Scientiae Mathematicae Japonicae Online, **Vol. 8**, (2003), 573–581

LIFTING OF SOME CHAOTIC MANIFOLDS ONTO TANGENT SPACES

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Received January 7, 2003; revised April 1, 2003

Abstract. The paper studies the relation between retraction of chaotic manifolds without conjugate points and their lifting on tangent spaces. Particular attention is given for folding manifolds as a mechanism for producing chaos and to their lifting on tangent spaces.

Introduction and Prelimenaries :

An n-simplex formed by a set of $(n+1)$-points (vertices) is the smallest closed set which contains these points, so a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron and so-on. Any$(r+1)$ or $(n+1)$ vertices of n-simplex, $0 \leq r \leq n$ can define an r-simplex, called an r-face of the given simplex. By n-dimensional polyhedron we mean a set of points in $\mathfrak{R}^n$ which can be decomposed in n-simplexes in such a way two simplexes of these decomposition have no common points or otherwise have a common face. A topological space M is called an n-dimensional manifold if it is homemorphic to a connected polyhedron and all of its points possess neighbourhoods that are homeomorphic to the interior of the n dimensional sphere. A chaotic manifold $M_{12\ldots\infty h}$ [4] is a manifold M_{0h} with infinite similar manifolds $M_{ih}, i = 1,\ 2,\ \ldots$, each has a physical character, such as magnetic field, electric charge, colour and so on. As biological example a nerve in human body is a geometric manifolds that carrying the temperature feeling, sign of weights, worried, excited, and so many other characters come from the brain, each of which on the nerve could represent a manifold in its own; all of these give a chaotic manifold. For simplicity during our work we restrict attention to two kinds of chaotic manifolds, the first consists of one fixed points in common as shown in Fig. (1)a and the second has no such fixed points as shown in Fig. (1)b.

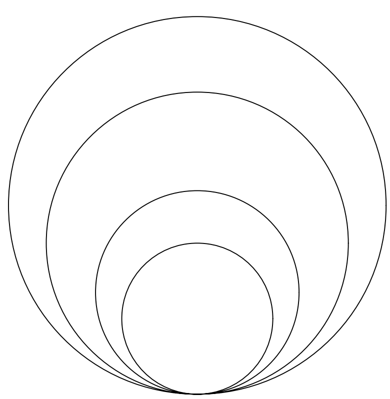

Fig. (1)a

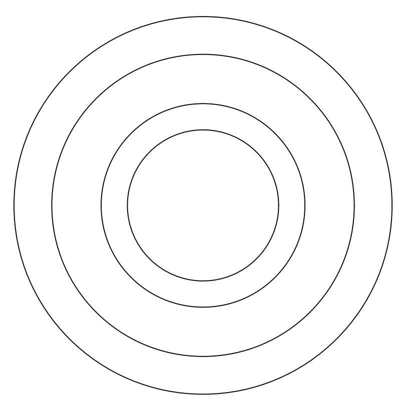

Fig. (1)b

2000 *Mathematics Subject Classification.* 51H10, 57N10.
Key words and phrases. Chaotic manifold, retraction, tangent space.

If $A \subset M$ and $f : M \to A$ is a continuous mapping such that $f|_A = \mathrm{id}_A$ i.e. f throws M onto A, keeping each point of A fixed, then one calls f is a retraction or a retracting mapping and A is a retract of M. Since M is Hausdoff and f is a retraction, A is closed in M [8]. By a topological folding of a smooth m-manifold M into a smooth n-manifold N[9] it is meant a mapping. $h : M \to N$ that satisfies (i) any piecewise geodesic path in $M; \gamma : [0.1] \to M$, the induced path $h \circ \gamma : [0,1] \to N$ is a piecewise geodesic and (ii) h does not preserve lengths. Some applications on folding in mathematics and physics are discussed in [1].

