$T-$FUZZY LEFT $K-$IDEALS OF SEMIRINGS

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Abstract. In this paper, we introduce the concept of $T-$fuzzy left $K-$ideals of semirings, and then some related results are obtained. Using a $t-$norm T, the direct product and $T-$product of $T-$fuzzy left $k-$ideals are discussed, and their properties are investigated.

1.Introduction and preliminaries A semiring S is an algebraic structure $(S, +, \cdot)$ in which $(S, +)$ is a commutative monoid and $(S, \cdot)$ is a semigroup such that $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ for all $a, b, c \in S$.Also $x \cdot 0 = 0 \cdot x = 0$ for all 0 is the additive identity of S. A subset I of S is called a left(right) ideal of S if I is closed under addition and $SI \subseteq I(IS \subseteq I)$. A left(right)ideal of S is called a left(right) $k-$ideal of S if $y, z \in I, x \in S, x + y = z$ implies $x \in I$. In 1965,L.A.Zadeh introduced the notion of a fuzzy subset A of a set X as a mapping from X into $[0, 1]$. Rosenfeld and Kuroki applied this concept in group theory and semigroup theory, respectively. A nonempty fuzzy subset A of a semiring S is called a fuzzy left (right)ideal of S if (i) $A(x + y) \geq min\{A(x), A(y)\}$ and (ii) $A(xy) \geq A(y)(A(xy) \geq A(x))$ for all $x, y \in S$. S.Ghosh([1]) introduced the concept of fuzzy $k-$ideals of semirings and obtain some results. In this paper,we introduce the concept of $T-$fuzzy left $k-$ideals of semirings and then some related results are obtained Using a $t-$norm T, the direct product and $T-$product of $T-$fuzzy left $k-$ideals of semirings are discussed, and their properties are investigated.

2.$T-$fuzzy left $k-$ideals

Definition 2.1([5]) By a $t-$norm T, we mean a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions:

(I) $T(x, 1) = x$

(II) $T(x, y) \leq T(x, z) \quad$ if $y \leq z$

(III) $T(x, y) = T(y, x)$

(IV) $T(x, T(y, z)) = T(T(x, y), z)$

for all $x, y, z \in [0, 1]$.

Note that every $t-$norm T has a useful property: $T(\alpha, \beta) \leq min\{\alpha, \beta\}$ for all $\alpha, \beta \in [0, 1]$.

Definition 2.2 Let μ be a fuzzy set of a semiring R. Then μ is called a $T-$fuzzy left $k-$ideal of R if for any $x, y, z \in R, x + y = z$ implies $\mu(x) \geq T(\mu(y), \mu(z))$.

$T-$fuzzy right $k-$ideal of a semiring is defined similarly. For any subset A of a semiring R, $\mathcal{X}_A$ will denote the characteristic function of A.

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Theorem 2.3 Let R and $I \subseteq S$.Then I is a left $k-$ideal of R if and only if $\mathcal{X}_I$ is a $T-$fuzzy left $k-$ideal of R.

Proof Let I be a left $k-$ideal of R. Then it's easy to show that $\mathcal{X}_I$ is a $T-$fuzzy left $k-$ideal of R. In fact, let $x, y, z \in R$ such that $x + y = z$.If $y, z \in I$, then $x \in I$ as I is a left $k-$ideal of R. Thus $\mathcal{X}_I(x) = 1 = T(\mathcal{X}_I(y), \mathcal{X}_I(z))$. Otherwise. $T(\mathcal{X}_I(y), \mathcal{X}_I(z)) = 0 \leq \mathcal{X}_I(x)$. Therefore, $\mathcal{X}_I$ is a $T-$fuzzy left $k-$ideal of R. Conversely,if $\mathcal{X}_I$ is a $T-$fuzzy left $k-$ideal of R. Let $x \in S, y, z \in I$ such that $x + y = z$. Then $\mathcal{X}_I(x) \geq T(\mathcal{X}_I(y), \mathcal{X}_I(z)) = 1$. Thus $\mathcal{X}_I(x) = 1$ and hence $x \in I$, as required.

