IMPLICATIVE BCS-ALGEBRA SUBREDUCTS OF SKEW BOOLEAN ALGEBRAS

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ABSTRACT. The variety SBA of skew Boolean algebras, introduced by Leech in [19], is a natural example of a binary discriminator variety. Central to the study of binary discriminator varieties is the variety iBCS of implicative BCS-algebras, first considered by the authors in [2]. In [2], it is shown that iBCS is generated (as a variety) by a certain three-element algebra $\mathbf{B}_2$, initially investigated by Blok and Raftery in [8]. In the first part of this paper, we show that the quasivariety $\mathbf{Q}(\mathbf{B}_2)$ generated by $\mathbf{B}_2$ is the class of all $\langle \backslash, 0\rangle$-subreducts of SBA. Using insights from the theory of skew Boolean algebras, we investigate $\mathbf{Q}(\mathbf{B}_2)$ in the second part of this paper, obtaining a fairly complete elementary theory. In particular, we characterise $\mathbf{Q}(\mathbf{B}_2)$ as a subclass of iBCS; provide a finite axiomatisation of $\mathbf{Q}(\mathbf{B}_2)$; describe the $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible algebras; and characterise the lattice of subquasivarieties of $\mathbf{Q}(\mathbf{B}_2)$. Collectively, the results may be understood as a generalisation to the 'non-commutative' situation of several well known theorems of classical algebraic logic connecting implicative BCK-algebras with (generalised) Boolean algebras.

1. Introduction

In their 1995 paper [8] on the quasivariety of BCK-algebras and its subvarieties, Blok and Raftery introduced the quasivariety $\mathbf{Q}(\mathbf{B}_2)$ generated by a certain three-element algebra $\mathbf{B}_2$. The properties of this quasivariety are exploited in the proofs of several deep results [13, 8, 24, 5] in the theory of BCK-algebras and in the theory of the varietal closure $\mathbf{H}(\mathsf{BCK})$ of BCK.

An implicative BCS-algebra is a non-commutative analogue of an implicative BCK-algebra. In [2] the authors considered the variety iBCS of all implicative BCS-algebras and showed that iBCS is generated (as a variety) by the algebra $\mathbf{B}_2$. In addition to their significance to the theory of BCK-algebras, implicative BCS-algebras play a central role in the theory of binary discriminator varieties, as evinced by the results of [2, 3, 4]. Binary discriminator varieties were introduced by Chajda, Halaš and Rosenberg [9] in 1999 in an attempt to generalise pointed ternary discriminator varieties to the $\mathbf{0}$-arithmetical case. In a binary discriminator variety, the term definable subreducts of the form $\langle \backslash, 0\rangle$, where $\backslash$ is the binary discriminator term, are implicative BCS-algebras. In [2] the variety of implicative BCS-algebras is shown to be the "pure" binary discriminator variety; that is, the variety generated by all algebras of the form $\langle A; \backslash, 0\rangle$, where $\backslash$ is the binary discriminator function on A and 0 is its associated constant.

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In [9] Chajda *et al* observe that any pointed ternary discriminator variety is a binary discriminator variety. One well known pointed ternary discriminator variety is the class GBA of all generalised Boolean algebras $\langle A; \wedge, \vee, \backslash, 0\rangle$. In his pioneering paper [16], Kalman showed in effect that an algebra $\langle A; \backslash, 0\rangle$ of type $\langle 2, 0\rangle$ is an implicative BCK-algebra if and only if it is isomorphic to a $\langle \backslash, 0\rangle$-subreduct of a generalised Boolean algebra. In the first part of this paper, we generalise Kalman's result to the variety of skew Boolean algebras. Skew Boolean algebras, introduced by Leech in [19] and also by Cornish in [10], are a non-commutative analogue of generalised Boolean algebras. The class of all skew Boolean algebras is a binary discriminator variety. We show that $\mathbf{Q}(\mathbf{B}_2)$ is the class of all $\langle \backslash, 0\rangle$-subreducts of the variety SBA of skew Boolean algebras, and thereby infer that $\mathbf{Q}(\mathbf{B}_2)$ is not a variety. For completeness, we also show that the variety $\mathbf{V}(\mathbf{B}_2)$ generated by $\mathbf{B}_2$ (*viz.*, iBCS) is the class of all $\langle \backslash, 0\rangle$-*reducts* of the variety PCSL of pseudo-complemented semilattices. The variety of pseudo-complemented semilattices is also a binary discriminator variety.