Two continuous mappings $f, g : X \to Y$ between two topological spaces are said to be homotopic, denoted by $f \simeq g$, if there exists a continuous mapping $F : X \times [0,1] \to Y$ such that for each $x \in X$, $F(x,0) = f(x)$, $F(x,1) = g(x)$ hold. The family of mappings $F(s,t)$ is called a homotopy connecting f and g. Two spaces X and Y are called homotopic or of the same homotopy type if there exist two continuous mappings $f_1 : X \to Y$ and $f_2 : Y \to X$ such that $f_2 \circ f_1 \simeq \mathrm{id}_x$ and $f_1 \circ f_2 \simeq \mathrm{id}_y$. Obviously, every two homeomorphic spaces are homotopic; whethere a simplex $[\sigma^n]$ and a space consisting of one single point are not homeomorphic but they are of the same homotopy type. Let M and $\tilde{M}$ be two arcwise connected, locally arcwise connected spaces. $\tilde{M}$ is called a covering of M if there exists a continuous mapping $p : \tilde{M} \to M$ (called covering map or admissable) such that for each $x \in M$, there exists an open set U of M, containing x such that $p^{-1}(U)$ is disjoint union of open sets each of which is mapped homeomorphically onto U by p. One of the most important result of covering is the following : if $p : \tilde{M} \to M$ is covereing and $\sigma : [0,1] \to M$ is a path beginning at $x_0 \in M$, then σ has a unique lifting path $\tilde{\sigma} : [0,1] \to \tilde{M}(p \circ \tilde{\sigma} = \sigma)$ beginning at $\tilde{x}_0$ such that $p(\tilde{x}_0) = x_0$ [8]. Moreover if σ_1 and σ_2 are two paths in M with $\sigma_1(0) = \sigma_2(0) = x_0$, $\sigma_1(1) = \sigma_2(1) = x_1$, and $\sigma_1 \simeq \sigma_2$, then their lifting paths $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ satisfy $\tilde{\sigma}_1(0) = \tilde{\sigma}_2(0)$, $\tilde{\sigma}_1(1) = \tilde{\sigma}_2(1)$ and $\tilde{\sigma}_1 \simeq \tilde{\sigma}_2$ [8].

It is known that [5] a tangent vector at a point p_0 in a smooth n-mainfold M is the best linear approximation of a smooth curve passing through $p_0 \in M$. Before going further we shall make the definition of tangent space $T_{p_0}(M)$ clear. Let $x^i = x^i(t)$ be a smooth curve at $t = t_0$ where the curve passing through p_0. At this point the curve has a tangent vector $\underline{a}$ with components a^i where $a^i = \dot{x}^i(t_0)$, $i = 1, \ \ldots, n$; the dot indicates differentiation with respect to t. When one changing to other coordinates ${x'}^1, \ldots, \ {x'}^n$ this curve is given by functions ${x'}^i = {x'}^i(x^1(t), \ldots, x^n(t))$ and the tangent vector has components ${a'}^i = {x'}^i(t_0)$ where ${x'}^i(t) = \frac{\partial}{\partial {x'}^i}\left[{x'}^i\left(x^1(t), \ldots, \ x^n(t)\right)\right]\dot{x}'^i(t)$. Thus ${a'}^i = \left(\frac{\partial {x'}^i}{\partial x^i}\right)_0 a^i$, where $\left(\frac{\partial {x'}^i}{\partial x^i}\right)_0 = \left(\frac{\partial {x'}^i}{\partial x^i}\right)_{p_0}$. Hence a tangent vector $\underline{a}$ at $p_0 \in M$ is a correspondence which associates with any local coordinate system $(x^i_1, \ \ldots, \ x^i_n)$ a set of numbers $(a^i_1, \ \ldots, \ a^i_n)$ that satisfying the relation : for each pair of local coordinate systems $a^i_k = \sum_{l=1}^n \frac{\partial x^i_k}{\partial x^j_l}(p_0)a^j_l$. The numbers $(a^i_1, \ \ldots, \ a^i_n)$ are called coordinates of the tangent vector $\underline{a}$ in the local coordinate systems $(x^i_1, \ \ldots, \ x^i_n)$. The set of all tangent vectors of M at p_0 is called a tangent space of M at p_0 and denoted by $T_{p_0}(M)$. Obviously $T_{p_0}(M)$ is a vector space over the field $\Re$ with dimension n.

In the sequel we shall use an important map; the exponential map, $\exp : T_{p_0}(M) \to M, \exp(a) = e^{|a|}$. This map sends each vector in $T_p(M)$ to its length in M[6].

