GENERALIZATIONS OF BCK-ALGEBRAS

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ABSTRACT. As a generalization of positive implicative BCK-algebra, the notion of generalized BCK-algebras is introduced. A method to make a $gBCK$-algebra from a quasi-ordered set is provided. The notion of generalized BCK-ideals of generalized BCK-algebras is introduced, and then the connections between such ideals and congruences are considered. Characterizations of generalized BCK-ideals are given. A generalized BCK-ideal generated by a set is established.

1. INTRODUCTION

In 1966, Y. Imai and K. Iséki [2] introduced a new notion, called a *BCK-algebra*. This notion is originated from two different ways: One of them is based on set theory; another is from classical and non-classical propositional calculi. As is well known, there is a close relationship between the notions of the set difference in set theory and the implication functor in logical systems. Then the following problems arise from this relationship. What is the most essential and fundamental common properties? Can we formulate a new general algebra from this viewpoint? How can we find an axiom system to establish a good theory of general algebras? To give an answer these problems, Y. Imai and K. Iséki introduced a notion of a new class of general algebras which is called a *BCK-algebra*. This name is taken from the BCK-system of C. A. Meredith. Since then many researchers studied several notions and properties of BCK-algebras. For the general development of BCK-algebras, the ideal theory plays an important role.

The aim of this paper is to construct a new algebra, called a *generalized BCK-algebra* (*$gBCK$-algebra* for short), which is a generalization of a positive implicative BCK-algebra. We give a method to make a $gBCK$-algebra from a quasi-ordered set. We introduce the notion of generalized BCK-ideals of generalized BCK-algebras, and then we consider the connections between such ideals and congruences. We provide characterizations of generalized BCK-ideals. We establish a generalized BCK-ideal generated by a set.

2. PRELIMINARIES

Recall that a *BCK-algebra* is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms: for every $x, y, z \in X$,

- $((x * y) * (x * z)) * (z * y) = 0$,
- $(x * (x * y)) * y = 0$,
- $x * x = 0$,
- $0 * x = 0$,
- $x * y = 0$ and $y * x = 0$ imply $x = y$.

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For any BCK-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x*y = 0$ is a partial order on X. A BCK-algebra X is said to be *positive implicative* if $(x*y)*z = (x*z)*(y*z)$ for all $x, y, z \in X$. A nonempty subset I of a BCK-algebra X is called an *ideal* of X if it satisfies

- $0 \in I$,
- $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

3. Generalized BCK-algebras

Definition 3.1. By a *generalized BCK-algebra* (*$gBCK$-algebra*, for short) we mean a triplet $(G, *, 0)$, where G is a nonempty set, $*$ is a binary operation on G and $0 \in G$ is a nullary operation, called *zero element*, such that

(G1) $x * 0 = x$,
(G2) $x * x = 0$,
(G3) $(x * y) * z = (x * z) * y$,
(G4) $(x * y) * z = (x * z) * (y * z)$.

Notice that $gBCK$-algebras are determined by identities, and thus the class of $gBCK$-algebras forms a variety.

Example 3.2. (1) Every positive implicative BCK-algebra is a $gBCK$-algebra.

(2) Let $G = \{0, a, b, c\}$ be a set with the following Cayley tables.

$*_1$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	0
c	c	0	0	0

$*_2$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

It is routine to check that $(G, *_1, 0)$ and $(G, *_2, 0)$ are $gBCK$-algebras, which are not BCK-algebras.

Example 3.2 tells us that a $gBCK$-algebra is a generalization of a positive implicative BCK-algebra, that is, Positive implicative BCK-algebras are stronger systems than $gBCK$-algebras.

Proposition 3.3. *Let G be a $gBCK$-algebra. Then*

(i) $0 * x = 0$,
(ii) $(x * y) * x = 0$.
(iii) $x * y = 0$ *implies* $(x * z) * (y * z) = 0$.

