and $t_3(S|r)$ are numerically evaluated via (4) with $x_0 = r = -80$. Furthermore, in columns four and six we have listed the approximate values $m_2(S|r) = 5\, t_1^2(S|r)/3$ and $m_3(S|r) = 61\, t_1^3(S|r)/15$. Note that again the goodness of the approximation of $t_2(S|r)$ by $m_2(S|r)$ and $t_3(S|r)$ by $m_3(S|r)$ improves as ϵ is decreased. Furthermore, we note that conditions (68) and (69) hold. Hence, from Theorem 3.3 one obtains (101). Furthermore, the FPT pdf $g(S, t|r)$ exhibits the exponential behavior (102). In Table 5 for the Ornstein Uhlenbeck process with $\vartheta = 5$, $\varrho = -70$ and $\sigma^2 = 400$, restricted to $I = [r, +\infty)$ with $r = -80$, we have listed the FPT moments $t_1(S|r)$, $t_2(S|r)$ and the coefficient of variation $CV(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 10, 20, \ldots, 210$. From Table 5 we see that the goodness of the approximation of $t_2(S|r)$ by $m_2(S|r)$ improves as ϵ increases; furthermore $CV(S|r)$ increases with S and it approach 1 for large thresholds.

5.3 Feller model

The Feller neuronal model [6] is defined as the diffusion process $X(t)$ characterized by the following drift and infinitesimal variance:

$$(104) \qquad A_1(x) = -\frac{1}{\vartheta}\,(x-\varrho), \qquad A_2(x) = 2\,\xi\,(x-r) \qquad (\varrho, r \in \mathbf{R}, \varrho > r, \vartheta > 0, \xi > 0).$$

S	$t_1(S\|r)$	$t_2(S\|r)$	$m_2(S\|r)$	$CV(S\|r)$
-70.	2.458883 E-1	1.005000 E-1	1.209221 E-1	8.137738 E-1
-60.	1.000222 E+0	1.667492 E+0	2.000889 E+0	8.165484 E-1
-50.	2.369550 E+0	9.435137 E+0	1.122954 E+1	8.248724 E-1
-40.	4.608184 E+0	3.617231 E+1	4.247072 E+1	8.386894 E-1
-30.	8.231322 E+0	1.175948 E+2	1.355093 E+2	8.576692 E-1
-20.	1.428070 E+1	3.622149 E+2	4.078767 E+2	8.809654 E-1
-10.	2.495717 E+1	1.135186 E+3	1.245721 E+3	9.069378 E-1
0.	4.520674 E+1	3.823049 E+3	4.087299 E+3	9.331113 E-1
10.	8.696040 E+1	1.44818 E+4	1.512422 E+4	9.565808 E-1
20.	1.812836 E+2	6.410183 E+4	6.572746 E+4	9.749534 E-1
30.	4.158379 E+2	3.414802 E+5	3.458423 E+5	9.873065 E-1
40.	1.059565 E+3	2.232766 E+6	2.245357 E+6	9.943763 E-1
50.	3.011874 E+3	1.810326 E+7	1.814277 E+7	9.978199 E-1
60.	9.558583 E+3	1.825973 E+8	1.827330 E+8	9.992570 E-1
70.	3.383678 E+4	2.289343 E+9	2.289855 E+9	9.997761 E-1
80.	1.334134 E+5	3.559614 E+10	3.559827 E+10	9.999401 E-1
90.	5.851153 E+5	6.847099 E+11	6.847197 E+11	9.999856 E-1
100.	2.851314 E+6	1.625993 E+13	1.625998 E+13	9.999969 E-1
110.	1.542584 E+7	4.759126 E+14	4.759129 E+14	9.999994 E-1
120.	9.259218 E+7	1.714662 E+16	1.714662 E+16	9.999998 E-1
130.	6.163155 E+8	7.596895 E+17	7.596895 E+17	1.

Table 5: Ornstein Uhlenbeck process with $\vartheta = 5$, $\varrho = -70$ and $\sigma^2 = 400$, restricted to $I = [r, +\infty)$ with $r = -80$. In columns two, three and five we have listed the FPT moments $t_1(S|r)$, $t_2(S|r)$ and the coefficient of variation $CV(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 10, 20, \dots, 210$; instead, in column four the approximate values $m_2(S|r) = 2\, t_1^2(S|r)$ are indicated.