The main results :

It is observed that the exponential map is monotonic and continuous but not epimorphic, and consequently the inverse map can not be defined because all the tangent vectors with length $k\pi$ (half of the cirumference) go to the south pole or what is called a conjugate point Q[6], see Fig. (2). Thus the map from a closed ball with centre p and redius k as a subset of $T_p(M)$ into the manifold M is defined exponentially takes all the boundary without conjugate points and hence exponential map can be formed from an open ball $B_k(p) \subset T_p(M)$ into M in such a way its inverse from M into $B_k(p)$ is well defined. It is also continuous and preserves lengths.

Now, one might ask : Does a retraction of chaotic manifolds produce a similar retraction on their tangent spaces. An effort is done to answer this question and we are in a position to give the answer.

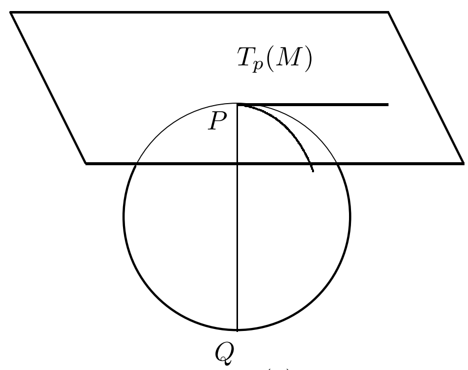

Fig. (2)

Theorem(1):

A retraction of a chaotic manifold induces a lift retraction on tangent spaces.

Proof:

Let $M_{012\ldots\infty h}$ be a chaotic manifold without conjugate points. Then the exponential map $\exp : T_p(M_{012\ldots\infty h}) \to M_{012\ldots\infty h}$ is defined as we discussed just before the theorem, and also $\exp^{-1}$ is well defined. Let A_0 be a retract of M_{0h} and $r_0 : M_{0h} \to A_0$ be the corresponding retraction with $\dim M_{0h} = \dim A_0$. Then, for each pure chaotic $M_{ih},\ i = 1,\ 2,\ \ldots$, there exists a retract A_i of M_{ih}. It follows that a sequence of retractions $r_1 : M_{ih} \to A_i$ is induced and each A_i is closed in M_{ih} since M_{ih} is Hausdorff and $\dim M_{ih} = \dim A_i$. By the continuity of $\exp^{-1}$ there induces a sequence of open balls $B_{ih} = \exp^{-1}(M_{ih})$ in the tangent space $T_{p_i}(M_{ih})$, where $p_i = p$ for all i if we consider the chaotic manifolds with a common point as in Fig. (3)a or otherwise p_i represents the corresponding point in each M_{ih} as in Fig. (3)b. In either cases, one can construct a sequence of open balls $\{B_{ih}\}$ such that

$$B_{0h} \subset B_{1h} \subset B_{2h} \subset \ldots \quad \subset B_{ih} \subset \ldots$$

This can be done since $\exp^{-1}$ preserves lengths and $M_{jh} \subset M_{ih}$ for $j < i$, and by noting that $T_{p_i}(M_{ih})$ is isomorphic to $T_{p_i}(M_{ih})$ for all i and j in the second case