The varieties SBA, PCSL and iBCS have all been studied extensively in the literature (see respectively [10, 19]; [12, 15]; and [2, 3]) and their respective structures are well understood. However, Blok and Raftery's study of $\mathbf{Q}(\mathbf{B}_2)$ in [8] was necessarily brief. In the second part of this paper, we exploit insights from the theory of skew Boolean algebras to present a fairly complete elementary theory for $\mathbf{Q}(\mathbf{B}_2)$. In particular, we characterise $\mathbf{Q}(\mathbf{B}_2)$ as a subclass of iBCS; present a finite axiomatisation of $\mathbf{Q}(\mathbf{B}_2)$; describe the $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible members of $\mathbf{Q}(\mathbf{B}_2)$; and characterise the lattice of subquasivarieties of $\mathbf{Q}(\mathbf{B}_2)$.

2. The Algebras $\mathbf{B}_2$ and $\mathbf{B}_1$

Let $\mathbf{A} := \langle A; 0\rangle$ with $0 \in A$ be a pointed set. The *binary discriminator* on $\mathbf{A}$ is the function $\backslash : A^2 \to A$ defined for all $a, b \in A$ by:

$$a\backslash b := \begin{cases} a & \text{if } b = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The binary discriminator arises naturally in universal algebraic logic as a generalisation of the (pointed) ternary fixedpoint discriminator of Blok and Pigozzi [7]. For details, see either Bignall and Spinks [2, 4] or Chajda *et al* [9].

Let $\mathbf{B}_2$ denote the algebra $\big\langle \{0,1,2\}; \backslash, 0\big\rangle$ of type $\langle 2, 0\rangle$ for which the operation $\backslash$ is the binary discriminator on $B_2 := \{0,1,2\}$. The algebra $\mathbf{B}_2$ generates the variety iBCS of *implicative BCS-algebras*, introduced by the authors in [2] in connection with the study of the *binary discriminator varieties* of Chajda *et al* [9]. In [2] it is shown that iBCS is axiomatised by the following identities:

$$x\backslash x \approx \mathbf{0} \tag{2.1}$$

$$(x\backslash y)\backslash z \approx (x\backslash z)\backslash y \tag{2.2}$$

$$(x\backslash z)\backslash(y\backslash z) \approx (x\backslash y)\backslash z \tag{2.3}$$

$$x\backslash(y\backslash x) \approx x. \tag{2.4}$$

The following easy consequences of (2.1)–(2.4), which will be needed in the sequel, are also established in [2]:

$$x\backslash\mathbf{0} \approx x \tag{2.5}$$

$$\mathbf{0}\backslash x \approx \mathbf{0} \tag{2.6}$$

$$\big(x\backslash(x\backslash y)\big)\backslash y \approx \mathbf{0} \tag{2.7}$$

$$x\backslash\big(y\backslash(z\backslash x)\big) \approx x\backslash y. \tag{2.8}$$

The algebra $\mathbf{B}_2$ possesses an important derived operation $\wedge$, where:

$$a \wedge b := a\backslash(a\backslash b)$$

for all $a, b \in B_2$. The operation $\wedge$ is the *dual binary discriminator* on A in the sense of Chajda *et al* [9]. In [2] it is shown that, for any implicative BCS-algebra $\mathbf{A}$, the term definable reduct $\langle A; \wedge, 0\rangle$ is a *left handed locally Boolean band*; that is, a left normal band with zero such that, for every $a \in A$, the *principal subalgebra* $(a] := \{a \wedge b : b \in A\}$ *generated by* a is a Boolean lattice with respect to the natural band partial ordering. The variety of implicative BCS-algebras thus satisfies the following useful identities, first established by the authors in [2]:

$$\mathbf{0} \wedge x \approx \mathbf{0} \tag{2.9}$$

$$(x\backslash y) \wedge z \approx (x \wedge z)\backslash(y \wedge z). \tag{2.10}$$