Let $f : R \to S$ be a mapping of semirings. For a fuzzy set μ in Y, the inverse image of μ under f, denoted by $f^{-1}(\mu)$, is defined by $f^{-1}(\mu x) = \mu(f(x))$ for all $x \in R$.

Theorem 2.4 Let $f : R \to S$ be a homomorphism of semirings. If μ is a $T-$fuzzy left $k-$ideal of S, then $f^{-1}(\mu)$ is a $T-$fuzzy left $k-$ideal of R.

Proof Let $x, y, z \in R$ such that $x+y = z$. Then $f(x)+f(y) = f(z)$,and that $f^{-1}(\mu(x) = \mu(f(x)) \geq T(\mu(f(y)), \mu(f(z))) = T(f^{-1}(\mu)(y), f^{-1}(\mu)(z))$. Hence $f^{-1}(\mu)$ is a $T-$fuzzy left $k-$ideal of R.

Lemma 2.5 ([5]) For all $\alpha, \beta, \gamma, \delta \in, [0, 1]$, $T(T(\alpha, \beta), T(\gamma, \delta)) = T(T(\alpha, \gamma), T(\beta, \delta))$

Theorem 2.6 Let T be a $t-$norm and let $R = R_1 \times R_2$ be the direct product of semirings R_1 and R_2. If $\mu_1(resp.\mu_2)$ is a $T-$fuzzy left $k-$ideal of $R_1(resp.R_2)$,then $\mu = \mu_1 \times \mu_2$ is a $T-$fuzzy left $k-$ideal of R defined by $\mu(x) = \mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2))$ for all $x = (x_1, x_2) \in R$.

Proof Let $x = (x_1, x_2), y = (y_1, y_2)$ and $z = (z_1, z_2)$ be such that $x + y = z$. Then

$$\begin{aligned}
\mu(x) &= (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2)) \\
&\geq T(T(\mu_1(y_1), \mu_1(z_1)), T(\mu_2(y_2), \mu_2(z_2))) \\
&= T(T(\mu_1(y_1), \mu_2(y_2)), T(\mu_1(z_1), \mu_2(z_2))) \\
&= T((\mu_1 \times \mu_2)(y_1, y_2), (\mu_1 \times \mu_2)(z_1, z_2)) \\
&= T(\mu(y), \mu(z))
\end{aligned}$$

Hence $\mu = \mu_1 \times \mu_2$ is a $T-$fuzzy left $k-$ideal of R.

We will generalize the idea to the product of n $T-$fuzzy left $k-$ideals, We first need to generalize the domain of T to $\prod_{i=1}^{n}[0, 1]$ as follows:

Definition 2.7([5]) The function $T_n : \prod_{i=1}^{n}[0, 1] \to [0, 1]$ is defined by

$$T_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = T(\alpha_i, T_{n-1}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n))$$

for all $1 \leq i \leq n$,where $n \geq 2, T_2 = T$,and $T_1 = id$(identity)

Lemma 2.8 ([5]) For every $\alpha_i, \beta_i \in [0, 1]$,where $1 \leq i \leq n$ and $n \geq 2$,

$$T_n(T(\alpha_1, \beta_1), T(\alpha_2, \beta_2), \ldots, T(\alpha_n, \beta_n)) = T(T_n(\alpha_1, \alpha_2, \ldots, \alpha_n), T_n(\beta - 1, \beta_2, \ldots \beta_n))$$

Theorem 2.9 Let $\{R_i\}_{i=1}^{n}$ be the finite collection of semirings and $R = \prod_{i=1}^{n} R_i$ the direct product semiring of $\{R_i\}$.Let μ_i be a $T-$fuzzy left $k-$ideal of R_i,where $1 \leq i \leq n$.Then $\mu = \prod_{i=1}^{n} \mu_i$ defined by

$$\mu(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \mu_i(x_1, x_2, \ldots, x_n) = T_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n))$$

is a $T-$fuzzy left $k-$ideal of the semiring R.