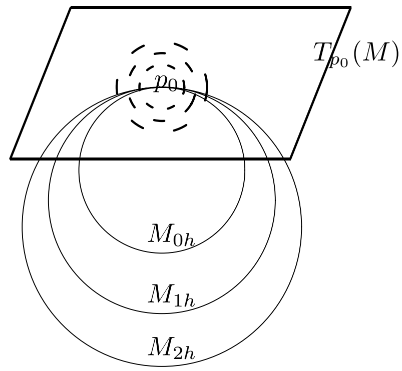

Fig. (3)a

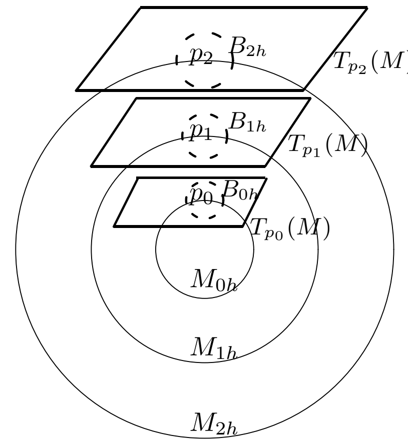

Fig. (3)b

Let us denote to $\exp^{-1}(A_i)$ by K_i. Then, we can define a sequence of retractions $\tilde{r_i} : B_{ih} \to K_i$ for which $\tilde{r_i} \circ \exp^{-1}(x_i) = \exp^{-1} \circ r_i(x_i)$ for all $x_i \in M_{ih}$, and for each i the following diagram commutes. This can be done because $\exp^{-1}$ is epimorphic as mentioned before. Thus, the lift retractions $\tilde{r_i}, i = 0,\ 1,\ 2,\ \ldots, \infty$ are contructed and our proof is completed.

$$\begin{tikzcd}
M_{ih} \arrow[r, "r_i"] \arrow[d, "\exp^{-1}"'] & A_i \arrow[d, "\exp^{-1}"] \\
B_{ih} \arrow[r, "\tilde{r_i}"'] & K_i
\end{tikzcd}$$

A notable consequence of the forgoing theorem is :

Corollary (1) :

Retracts of a chaotic manifold forms a chaotic manifold.

Proof :

Let A_0 be a retract of M_{0h} with a corresponding retraction $r_0 : M_{oh} \to A_0$. Then, there is a retraction $\tilde{r_0}$ such that $\tilde{r_0} \circ \exp^{-1}(x) = \exp^{-1} \circ r_0(x)$ for all $x \in M_0$. Also, for the pure chaotic manifolds $M_{ih},\ i = 0,\ 1,\ 2,\ \ldots,\ \infty$, the retracts A_i of M_i satisfy $\tilde{r_i} \circ \exp^{-1}(x_i) = \exp^{-1} \circ r_i(x_i),\ x_i \in M_{ih}$. Thus the chaotic manifold $A_{012\ldots\infty h}$, as shown in Fig. (4), is formed as we wished to prove.

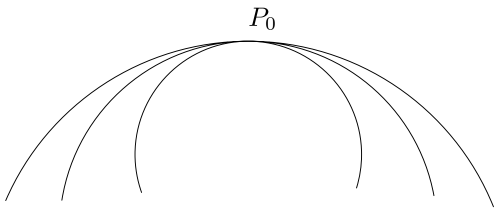

Fig. (4)a

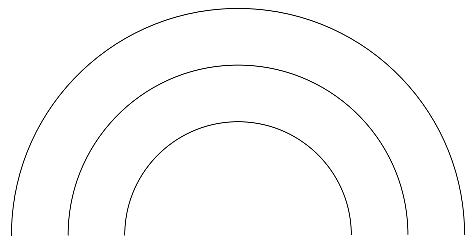

Fig. (4)b

The second question which naturally arises now is the following say : What can one say about lifted folding of a chaotic manifold to its tangent spaces. This will be our study next.

Theorem (2) :

The folding of a chaotic manifold M into itself induces a folding of $T_p(M)$ into itself.

Proof :

Let $f_0 : M_{oh} \to M_{0h}$ be a topological (isometric) folding of the essential (geometric) manifold M_{0h} into itself. Then, using lifting process, we gain a folding $\tilde{f}_0 : \exp^{-1}(M_{0h}) \subset T_p(M_{0h}) \to \exp^{-1}(M_{0h})$ which makes the following diagram commutes. The map $\tilde{f}_0$ is well defined and continuous because $\exp^{-1}$ and f_0 are so, and since $\exp^{-1}$ preserves lengths, $\tilde{f}_0$ is a topological (isometric) folding. The folding of M_{oh} induces a sequence of foldings for all pure chaotic manifolds M_{ih} into itself $i = 1,\ 2,\ \ldots, \infty$. Thus, we have a sequence of topological (isometric) foldings $\{f_1 : M_{ih} \to M_{ih}\}$. Each of theses pure chaoticness can be lifted into a manifold $\exp^{-1}(M_{ih}) \subset T_p(M_{ih})$ and moreover a folding $\tilde{f}_i$ is constructed in such a way $\tilde{f}_i \circ \exp^{-1}(x_i) = \exp^{-1} \circ f_i(x_i),\ x_i \in M_{ih}$. Because $\exp^{-1}$ preserves length, so does each $\tilde{f}_i$. Hence the set $\{\tilde{f}_i\}$ consists of topological (isometric) foldings as we wanted to show.

$$\begin{tikzcd} M_{0h} \arrow[r, "f_0"] \arrow[d, "\exp^{-1}"'] & M_{0h} \arrow[d, "\exp^{-1}"] \\ T_p(M_{0h}) \arrow[r, "\tilde{f}_0"'] & T_p(M_{0h}) \end{tikzcd}$$

Theorem (3) :

The limit of foldings of chaotic manifold induces a limit of foldings of tangent spaces.