The algebra $\mathbf{B}_2$ has just two non-trivial subalgebras, both of which are isomorphic to $\mathbf{B}_1 := \big\langle \{0,1\}; \backslash, 0\big\rangle$. It is well known [1, 16] that $\mathbf{B}_1$ generates the class iBCK of all *implicative BCK-algebras* as a variety; and that, relative to iBCS, iBCK is axiomatised by the *commutative* identity:

$$x\backslash(x\backslash y) \approx y\backslash(y\backslash x). \tag{2.11}$$

Implicative BCK-algebras are an important subclass of the quasivariety BCK of all BCK-algebras [11, 14]. They have been studied extensively in the literature (see for instance [1, 14, 16, 23]) and their properties are well understood. In particular, it is known that $\mathbf{B}_1$ is, to within isomorphism, the only subdirectly irreducible implicative BCK-algebra; and further, that every *bounded* implicative BCK-algebra is order isomorphic to a Boolean lattice. For details, see respectively Kalman [16, Lemma 1] and Iséki and Tanaka [14, Theorem 12].

Let $\mathbf{B} := \langle B; \backslash, 0\rangle$ be a non-trivial bounded implicative BCK-algebra and let $\hat{\mathbf{B}}$ denote the implicative BCS-algebra obtained from $\mathbf{B}$ upon replacing the unit element of B with a two-element maximal clique $\{m_1, m_2\}$. In [2], the authors proved that, to within isomorphism, a non-trivial implicative BCS-algebra $\mathbf{A}$ is subdirectly irreducible if and only if $\mathbf{A}$ is isomorphic to $\mathbf{B}_1$ or $\mathbf{A}$ is isomorphic to $\hat{\mathbf{B}}$ for some non-trivial bounded implicative BCK-algebra $\mathbf{B}$.

Example 2.1. The five-element subdirectly irreducible implicative BCS-algebra is the algebra with base set $\{0, 1, 2, m_1, m_2\}$ and whose operation $\backslash$ is determined by the following table:

$\backslash$	0	1	2	m_1	m_2
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
m_1	m_1	2	1	0	0
m_2	m_2	2	1	0	0

■

3. Implicative BCS-Algebra Subreducts

Let $\mathcal{L}$ be a language of algebras. Following Blok and Pigozzi [6, p. 8], we call any algebra of the form $\mathbf{A} := \langle A; f^{\mathbf{A}}\rangle_{f \in \mathcal{L}}$ an $\mathcal{L}$*-algebra*. Let $\mathcal{L}'$ be a sublanguage of $\mathcal{L}$. The $\mathcal{L}'$*-reduct* of $\mathbf{A}$ is the algebra $\mathbf{A}' := \langle A; f^{\mathbf{A}}\rangle_{f \in \mathcal{L}'}$; any subalgebra of $\mathbf{A}'$ is called an $\mathcal{L}'$*-subreduct* of $\mathbf{A}$. The following theorem is usually attributed to Mal'cev.

Theorem 3.1. *(cf. [22, Chapter 5]) Let $\mathcal{L}$ be a language of algebras and let $\mathbf{A}$ be an $\mathcal{L}$-algebra. Let $\mathcal{L}'$ be a sublanguage of $\mathcal{L}$ and let $\mathbf{B}$ be the $\mathcal{L}'$-reduct of $\mathbf{A}$. Then the quasivariety $\mathsf{Q}(\mathbf{B})$ generated by $\mathbf{B}$ is the class of all $\mathcal{L}'$-subreducts of $\mathsf{Q}(\mathbf{A})$.* ∎