Proof. (i) Putting $x = y = z$ in (G4) and using (G2), we have

$$0 = 0 * 0 = (x * x) * (x * x) = (x * x) * x = 0 * x.$$

(ii) Replacing z by x in (G4) and using (G2) and (i), we get

$$(x * y) * x = (x * x) * (y * x) = 0 * (y * x) = 0.$$

(iii) Assume that $x * y = 0$. Then

$$(x * z) * (y * z) = (x * y) * z = 0 * z = 0.$$

This completes the proof. □

A reflexive and transitive relation R on a set G is called a *quasi-ordering* of G, and the couple (G, R) is then called a *quasi-ordered set* (see [1, p. 20]).

Proposition 3.4. *Let R_G be a relation on a gBCK-algebra G defined by*

$$(x, y) \in R_G \text{ if and only if } y * x = 0.$$

Then R_G is a quasi-ordering of G. Moreover,

(i) $(x, 0) \in R_G,\ \forall x \in G$.

(ii) *If $x \in G$ is such that $(0, x) \in R_G$, then $x = 0$.*

We then call R_G the *induced quasi-ordering* of a $gBCK$-algebra G.

Proof. Since $x * x = 0$ for all $x \in G$, we have $(x, x) \in R_G$, that is, R_G is reflexive. Let $x, y, z \in G$ be such that $(x, y) \in R_G$ and $(y, z) \in R_G$. Then $y * x = 0$ and $z * y = 0$. It follows from Proposition 3.3(i), (G1) and (G4) that

$$0 = 0 * x = (z * y) * x = (z * x) * (y * x) = (z * x) * 0 = z * x$$

so that $(x, z) \in R_G$, that is, R_G is transitive. Hence R_G is a quasi-ordering of G. Moreover,

(i) follows from Proposition 3.3(i).

(ii) Let $x \in G$ be such that $(0, x) \in R_G$. Then $x = x * 0 = 0$. This completes the proof. □

Lemma 3.5. *Let R_G be the induced quasi-ordering of a gBCK-algebra G. Let $x, y \in G$ be such that $(x, y) \in R_G$. Then $(x * z, y * z) \in R_G$ and $(z * y, z * x) \in R_G$ for all $z \in G$.*

Proof. Let $x, y, z \in G$ be such that $(x, y) \in R_G$. Then $y * x = 0$, and so

$$(y * z) * (x * z) = (y * x) * z = 0 * z = 0,$$

and

$$\begin{aligned}(z * x) * (z * y) &= \Big(z * (z * y)\Big) * x = (z * x) * \Big((z * y) * x\Big) \\ &= (z * x) * \Big((z * x) * (y * x)\Big) \\ &= (z * x) * \Big((z * x) * 0\Big) \\ &= (z * x) * (z * x) = 0.\end{aligned}$$

Hence $(x * z, y * z) \in R_G$ and $(z * y, z * x) \in R_G$ for all $z \in G$. □

Proposition 3.6. *Let R_G be the induced quasi-ordering of a gBCK-algebra G. Then*

(i) $\Big(y,\ x * (x * y)\Big) \in R_G,\ \forall x, y \in G$.

(ii) $\Big(x * y,\ (x * z) * (y * z)\Big) \in R_G,\ \forall x, y, z \in G$.

Proof. (i) is by (G2) and (G3).

(ii) Proposition 3.3(ii) implies that $(x,\ x * z) \in R_G$ for all $x, z \in G$. It follows from (G3), (G4) and Lemma 3.5 that

$$\Big(x * y,\ (x * z) * (y * z)\Big) = \Big(x * y,\ (x * y) * z\Big) = \Big(x * y,\ (x * z) * y\Big) \in R_G$$

for all $x, y, z \in G$. □

For every quasi-ordering R of G, denote by $\mathfrak{E}_R$ the relation on G given by

$$(x, y) \in \mathfrak{E}_R \text{ if and only if } (x, y) \in R \text{ and } (y, x) \in R.$$

