A NEW PROOF OF L.K. HUA'S THEOREM ON HOMOMORPHISMS *

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Abstract. In this paper, we provide a new proof of a well known theorem of L.K. Hua in 1949. The approach is different from Hua.

In 1949, L.K. Hua proved the following interesting theorem for ring homomorphisms [1]:
Hua's Theorem: *Let f be a mapping which maps a right R into another ring $R^{'}$. If f satisfies the following conditions:*

$$(*) \qquad (\forall a, b \in R) \begin{cases} f(a) + f(b) = f(a+b) \\ f(ab) = f(a)f(b) \quad \text{or} \quad f(b)f(a) \end{cases}$$

then f is either a homoromphism or an anti-homorphism from R to $R^{'}$.

This theorem has now been included on some texts of algebra as an exercise, for example, Jocobson [2].

We now provide a new and elegant proof of the above theorem by using group theory. We divide the proof into three steps.

Step 1: We first recall a crucial fact. Let G_1 and G_2 be proper subgroups of a group G, denoted by $G_1 \lneq G$ and $G_2 \lneq G$. Then

$$(\dagger) \qquad G_1 \cup G_2 \overset{\subset}{\neq} G$$

Step 2: For any $a \in R$, let

$$S_a = \{b \in R | f(ab) = f(a)f(b)\},$$

and

$$T_a = \{b \in R | f(ab) = f(b)f(a)\}.$$

Then, it is easy to see that S_a and T_a are both subgroups of the additive group of the ring R, that is, $S_a \leq (R, +)$ and $T_a \leq (R, +)$. By the conditions $(*)$ of f, we see that $S_a \cup T_a = (R, +)$, and thereby, by $(\dagger)$ stated in Step 1, we have $S_a = (R, +)$ or $T_a = (R, +)$.

Step 3: We consider the following sets

$$S = \{a \in R | S_a = R\},$$

and

$$T = \{a \in R | T_a = R\}.$$

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Now, we can easily see that $S \leq (R, +)$ and $T \leq (R, +)$. Thereby, by the condition (*) of f, we see that

$$S \cup T = (R, +),$$

and hence by (†) in Step 1, we have

$$S = (R, +) \quad \text{or} \quad T = (R, +).$$

This means that

$$(\forall a, b \in R) f(ab) = f(a)f(b),$$

or

$$(\forall a, b \in R), f(ab) = f(b)f(a).$$

In otherwords, f is either a homomorphism or an anti-homomorphism. Thus, Hua's theorem is proved.

Remark C. Reis and J.H. Shyr have removed the above Hua's theorem from rings to free semigroups in 1977 [2]. They have proved the following interesting theorem:

Let X be a non-empty set and X^+ be the free semigroup generated by X together with a transformation f defined by

$$(\triangle) \qquad (\forall = u, v \in X^+) f(uv) = f(u)f(v) \quad \text{or} \quad f(v)f(u).$$

Then f is either a endomomorpism or an anti-endomomorphism on X^+. However, we remark here that Hua's theorem does not generally hold for arbitrary semigroups. The following is an example.

Example. Let $L = \{f, g, h\}$ be a left zero semigroup. Form the semigroup S^1 by adjoining an identity element e to L. Then, we have the following Cayley table for S^1.

	e	f	g	h
e	e	f	g	h
f	f	f	f	f
g	g	g	g	g
h	h	h	h	h

Define a transformation F from $S \longrightarrow S$ by

$$F : \begin{cases} e \longrightarrow f, \\ x \longrightarrow x, \quad \text{for} \quad x = f, g, h. \end{cases}$$

Then, it is easy to see that F satisfies the so-called Hua's condition ($\triangle$) given by Reis and Shyr in [3], but F is clearly neither a endomomorphism nor an anti-endomomorphism in S. This is because we always have

$$F(g) = F(eg) = F(g)F(e) \neq F(e)F(g),$$

and
$$F(g) = F(ge) = F(g)F(e) \neq F(e)F(g).$$

In otherwords, Hua's theorem does not generally hold for semigroups.

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