The state space for the underlying stochastic process is $[r, +\infty)$, implying the existence of a 'reversal hyperpolarization potential' in r. Boundary $x = r$ is regular if $\varrho - r < \xi\,\vartheta$ and entrance if $\varrho - r \geq \xi\,\vartheta$; instead, boundary $+\infty$ is natural. Similarly to the Ornstein Uhlenbeck model, for the Feller model (104) the FPT problem is in general still unsolved. However, the mean of the firing time can be calculated:

$$t_1(S|r) = \frac{\vartheta}{\varrho - r}\left[S - r + \sum_{k=1}^{+\infty}\left(\frac{1}{\vartheta}\right)^k \frac{(S-r)^{k+1}}{k+1}\left\{\prod_{i=1}^{k}\left(\frac{\varrho - r}{\vartheta} + \xi\, i\right)\right\}^{-1}\right].$$

Relations (9) and (10) hold with $\nu = 1 - \vartheta\xi/[2\,(\varrho - r)]$. Moreover, from Theorem 2.1 we obtain:

$$\lim_{S \downarrow r} \frac{t_n(S|r)}{[t_1(S|r)]^n} = u_n \qquad (n = 0, 1, \dots),$$

where u_n are given in (20). In Table 6 for the Feller process with $\vartheta = 5$, $\varrho = -70$, $\xi = 1$ and $r = -80$, restricted to $I = [-80, +\infty)$, we have listed the FPT moments $t_1(S|r)$, $t_2(S|r)$ and $t_3(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 0.1, 0.2, \dots, 1$. The FPT moments $t_1(S|r)$, $t_2(S|r)$ and $t_3(S|r)$ are numerically evaluated via (4) with $x_0 = r = -80$. In columns four and six we have listed the approximate values $m_2(S|r) = [1 + 4\,(1-\nu)]\, t_1^2(S|r)/[1 + 2\,(1-\nu)]$ and $m_3(S|r) = [1 + 12\,(1-\nu) + 48\,(1-\nu)^2]\, t_1^3(S|r)/\{[1 + 2\,(1-\nu)]\,[1 + 4\,(1-\nu)]\}$, with $\nu = 1 - \vartheta\xi/[2\,(\varrho - r)] = 3/4$. Note that the goodness of the approximation of $t_2(S|r)$ by $m_2(S|r)$ and $t_3(S|r)$ by $m_3(S|r)$ improves as ϵ decreases. Furthermore, we note that conditions (68) and (69) hold. Hence, from Theorem 3.3 one has (101) and the FPT pdf $g(S, t|r)$ exhibits the exponential behavior (102). In Table 7 for the Feller process with $\vartheta = 5$, $\varrho = -70$, $\xi = 2$ and $r = -80$, restricted to $I = [-80, +\infty)$, we have listed

S	$t_1(S\|r)$	$t_2(S\|r)$	$m_2(S\|r)$	$t_3(S\|r)$	$m_3(S\|r)$
-79.9	5.016722 E-2	3.358470 E-3	3.355667 E-3	2.952781 E-4	2.946029 E-4
-79.8	1.006711 E-1	1.353553 E-2	1.351290 E-2	2.391571 E-3	2.380629 E-3
-79.7	1.515151 E-1	3.068618 E-2	3.060912 E-2	8.172159 E-3	8.116052 E-3
-79.6	2.027027 E-1	5.496881 E-2	5.478449 E-2	1.961330 E-2	1.943368 E-2
-79.5	2.542372 E-1	8.654535 E-2	8.618205 E-2	3.878788 E-2	3.834369 E-2
-79.4	3.061222 E-1	1.255813 E-1	1.249477 E-1	6.786924 E-2	6.693623 E-2
-79.3	3.583613 E-1	1.722457 E-1	1.712304 E-1	1.091351 E-1	1.073841 E-1
-79.2	4.109581 E-1	2.267115 E-1	2.251821 E-1	1.649718 E-1	1.619457 E-1
-79.1	4.639162 E-1	2.891554 E-1	2.869577 E-1	2.378782 E-1	2.329675 E-1
-79.0	5.172393 E-1	3.597580 E-1	3.567154 E-1	3.304704 E-1	3.228877 E-1

Table 6: Feller process with $\vartheta = 5$, $\varrho = -70$, $\xi = 1$ and $r = -80$, restricted to $I = [-80, +\infty)$, with $x = -80$ regular boundary. In columns two, three and five we have listed the FPT moments $t_1(S|r)$, $t_2(S|r)$ and $t_3(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 0.1, 0.2, \ldots, 1$; instead, in columns four and six the approximate values $m_2(S|r) = [1 + 4\,(1-\nu)]\, t_1^2(S|r)/[1 + 2\,(1-\nu)]$ and $m_3(S|r) = [1 + 12\,(1-\nu) + 48\,(1-\nu)^2]\, t_1^3(S|r)/\{[1 + 2\,(1-\nu)]\,[1 + 4\,(1-\nu)]\}$, with $\nu = 3/4$ have been indicated.