Obviously, $\mathfrak{E}_R$ is an equivalence relation on G, which is called an *equivalence relation induced by R*. Denote by $[a]_{\mathfrak{E}_R}$ the equivalence class containing a and by $G/\mathfrak{E}_R$ the set of all equivalence classes of X with respect to $\mathfrak{E}_R$, that is,

$$[a]_{\mathfrak{E}_R} = \{x \in G \mid (x, a) \in \mathfrak{E}_R\}$$

and

$$G/\mathfrak{E}_R = \Big\{[a]_{\mathfrak{E}_R} \mid a \in G\Big\}.$$

Define a relation $\leq_R$ on $G/\mathfrak{E}_R$ by

$$[a]_{\mathfrak{E}_R} \leq_R [b]_{\mathfrak{E}_R} \text{ if and only if } (a,b) \in R.$$

Then $\leq_R$ is a partial order on $G/\mathfrak{E}_R$, and so $(G/\mathfrak{E}_R, \leq_R)$ is a poset, which is called a *poset assigned to the quasi-ordered set* (G, R). A relation R on G is said to be *compatible* if $(x * u, y * v) \in R$ whenever $(x, y) \in R$ and $(u, v) \in R$ for all $x, y, u, v \in G$. A compatible equivalence relation on G is called a *congruence relation* on G. The set

$$[0]_R = \{x \in G \mid (x, 0) \in R\}$$

is called the *kernel* of R.

Theorem 3.7. *Let R_G be the induced quasi-ordering of a gBCK-algebra G and let $\Theta = \mathfrak{E}_{R_G}$ be the equivalence relation induced by R_G. Then*

(i) *Θ is a congruence relation on G with kernel $[0]_\Theta = \{0\}$.*

(ii) *the quotient algebra $(G/\Theta, \bullet, [0]_\Theta)$ is a gBCK-algebra, where the operation $\bullet$ on G/Θ is defined by*

$$[a]_\Theta \bullet [b]_\Theta = [a * b]_\Theta.$$

Proof. (i) Note that Θ is an equivalence relation on G. Let $x, y, u, v \in G$ be such that $(x, y) \in \Theta$ and $(u, v) \in \Theta$. Then $(x, y) \in R_G$, $(y, x) \in R_G$, $(u, v) \in R_G$, and $(v, u) \in R_G$. Using Lemma 3.5, we obtain $(x*u, x*v) \in R_G$ and $(x*v, y*v) \in R_G$. By the transitivity of R_G, we get $(x*u, y*v) \in R_G$. Similarly, we have $(y*v, x*u) \in R_G$. Hence $(x*u, y*v) \in \Theta$, that is, Θ is a congruence relation on G. Now if $x \in [0]_\Theta$, then $(x, 0) \in \Theta$ and so $(0, x) \in R_G$. It follows from Proposition 3.4(ii) that $x = 0$. Hence $[0]_\Theta = \{0\}$.

(ii) is straightforward. □

Let G be a $gBCK$-algebra and $\emptyset \neq K \subseteq G$. Denote by θ_K the relation on G given by

$$(x, y) \in \theta_K \text{ if and only if } x * y \in K \text{ and } y * x \in K.$$

Lemma 3.8. *If θ_K is reflexive for every nonempty subset K of a gBCK-algebra G, then $[0]_{\theta_K} = K$.*

Proof. Suppose that θ_K is reflexive for every nonempty subset K of G. Then $0 = x*x \in K$. If $a \in K$, then $a * 0 = a \in K$ and $0 * a = 0 \in K$. Hence $(a, 0) \in \theta_K$, that is, $a \in [0]_{\theta_K}$. Conversely if $a \in [0]_{\theta_K}$, then $(a, 0) \in \theta_K$ and hence $a = a*0 \in K$. Therefore $[0]_{\theta_K} = K$. □

Lemma 3.9. *Let K be a nonempty subset of a gBCK-algebra G. Assume that the relation θ_K is an equivalence relation on G. Then*

$$a \in K,\ a * b \in K \text{ and } b * a = 0 \text{ imply } b \in K.$$

Proof. Suppose that $a \in K$, $a * b \in K$ and $b * a = 0$. Then $b * a = 0 \in [0]_{\theta_K} = K$, and so $(a, b) \in \theta_K$. Since θ_K is an equivalence relation on G, a and b belong to the same class of θ_K. Hence $a \in K = [0]_{\theta_K}$ implies $b \in [0]_{\theta_K} = K$. This completes the proof. □

We provide a method to make a $gBCK$-algebra from a quasi-ordered set.