Proof :

Suppose that $M^n_{012\ldots\infty h}$ is an n-dimensional chaotic manifold without conjugate points and suppose $f^1_0 : M^n_{0h} \to M^n_{0h}$ is a topological (isometric) folding with $f^1_0(M^{n-1}_{0h}) = M^{n-1}_{0h}$.

Then there exists a lifted topological (isometric) folding $\tilde{f_0}^1 : B_{0h} \to B_{0h}$ such that $\exp^{-1} \circ f_0^1 \equiv \tilde{f}_0^1 \circ \exp^{-1}$, where B_{0h} is an open ball in $T_p(M_{0h})$, M_{0h}^n is the geometric manifold and p is either the fixed (common) point in the given chaotic manifold as shown in Fig. (3)a or otherwise p is chosen as pictured in Fig. (3)b. Now if

$$f_0^2 : f_0^1(M_{0h}^n) \longrightarrow f_0^1(M_{0h}^n)$$

$$f_0^3 : f_0^2 f_0^1(M_{0h}^n) \longrightarrow f_0^2 f_0^1(M_{0h}^n)$$

$$\vdots$$

$$f_0^m : f_0^{m-1} f_0^{m-2} \ldots f_0^2 f_0^1(M_{0h}^n) \longrightarrow f_0^{m-1} f_0^{m-2} \ldots f_0^2 f_0^1(M_{0h}^n)$$

such that $f_0^2(M_{0h}^{n-1}) = f_0^3(M_{0h}^{n-1}) = \ldots = f_0^m(M_{0h}^{n-1}) = M_{0h}^{n-1}$, then $\lim_{m\to\infty} f_0^m(M_{0h}^n) = M_{0h}^{n-1}$. By using the lifting process into tangent spaces, one can induce a sequence of lifted topological (isometric) foldings $\{\tilde{f}_0^m\}$ on open balls of $T_p(M_{0h}^n)$ with $\tilde{f}_0^m \circ \exp^{-1} \equiv \exp^{-1} \circ f_0^m$. If each pure chaotic manifold M_{ih}^n, $i = 1, 2, \ldots, \infty$, has a sequence of topological (isometric) foldings $\{f_i^m\}$ where

$$f_i^1 : M_{ih}^n \longrightarrow M_{ih}^n$$

$$f_i^2 : f_i^1(M_{ih}^n) \longrightarrow f_i^1(M_{ih}^n)$$

$$\vdots$$

$$f_i^m : f_i^{m-1} f_i^{m-2} \ldots f_i^1(M_{ih}^n) \longrightarrow f_i^{m-1} f_i^{m-2} \ldots f_i^1(M_{ih}^n)$$

such that $f_i^1(M_{ih}^{n-1}) = f_i^2(M_{ih}^{n-1}) = \ldots = f_i^m(M_{ih}^{n-1}) = M_{ih}^{n-1}$, then $\lim_{m\to\infty} f_i^m(M_{ih}^n) = M_{ih}^{n-1}$. Using the lifted process into tangent spaces we induce a sequence of lifted topological (isometric) foldings $\{\tilde{f}_i^m\}$ on open balls of $T_p(M_{ih}^n)$ with $\tilde{f}_i^m \circ \exp^{-1} \equiv \exp^{-1} \circ f_i^m$ for all $i = 1, 2, \ldots, \infty$. Thus $\lim_{m\to\infty} \tilde{f}_i^m \circ \exp^{-1} \equiv \exp^{-1} \circ \lim_{m\to\infty} f_i^m$.

Corollary (2) :

The end of the limits of foldings of a chaotic manifold into itself induce one point or a sequence of points.