Proof Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ be such that $x = y + z$. Then $x_i = y_i + z_i$ for all $1 \leq i \leq n$. We have

$$\begin{aligned}
\mu(x) &= \mu(x_1, x_2, \ldots, x_n) = T_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n) \\
&\geq T_n(T(\mu_1(y_1), \mu_1(z_1)), T(\mu_2(y_2), \mu_2(z_2)), \ldots, T(\mu_n(y_n), \mu_n(z_n))) \\
&= T(T_n(\mu_1(y_1), \mu_2(y_2), \ldots, \mu_n(y_n)), T_n(\mu_1(z_1), \mu_2(z_2), \ldots, \mu_n(z_n))) \\
&= T(\mu(y_1, y_2, \ldots, y_n), \mu(z_1, z_2, \ldots, z_n)) \\
&= T(\mu(y), \mu(z))
\end{aligned}$$

Hence μ is a $T-$fuzzy left $k-$ideal of R.

Definition 2.10 Let μ and ν be fuzzy sets in R. then the product of μ and ν, written $[\mu \cdot \nu]_T$, is defined by $[\mu \cdot \nu]_T(x) = T(\mu(x), \nu(x))$ for all $x \in R$.

Theorem 2.11 Let μ and ν be $T-$fuzzy left $k-$ideals of semiring R.If T^* is a $t-$norm which dominates T, that is, $T^*(T(\alpha,\beta), T(\gamma,\delta)) \geq T(T^*(\alpha,\gamma), T^*(\beta,\delta))$ for all $\alpha,\beta,\gamma,\delta \in [0,1]$,then T^*-product of μ and $\nu, [\mu \cdot \nu]_{T^*}$ is a $T-$fuzzy left $k-$ideal of R.

Proof Let $x, y, z \in R$ be such that $x + y = z$,

$[\mu \cdot \nu]_T * (x) = T^*(\mu(x), \nu(x))$

$\geq T^*(T(\mu(y), \mu(z)), T(\nu(y), \nu(z)))$

$\geq T(T^*(\mu(y), \nu(y)), T^*(\mu(y), \nu(z)))$

$= T([\mu \cdot \nu]_T * (y), [\mu \cdot \nu]_T * (z))$

Hence $[\mu \cdot \nu]_{T^*}$ is a $T-$fuzzy left $k-$ideal of R.

Let $f : R \to S$ be an onto homomorphism of semirings. Let T and T^* be $t-$norms such that T^* dominates T. If μ and ν are $T-$fuzzy left $k-$ideal of S, then the T^*-product of μ and $\nu, [\mu\cdot\nu]_{T^*}$ is a $T-$fuzzy left $k-$ideal of S. Since every onto homomorphic inverse image of a $T-$fuzzy left $k-$ideal is a $T-$fuzzy left $k-$ideal, the inverse images $f^{-1}(\mu), f^{-1}(\nu)$, and $f^{-1}([\mu\cdot\nu]_{T^*})$ are $T-$fuzzy left $k-$ideals of R. The next theorem provides that the relation between $f^{-1}([\mu\cdot\nu])_{T^*})$ and T^*-product $[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Theorem 2.12 Let $f : R \to S$ be an onto homomorphism of semirings. Let T^* be a $t-$norm such that T^* dominates T. Let μ and ν be $T-$fuzzy left $k-$ideals of S. If $[\mu\cdot\nu]_{T^*}$ is the T^*-product of μ and ν, and $[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}$ is the T^*-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$, then

$f^{-1}([\mu\cdot\nu]_{T^*}) = [f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}$

Proof Let $x \in R$, then

$f^{-1}([\mu\cdot\nu]_T * (f(x)))$

$=T^*(\mu(f(x)), \nu(f(x)))$

$=T^*(f^{-1}(\mu)(x), f^{-1}(\nu)(x))$

$=[f^{-1}(\mu)\cdot f^{-1}(\nu)]_T * (x)$

3.Chain conditions

Let μ be a fuzzy set in X. For $\alpha \in [0,1]$, the set $U(\mu;\alpha) = \{x \in X | \mu(x) \geq \alpha\}$ is called a level set of μ.