Theorem 3.10. *Let (G, R) be a quasi-ordered set. Suppose $0 \notin G$ and let $G_0 = G \cup \{0\}$. Define a binary operation $*$ on G_0 as follows:*

$$x * y = \begin{cases} 0 & \text{if } (x, y) \in R \\ x & \text{otherwise.} \end{cases}$$

Then $(G_0, *, 0)$ is a gBCK-algebra.

Proof. Since R is reflexive, obviously $x * x = 0$ for all $x \in G$. Since $(x, 0) \notin R$ for every $x \in G$, we have $x * 0 = x$ for all $x \in G$. Note that $0 * x = 0$ for all $x \in G$. Assume that $(x, y) \notin R$ and $(x, z) \notin R$. Then

$$(x * y) * z = x * z = x = x * y = (x * z) * y.$$

If $(x, y) \in R$ and $(x, z) \notin R$, then

$$(x * y) * z = 0 * z = 0 = x * y = (x * z) * y.$$

Suppose that $(x, y) \notin R$ and $(x, z) \in R$. Then

$$(x * y) * z = x * z = 0 = 0 * y = (x * z) * y.$$

If $(x, y) \in R$ and $(x, z) \in R$, then

$$(x * y) * z = 0 * z = 0 = 0 * y = (x * z) * y.$$

This proves the condition (G3) holds. To verify the condition (G4), we consider the following cases:

(1) $(x, y) \in R$ and $(y, z) \in R$.
(2) $(x, y) \notin R$ and $(y, z) \in R$.
(3) $(x, y) \in R$ and $(y, z) \notin R$.
(4) $(x, y) \notin R$ and $(y, z) \notin R$.

For the case (1), we have $(x, z) \in R$, and so

$$(x * y) * z = 0 * z = 0 = 0 * 0 = (x * z) * (y * z).$$

Case (2) implies that

$$(x * y) * z = x * z = (x * z) * 0 = (x * z) * (y * z).$$

For the case (3), we get first $(x * y) * z = 0 * z = 0$. If $(x, z) \in R$, then

$$(x * z) * (y * z) = 0 * (y * z) = 0 = (x * y) * z;$$

if $(x, z) \notin R$ then

$$(x * z) * (y * z) = x * y = 0 = (x * y) * z.$$

For the case (4), if $(x, z) \in R$ then

$$(x * y) * z = x * z = 0 = 0 * y = (x * z) * (y * z).$$

If $(x, z) \notin R$, then

$$(x * y) * z = x * z = x = x * y = (x * z) * (y * z).$$

Hence the condition (G4) is true. This completes the proof. □

4. Generalized BCK-ideals

In what follows let G denote a $gBCK$-algebra unless otherwise specified.

Definition 4.1. Let G be a $gBCK$-algebra. A nonempty subset I of G is called a *generalized BCK-ideal* (*gBCK-ideal*, for short) of G if it satisfies the following conditions:

(I1) $x \in G$ and $a \in I$ imply $a * x \in I$,
(I2) $x \in G$ and $a, b \in I$ imply $x * \Big((x * a) * b\Big) \in I$.

Combining (G2) and (I1), we obtain that every $gBCK$-ideal contains the zero element 0.

Example 4.2. Let $G = \{0, a, b, c, d\}$ be a set with the following Cayley table.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

It is routine to check that $(G, *, 0)$ is a $gBCK$-algebra and the sets $I_1 = \{0, a, d\}$ and $I_2 = \{0, b, d\}$ are $gBCK$-ideals of G.