S	$t_1(S\|r)$	$t_2(S\|r)$	$m_2(S\|r)$	$CV(S\|r)$
-70.	6.589510 E+0	6.901781 E+1	8.684329 E+1	7.677756 E-1
-60.	1.841936 E+1	5.721140 E+2	6.785454 E+2	8.284294 E-1
-50.	4.129002 E+1	3.034578 E+3	3.409732 E+3	8.831485 E-1
-40.	8.833682 E+1	1.450941 E+4	1.560679 E+4	9.270233 E-1
-30.	1.899931 E+2	6.923538 E+4	7.219476 E+4	9.581319 E-1
-20.	4.181039 E+2	3.419298 E+5	3.496218 E+5	9.777516 E-1
-10.	9.449081 E+2	1.766021 E+6	1.785703 E+6	9.889170 E-1
0.	2.188616 E+3	9.529990 E+6	9.580082 E+6	9.947576 E-1
10.	5.175519 E+3	5.344454 E+7	5.357200 E+7	9.976179 E-1
20.	1.244675 E+4	3.095179 E+8	3.098430 E+8	9.989505 E-1
30.	3.034216 E+4	1.840462 E+9	1.841293 E+9	9.995484 E-1
40.	7.478235 E+4	1.118267 E+10	1.118480 E+10	9.998092 E-1

Table 7: Feller process with $\vartheta = 5$, $\varrho = -70$, $\xi = 2$ and $r = -80$, restricted to $I = [-80, +\infty)$. Columns two, three and five list the FPT moments $t_1(S|r)$, $t_2(S|r)$ and the coefficient of variation $CV(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 10, 20, \ldots, 120$; instead, column four indicates the approximate values $m_2(S|r) = 2\, t_1^2(S|r)$.

the FPT moments $t_1(S|r)$, $t_2(S|r)$ and the coefficient of variation $CV(S|r)$ through the thresholds $S = -80 + \epsilon$, with $\epsilon = 10, 20, \ldots, 120$. Instead, in columns four we have listed the approximate values $m_2(S|r) = 2\, t_1^2(S|r)$. Similarly to the Ornstein Uhlenbeck model, from Table 7 we see that the goodness of the approximation of $t_2(S|r)$ by $m_2(S|r)$ improves as ϵ increases; furthermore $CV(S|r)$ increases with S and it approach 1 for large thresholds.

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References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions. Dover Publications Inc., New York (1972).

[2] L. Arnold, Stochastic Differential Equations: Theory and Applications. Wiley & Sons, New York (1974).

[3] A. Erdèlyi, W. Magnus, F. Oberthettinger and F.G. Tricomi, Tables of Integral Transforms. **Vol. 1**. Mc Graw–Hill, New York (1954).

[4] G.L. Gerstein, and B. Mandelbrot B., Random walk models for the spike activity of a single neuron. Biophys. J. **4** (1964), 41–68.

[5] V. Giorno, A.G. Nobile, L.M. Ricciardi and L. Sacerdote, Some remarks on the Rayleigh process. J. Appl. Prob. **23** (1986), 398–408.

[6] V. Giorno, P. Lánský, A.G. Nobile and L.M. Ricciardi, Diffusion approximation and first–passage–time problem for a model neuron. III. A birth–and–death process approach. Biol. Cybern. **58** (1988), 387–404.

[7] V. Giorno, A.G. Nobile and L.M. Ricciardi, On the first passage time moments for a class of specially confined diffusion processes. In: Cybernetics and Systems 2000, Vol. 1 (Trappl., R., ed.) (2000), pp. 312-317.

[8] I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Series and Products. Academic Press, New York (1980).

[9] S. Karlin and H.M. Taylor, A Second Course in Stochastic Processes. Academic Press, New York (1981).

[10] L.M. Ricciardi, A. Di Crescenzo, V. Giorno and A.G. Nobile, An outline of theoretical and algorithmic approaches to first passage time problems with applications to biological modeling. Math. Japonica **50**, No. 2 (1999), 247–322.

[11] L.M. Ricciardi and P. Lánský, Diffusion models of neuron activity. In: The Handbook of Brain Theory and Neural Networks (M.A. Arbib, ed.). The MIT Press, Cambridge. (2002), 343-348.

[12] A.J.F. Siegert, On the first passage time probability problem. Phys. Rev. **81** (1951), 617-623.

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