Theorem 3.1 Let μ be a $T-$fuzzy left $K-$ideal of R and Let $\alpha \in [0,1]$ be such that $T(\alpha,\alpha) = \alpha$.Then the nonempty $U(\mu;\alpha)$ is a left $k-$ideal of R.

Proof Let $x \in R, y, z \in U(\mu;\alpha)$ be such that $x = y + z$. Then $\mu(y) \geq \alpha$ and $\mu(z) \geq \alpha$. Hence $\mu(x) \geq T(\mu(y), \mu(z)) \geq T(\alpha,\alpha) = \alpha$, which implies that $x \in U(\mu,\alpha)$. Therefore $U(\mu;\alpha)$ is a left $k-$ideal of R.

Definition 3.2 A semiring R is said to satisfy the $K-$ascending($resp.K-$descending) chain condition(briefly,$K-ACC(resp.K-DCC)$) if for every ascending ($resp.$descending)sequence $A_1 \subseteq A_2 \subseteq \ldots (resp.A_1 \supseteq A_2 \supseteq \ldots)$ of left $K-$ideals of R, there exists a natural number n such that $A_n = A_k$ for all $n \geq k$.If R satisfies the $K-ACC$, we say R is $K-$Noetherian semiring.

Theorem 3.3 Let $\{A_k|k \in N\}$ be a family of left $k-$ideals of a semiring R which is nested, i.e.,$A_1 \supset A_2 \supset \ldots$. Let μ be a fuzzy set in R defined by

$$\mu(x) = \begin{cases} \frac{k}{k+1} & \text{if} \quad x \in A_k/A_{k+1}, k = 0, 1, 2 \ldots \\ 1 & \text{if} \quad x \in \bigcap_{k=0}^{\infty} A_k \end{cases}$$

for all $x \in X$.where A_0 stands for R.Let $\alpha \in [0, 1]$,such that $T(\alpha, \alpha) = \alpha$. Then μ is a $T-$fuzzy left $k-$ideal of R.

Proof Let $x, y, z \in R$ be such that $x + y = z$. Suppose that $y \in A_k/A_{k+1}$ and $z \in A_r/A_{r+1}$ for $k = 0, 1, 2, \ldots; r = 0, 1, 2, \ldots$. Without loss of generality, we may assume that $k \leq r$.Then obviously $z \in A_k$.Since A_k is a left $k-$ideal of R, it follows that $x \in A_k$,so that $\mu(x) \geq \frac{k}{k+1} = T(\mu(y), \mu(z))$. If $y \in \bigcap_{k=0}^{\infty} A_k$ and $z \in \bigcap_{k=0}^{\infty} A_k$, then $x \in \bigcap_{k=0}^{\infty} A_k$. Hence $\mu(x) = 1 = T(\mu(y), \mu(z))$. If $y \notin \bigcap_{k=0}^{\infty} A_k$ and $z \in \bigcap_{k=0}^{\infty} A_k$,then there exists $n \in N$, such that $y \in A_n/A_{n+1}$. It follows that $x \in A_n$, so that $\mu(x) \geq \frac{n}{n+1} = T(\mu(y), \mu(z))$. Finally, assume that $y \in \bigcap_{k=0}^{\infty} A_k$ and $z \notin \bigcap_{k=0}^{\infty} A_k$.Then $z \in A_n/A_{n+1}$ for some $n \in N$.Hence $x \in A_n$, and thus $\mu(x) \geq \frac{n}{n+1} = T(\mu(y), \mu(z))$. Therefore, is a $T-$fuzzy left $k-$ideal of R.

Theorem 3.4 Let R be a semiring satisfying $K - DCC$ and let μ be a $T-$fuzzy left $k-$ideal of R. Let $\alpha \in [0, 1]$ be such that $T(\alpha, \alpha) = \alpha$.If a sequence of elements of $Im\mu$ is strictly increasing, then μ has finite number of values.