Proposition 4.3. *Let I be a $gBCK$-ideal of G. If $b \in I$ and $x * b = 0$ for $x \in G$, then $x \in I$.*

Proof. Let $b \in I$ and $x * b = 0$ for $x \in G$. Putting $b = 0$ in (I2) and using (G1), we obtain $x*(x*a) \in I$ for all $a \in I$ and $x \in G$. It follows from (G1) that $x = x*0 = x*(x*b) \in I$. □

Corollary 4.4. *Let I be a $gBCK$-ideal of G and let R_G be the induced quasi-ordering of G. If $b \in I$ and $(b, x) \in R_G$ for $x \in G$, then $x \in I$.*

Theorem 4.5. *Let K be a $gBCK$-ideal of a $gBCK$-algebra G. Then the relation θ_K is a congruence relation on G with the kernel $[0]_{\theta_K} = K$.*

Proof. Since $x * x = 0$ for all $x \in G$, we know that θ_K is reflexive. Obviously, θ_K is symmetric. Let $x, y, z \in G$ be such that $(x, y) \in \theta_K$ and $(y, z) \in \theta_K$. Then $x * y \in K$, $y * x \in K$, $y * z \in K$, and $z * y \in K$. By (I1), we have $(z * y) * x \in K$. It follows from (G1), (G2), (G4) and (I2) that

$$z * x = (z * x) * 0 = (z * x) * \Big(((z * x) * (y * x)) * ((z * y) * x)\Big) \in K.$$

Similarly, we get $x * z \in K$. Hence $(x, z) \in \theta_K$, showing that θ_K is an equivalence relation on G. By Lemma 3.8, $[0]_{\theta_K} = K$. Now it remains to show that θ_K is compatible. Let $x, y, u, v \in G$ be such that $(x, y) \in \theta_K$ and $(u, v) \in \theta_K$. Then $x * y \in K$, $y * x \in K$, $u * v \in K$, and $v * u \in K$. Using (G4) and (I1), we obtain

$$(x * u) * (y * u) = (x * y) * u \in K \ \text{ and } \ (y * u) * (x * u) = (y * x) * u \in K.$$

Hence

$$(x * u,\ y * u) \in \theta_K \tag{4.1}$$

On the other hand,

$$\begin{aligned} (y * u) * (y * v) &= \Big(y * (y * v)\Big) * u \\ &= (y * u) * \Big((y * v) * u\Big) \\ &= (y * u) * \Big((y * u) * (v * u)\Big), \end{aligned}$$

and hence

$$\begin{aligned} \Big((y * u) * (y * v)\Big) * (v * u) &= \Big((y * u) * ((y * u) * (v * u))\Big) * (v * u) \\ &= \Big((y * u) * (v * u)\Big) * \Big((y * u) * (v * u)\Big) \\ &= 0. \end{aligned}$$

Since $v * u \in K$, it follows from Proposition 4.3 that $(y * u) * (y * v) \in K$. Similarly, $(y * v) * (y * u) \in K$. Thus

$$(y * u,\ y * v) \in \theta_K. \tag{4.2}$$

Combining (4.1) and (4.2), and using the transitivity of θ_K, we conclude that $(x*u,\ y*v) \in \theta_K$, that is, θ_K is compatible. This completes the proof. □

Theorem 4.6. *Let R be a reflexive and compatible relation on G. Then the kernel $[0]_R$ is a $gBCK$-ideal of G.*

Proof. Let $x \in G$ and $a \in [0]_R$. Then $(a,\ 0) \in R$. Since R is reflexive and compatible, it follows from Proposition 3.3(i) that

$$(a * x,\ 0) = (a * x,\ 0 * x) \in R$$

so that $a * x \in [0]_R$. Now let $x \in G$ and $a, b \in [0]_R$. Using (G1), Proposition 3.3(i) and the reflexivity and compatibility of R, we have

$$\Big(x * ((x * a) * b),\ 0\Big) = \Big(x * ((x * a) * b),\ x * ((x * 0) * 0)\Big) \in R,$$

proving $x * \Big((x * a) * b\Big) \in [0]_R$. Hence $[0]_R$ is a $gBCK$-ideal of G. □

Lemma 4.7. *Let I be a nonempty subset of G such that*

(I3) $0 \in I$,

(I4) $x * y \in I$ *and* $y \in I$ *imply* $x \in I$.