Proof :

Let $f^m : M^n \to M^n$, $n = 1, 2, 3, \ldots$, be a set of foldings where $n = 1, 2, 3, \ldots$ and M^n is an n-dimensional chaotic manifold. Then $\lim_{m\to\infty} f^m(M^n) = M^{n-1}$. If $h^m : M^{n-1} \to M^{n-1}$, $m = 1, 2, \ldots$ such that $\lim_{m\to\infty} h^m(M^{n-1}) = M^{n-1}$ and thus $H^m : M^1 \to M^1$ satisfies $\lim_{m\to\infty} H^m(M^1) = p_0$, where p_0 is the fixed point in the chaotic manifold without conjugate points of the first kind. For the second kind, the chaotic manifolds without conjugate points and have no common point, we gain a sequence of points $p_0, p_1, \ldots$ each point p_i corresponds to M_{ih}^n in the chaotic manifold $M_{012\ldots\infty h}^n$, see Fig. (3)a and (3)b.

So from the results which are induced above, for foldings, one can get the following chains :

$$\begin{tikzcd}
M_{ih}^{n} \arrow[r, "f_{ih}^{1}"] \arrow[d, "\exp^{-1}"'] & A_{ih}^{1} \arrow[r, "f_{ih}^{2}"] \arrow[d, "\exp^{-1}"'] & A_{ih}^{2} \arrow[r, "f_{ih}^{3}"] \arrow[d, "\exp^{-1}"'] & {} & \cdots\cdots & {} \arrow[r, "\lim\limits_{m\to\infty} f_{ih}^{m}"] & M_{ih}^{n-1} \arrow[d, "\exp^{-1}"] \\
T_p(M_{ih}) \arrow[r, "\tilde{f}_{ih}^{1}"'] & T_p(A_{ih}^{1}) \arrow[r, "\tilde{f}_{ih}^{2}"'] & T_p(A_{ih}^{2}) \arrow[r, "\tilde{f}_{ih}^{3}"'] & {} & \cdots\cdots & {} \arrow[r, "\lim\limits_{m\to\infty} \tilde{f}_{ih}^{m}"'] & T_p(M_{ih}^{n-1})
\end{tikzcd}$$

$$i = 0,\ 1,\ 2,\ \ldots$$

$$\begin{tikzcd}
M_{ih}^{n-1} \arrow[r, "f_{ih}^{1}"] \arrow[d, "\exp^{-1}"'] & C_{ih}^{1} \arrow[r, "f_{ih}^{2}"] \arrow[d, "\exp^{-1}"'] & C_{ih}^{2} \arrow[r, "f_{ih}^{3}"] \arrow[d, "\exp^{-1}"'] & {} & \cdots\cdots & {} \arrow[r, "\lim\limits_{m\to\infty} f_{ih}^{m}"] & M_{ih}^{n-2} \arrow[d, "\exp^{-1}"] \\
T_p(M_{ih}^{n-1}) \arrow[r, "\tilde{f}_{ih}^{1}"'] & T_p(C_{ih}^{1}) \arrow[r, "\tilde{f}_{ih}^{2}"'] & T_p(C_{ih}^{2}) \arrow[r, "\tilde{f}_{ih}^{3}"'] & {} & \cdots\cdots & {} \arrow[r, "\lim\limits_{m\to\infty} \tilde{f}_{ih}^{m}"'] & T_p(M_{ih}^{n-2})
\end{tikzcd}$$

$$i = 0,\ 1,\ 2,\ \ldots$$

$$\begin{tikzcd}
M_{ih}^{1} \arrow[r, "f_{ih}^{1}"] \arrow[d, "\exp^{-1}"'] & K_{ih}^{1} \arrow[r, "f_{ih}^{2}"] \arrow[d, "\exp^{-1}"'] & K_{ih}^{2} \arrow[r, "f_{ih}^{3}"] \arrow[d, "\exp^{-1}"'] & {} & \cdots\cdots & {} \arrow[r, "\lim\limits_{m\to\infty} f_{ih}^{m}"] & 0 \arrow[d, "\exp^{-1}"] \\
T_p(M_{ih}^{1}) \arrow[r, "\tilde{f}_{ih}^{1}"'] & T_p(K_{ih}^{1}) \arrow[r, "\tilde{f}_{ih}^{2}"'] & T_p(K_{ih}^{2}) \arrow[r, "\tilde{f}_{ih}^{3}"'] & {} & \cdots\cdots & {} \arrow[r, "\lim\limits_{m\to\infty} \tilde{f}_{ih}^{m}"'] & 0
\end{tikzcd}$$