Proof Let $\{t_k\}$ be a strictly increasing sequence of elements of $Im\mu$.The $0 \leq t_1 < t_2 < \ldots \leq 1$.Then $U(\mu; t_r) = \{x \in R|\mu(x) \geq t_r\}$ is a left $k-$ideal of R for all $r = 2, 3, \ldots$ from Theorem 3.1. Let $x \in U(\mu; t_r)$.Then $\mu(x) \geq t_r \geq t_{r-1}$, and so $x \in U(\mu; t_{r-1})$.Hence $U(\mu; t_r) \subseteq U(\mu; t_{r-1})$. Since $t_{r-1} \in Im\mu$,there exists $x_{r-1} \in X$ such that $\mu_{r-1} = t_{r-1}$. It follows that $x_{r-1} \in U(\mu; t_{r-1})$,but $x_{r-1} \notin U(\mu; t_r)$. Thus $U(\mu; t_r) \subset U(\mu; t_{r-1})$,and so we obtain a strictly descending sequence $U(\mu; t_1) \supset U(\mu; t_2) \supset U(\mu; t_3) \supset \ldots$ of left $k-$ideals of R which is not terminating. This contradicts the assumption that X satisfies the $K-DCC$. Consequently, μ has finite number of values.

Theorem 3.5 Let R be a semiring and let $\alpha \in [0, 1]$ be such that $(\alpha, \alpha) = \alpha$. The following are equivalent:

(i) R is a $k-$Noetherian semiring.

(ii) The set of values of any left $k-$ideal of R is a well-ordered subset of $[0, 1]$.

Proof $(i) \Rightarrow (ii)$ let μ be a left $k-$ideal of R. Assume that the set of values of μ is not a well-ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequence $\{t_k\}$ such that $\mu(x_k) = t_k$.It follows that $U(\mu; t_1) \subset U(\mu; t_2) \subset \ldots$ is a strictly ascending chain of left $k-$ideals of R, where $U(\mu; t_r) = \{x \in R|\mu(x) \geq t_r\}$ for every $r = 1, 2, \ldots$. This contradicts the assumption that R is $K-$Noetherian.

$(ii) \Rightarrow (i)$ Assume that the condition (i) is satisfied and R is not $K-$Noetherian. Then there exists a strictly ascending chain.

$(*)$ $A_1 \subset A_2 \subset \ldots$

of left $k-$ideals of R. Let $A = \bigcup_{k \in N} A_k$. Then A is a left $k-$ideal of R. Define a fuzzy set in R by

$$\nu(x) = \begin{cases} 0 & \text{if} \quad x \notin A \\ \frac{1}{r} & \text{where } r = min\{k \in N|x \in A_k\} \end{cases}$$

we claim that ν is a left $K-$ideal of R. Let $x, y, z \in R$ be such that $x+y = z$. If $y \in A_k/A_{k-1}$ and $z \in A_k/A_{k-1}$, then $x \in A_k$. It follows that $\nu(x) \geq \frac{1}{k} = T(\nu(y), \nu(z))$. Suppose that $y \in A_k$ and $z \in A_k/A_r$ for all $r < k$. Sine A_k is a left $k-$ideal of R, it follows that $x \in A_k$. Hence $\nu(x) \geq \frac{1}{k} \geq \frac{1}{k+1} \geq \nu(z)$,and so $\nu(x) \geq min\{\nu(y), \nu(z)\} \geq T(\nu(y), \nu(z))$. Similarly for the case $y \in A_k/A_r$ and $z \in A_k$,we have $\nu(x) \geq min\{\nu(y), \nu(z)\} \geq T(\nu(y), \nu(z))$. Thus ν is a left $k-$ideal of R. Since the chain $(*)$ is not terminating,ν has a strictly descending sequence of values. This contradicts the assumption that the value set of any left $k-$ideal is well-ordered. Therefore R is $k-$Noetherian. This completes the proof.

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