If $w \in I$ then $x * (x * w) \in I$ for all $x \in G$.

Proof. If $w \in I$, then $\big(x * (x * w)\big) * w = (x * w) * (x * w) = 0 \in I$. It follows from (I4) that $x * (x * w) \in I$. □

We now give characterizations of $gBCK$-ideals.

Theorem 4.8. *Let I be a nonempty subset of G. Then I is a $gBCK$-ideal of G if and only if it satisfies* (I3) *and* (I4).

Proof. Let I be a $gBCK$-ideal of G. It is sufficient to show that I satisfies the condition (I4). Let $x, y \in G$ be such that $x*y \in I$ and $y \in I$. Then $b := x*(x*y) = x*\big((x*y)*0\big) \in I$. It follows from taking $a = x * y$ that

$$\begin{aligned} x = x * 0 &= x * \Big(\big(x * (x * y)\big) * \big(x * (x * y)\big)\Big) \\ &= x * \big((x * a) * b\big) \in I, \end{aligned}$$

which proves (I4). Conversely assume that I satisfies (I3) and (I4). Let $x \in G$ and $a \in I$. Then $(a * x) * a = 0 \in I$ by Proposition 3.3(ii) and (I3), and so $a * x \in I$ by (I4). Now let $x \in G$ and $a, b \in I$. Then

$$\Big(x * \big((x * a) * b\big)\Big) * a = (x * a) * ((x * a) * b) \in I$$

by Lemma 4.7. It follows from (I4) that $x * \big((x * a) * b\big) \in I$, which proves (I2). Hence I is a $gBCK$-ideal of G. □

Theorem 4.9. *Let I be a nonempty subset of G. Then I is a $gBCK$-ideal of G if and only if for any $a, b \in I$, $(x * a) * b = 0$ implies $x \in I$.*

Proof. The necessity is straightforward. Suppose that for any $a, b \in I$, $(x*a)*b = 0$ implies $x \in I$. Since $(0*x)*x = 0$ for all $x \in G$, obviously $0 \in I$. Let $x, y \in G$ be such that $x*y \in I$ and $y \in I$. Then

$$\big(x * (x * y)\big) * y = (x * y) * (x * y) = 0 \in I,$$

and so $x \in I$ by assumption. Hence I is a $gBCK$-ideal of G. □

Denote by $\downarrow\!a$ the set of all elements $x \in G$ satisfying $x * a = 0$, that is,

$$\downarrow\!a = \{x \in G \mid x * a = 0\}.$$

Note that, for some element a of a BCK-algebra G, the set $\downarrow a$ may not be an ideal in general.

Theorem 4.10. *For any $a \in G$, the set $\downarrow\!a$ is a $gBCK$-ideal of G.*

Proof. Obviously $0 \in \downarrow\!a$. Let $x, y \in G$ be such that $x*y \in \downarrow\!a$ and $y \in \downarrow\!a$. Then $(x*y)*a = 0$ and $y * a = 0$. It follows from (G4) and (G1) that

$$0 = (x * y) * a = (x * a) * (y * a) = (x * a) * 0 = x * a$$

so that $x \in \downarrow\!a$. Hence $\downarrow\!a$ is a $gBCK$-ideal of G. □

Theorem 4.11. *Let I be a nonempty subset of a $gBCK$-algebra G. Then the set J consists of all $x \in G$ such that the identity*

$$\big(\cdots((x * a_0) * a_1) * \cdots\big) * a_n = 0 \text{ for some } a_0, a_1, \cdots, a_n \in I$$

is the smallest $gBCK$-ideal of G containing I.