$$i = 0,\ 1,\ 2,\ \ldots$$

The corresponding chains for retractions are :

$$\begin{tikzcd}
M_{ih} \arrow[r, "r_{ih}^{1}"] \arrow[d, "\exp^{-1}"'] & A_{ih}^{1} \arrow[r, "r_{ih}^{2}"] \arrow[d, "\exp^{-1}"'] & A_{ih}^{2} \arrow[r, "r_{ih}^{3}"] \arrow[d, "\exp^{-1}"'] & {} & \cdots\cdots & {} \arrow[r, "\lim\limits_{m\to\infty} r_{ih}^{m}"] & M_{ih}^{n-1} \arrow[d, "\exp^{-1}"] \\
T_p(M_{ih}) \arrow[r, "\tilde{r}_{ih}^{1}"'] & T_p(A_{ih}^{1}) \arrow[r, "\tilde{r}_{ih}^{2}"'] & T_p(A_{ih}^{2}) \arrow[r, "\tilde{r}_{ih}^{3}"'] & {} & \cdots\cdots & {} \arrow[r, "\lim\limits_{m\to\infty} \tilde{r}_{ih}^{m}"'] & T_p(M_{ih}^{n-1})
\end{tikzcd}$$

$$i = 0,\ 1,\ 2,\ \ldots$$

$$\begin{tikzcd}
M_{ih}^{n-1} \arrow[r, "r_{ih}^1"] \arrow[d, "\exp^{-1}"'] & C_{ih}^1 \arrow[r, "r_{ih}^2"] \arrow[d, "\exp^{-1}"'] & C_{ih}^2 \arrow[r, "r_{ih}^3"] \arrow[d, "\exp^{-1}"'] & \cdots\cdots \arrow[r, "\lim_{m\to\infty} f_{ih}^m"] & M_{ih}^{n-2} \arrow[d, "\exp^{-1}"] \\
T_p(M_{ih}^{n-1}) \arrow[r, "\tilde{r}_{ih}^1"'] & T_p(C_{ih}^1) \arrow[r, "\tilde{r}_{ih}^2"'] & T_p(C_{ih}^2) \arrow[r, "\tilde{r}_{ih}^3"'] & \cdots\cdots \arrow[r, "\lim_{m\to\infty} \tilde{r}_{ih}^m"'] & T_p(M_{ih}^{n-2})
\end{tikzcd}$$

$i = 0,\ 1,\ 2,\ \ldots$

$$\begin{tikzcd}
M_{ih}^{1} \arrow[r, "r_{ih}^1"] \arrow[d, "\exp^{-1}"'] & K_{ih}^1 \arrow[r, "r_{ih}^2"] \arrow[d, "\exp^{-1}"'] & K_{ih}^2 \arrow[r, "r_{ih}^3"] \arrow[d, "\exp^{-1}"'] & \cdots\cdots \arrow[r, "\lim_{m\to\infty} r_{ih}^m"] & 0 \arrow[d, "\exp^{-1}"] \\
T_p(M_{ih}^{1}) \arrow[r, "\tilde{r}_{ih}^1"'] & T_p(K_{ih}^1) \arrow[r, "\tilde{r}_{ih}^2"'] & T_p(K_{ih}^2) \arrow[r, "\tilde{r}_{ih}^3"'] & \cdots\cdots \arrow[r, "\lim_{m\to\infty} \tilde{f}_{ih}^m"'] & 0
\end{tikzcd}$$

$i = 0,\ 1,\ 2,\ \ldots$

Conclusion :

This article discusses the lifting of foldings of a chaotic manifolds into their into their tangent spaces, and the lifting of retractions of chaotic manifolds into their tangents. The relation between the two liftings are formulated in the form of chains of commutative diagrams that are given in the last paragraph. The limits of the two chains of folding and retraction are idential. The work is thus a contribution to the topological foundation of chaos theory which have many applications in science and technology [10, 11].

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