Proof. By means of Proposition 3.3(i), we have $0 \in J$. Let $x, y \in G$ be such that $x * y \in J$ and $y \in J$. Then there are $b_0, b_1, \cdots, b_m, c_0, c_1, \cdots, c_n \in I$ such that

$$\big(\cdots(((x * y) * b_0) * b_1) * \cdots\big) * b_m = 0 \tag{4.3}$$

and

$$\big(\cdots((y * c_0) * c_1) * \cdots\big) * c_n = 0. \tag{4.4}$$

Using (G3) repeatedly in (4.3) induces

$$\big((\cdots((x * b_0) * b_1) * \cdots) * b_m\big)*y = 0. \tag{4.5}$$

Combining (G1), (G4), (4.4), (4.5) and Proposition 3.3(i), we get

$$\begin{aligned} 0 &= \big((\cdots(((\cdots((x * b_0) * b_1) * \cdots) * b_m) * c_0) * \cdots) * c_n\big) \\ &\qquad *\big((\cdots(((y * c_0) * c_1) * \cdots) * c_n\big) \\ &= \big(\cdots(((\cdots((x * b_0) * b_1) * \cdots) * b_m) * c_0) * \cdots\big) * c_n, \end{aligned}$$

which shows $x \in J$. Using Theorem 4.8, we obtain that J is a $gBCK$-ideal of G. Let K be a $gBCK$-ideal of G containing I. Let $z \in J$. Then

$$\big(\cdots((z * a_0) * a_1) * \cdots\big) * a_n = 0 \text{ for some } a_0, a_1, \cdots, a_n \in I,$$

and hence $\big(\cdots((z * a_0) * a_1) * \cdots\big) * a_n = 0 \in K$. Since $a_0, a_1, \cdots, a_n \in I \subseteq K$ and K is a $gBCK$-ideal, it follows from (I4) that $z \in K$. Therefore $J \subseteq K$, and the proof is complete. □

The $gBCK$-ideal J in Theorem 4.11 is called the *$gBCK$-ideal generated by I*, and is denoted by $\langle I\rangle$, that is,

$$\langle I\rangle = \{x \in G \mid \big(\cdots((x * a_0) * a_1) * \cdots\big) * a_n = 0,\, a_0, a_1, \cdots a_n \in I\}.$$

Theorem 4.12. *Let I be a $gBCK$-ideal of G and w be an element of G. Then*

$$\langle I \cup \{w\}\rangle = \{x \in G \mid x * w^n \in I \text{ for some non-negative integer } n\},$$

where $x * w^n$ means $\big(\cdots((x * w) * w) * \cdots\big) * w$ in which w occurs n-times, and $x * w^0 = x$.

Proof. Let $U := \{x \in G \mid x * w^n \in I \text{ for some non-negative integer } n\}$, and let $x \in \langle I \cup \{w\}\rangle$. Then there exist $a_0, a_1, \cdots, a_k \in I$ such that

$$\big(\cdots(((\cdots((x * a_0) * a_1) * \cdots) * a_k) * w) * \cdots\big) * w = 0, \tag{4.6}$$

where w can be repeated n-times. Applying (G3) to (4.6) implies

$$\big(\cdots((x * w^n) * a_0) * \cdots\big) * a_k = 0 \in I,$$

and so $x * w^n \in I$ by (I4). Hence $x \in U$, which proves $\langle I \cup \{w\}\rangle \subseteq U$. Conversely, let $y \in U$. Then $y * w^n \in I$ for some non-negative integer n. Since $I \subseteq \langle I \cup \{w\}\rangle$, it follows that $(y * w^{n-1}) * w = y * w^n \in \langle I \cup \{w\}\rangle$ so from (I4) that $y * w^{n-1} \in \langle I \cup \{w\}\rangle$ because $w \in \langle I \cup \{w\}\rangle$ and $\langle I \cup \{w\}\rangle$ is a $gBCK$-ideal. Continuing this process, we conclude that $y = y * w^0 \in \langle I \cup \{w\}\rangle$, and so $U \subseteq \langle I \cup \{w\}\rangle$. This completes the proof. □

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