A *skew Boolean algebra* is an algebra $\mathbf{A} := \langle A; \wedge, \vee, \backslash, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that: (i) the reduct $\langle A; \wedge, \vee, 0\rangle$ is a *symmetric skew lattice with zero* in the sense of Leech [18]; (ii) the reduct $\langle A; \backslash, 0\rangle$ is an implicative BCS-algebra; and (iii) $\mathbf{A} \models x \wedge y \wedge x \approx x\backslash(x\backslash y)$. By Leech [19, Theorem 1.8] the class SBA of all skew Boolean algebras is a variety. Skew Boolean algebras were introduced by Leech [19] in connection with the study of normal bands of idempotents in rings. The class LSBA of all *left handed skew Boolean algebras* is the subvariety of SBA axiomatised relative to SBA by the identity $x \wedge y \wedge x \approx x \wedge y$. Left handed skew Boolean algebras were introduced independently by Cornish in [10]. The class GBA of all *generalised Boolean algebras* is the subvariety of all commutative skew Boolean algebras; it is easy to see that GBA is the smallest non-trivial subvariety of SBA. For a further discussion and references, see the survey paper [20].

Example 3.2. The three-element left handed skew Boolean algebra, in symbols $\mathbf{3}_L$, is the algebra with base set $\{0, 1, 2\}$ and whose operations $\wedge$, $\vee$ and $\backslash$ are determined by the following tables:

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	1	2

$\backslash$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

By Cornish [10, Theorem 4.10], $\mathbf{3}_L$ and its two-element generalised Boolean subalgebra $\mathbf{2} := \big\langle \{0, 1\}; \wedge, \vee, \backslash, 0\big\rangle$ are, to within isomorphism, the only subdirectly irreducible left handed skew Boolean algebras. ∎

By the preceding example, LSBA is generated (as a variety) by $\mathbf{3}_L$. Since the reduct $\big\langle \{0, 1, 2\}; \backslash, 0\big\rangle$ of $\mathbf{3}_L$ is the implicative BCS-algebra $\mathbf{B}_2$, the variety of left handed skew Boolean algebras is a binary discriminator variety with binary discriminator term $x\backslash y$.

Theorem 3.3. *$\mathsf{Q}(\mathbf{B}_2)$ is the class of all $\langle\backslash, 0\rangle$-subreducts of LSBA.*

Proof. Since the implicative BCS-algebra reduct of $\mathbf{3}_L$ is just $\mathbf{B}_2$, the quasivariety $\mathsf{Q}(\mathbf{B}_2)$ generated by $\mathbf{B}_2$ is the class of $\langle\backslash, 0\rangle$-subreducts of $\mathsf{Q}(\mathbf{3}_L)$ by Theorem 3.1. Since $\mathbf{3}_L$ is finite, $\mathsf{Q}(\mathbf{3}_L) = \mathsf{ISP}(\mathbf{3}_L)$. But $\mathsf{ISP}(\mathbf{3}_L) = \mathsf{LSBA}$ since $\mathbf{2}$ is (to within isomorphism) the only non-trivial subalgebra of $\mathbf{3}_L$. Hence $\mathsf{Q}(\mathbf{B}_2)$ is the class of all $\langle\backslash, 0\rangle$-subreducts of LSBA. ∎

Because of Leech [19, Theorem 1.13], an obvious modification of the remarks immediately preceding Theorem 3.3 shows that SBA is also a binary discriminator variety with binary discriminator term $x\backslash y$.

Theorem 3.4. *$\mathsf{Q}(\mathbf{B}_2)$ is the class of all $\langle\backslash, 0\rangle$-subreducts of SBA.*

Proof. Any skew Boolean algebra $\mathbf{A} := \langle A; \wedge, \vee, \backslash, 0\rangle$ has a term definable left handed skew Boolean algebra reduct $\mathbf{A}_L := \langle A; \wedge_L, \vee_L, \backslash, 0\rangle$, where for all $a, b \in A$, $a \wedge_L b := a \wedge b \wedge a$ and $a \vee_L b := b \vee a \vee b$. Since the operation $\backslash$ is the same on these two algebras, $\mathbf{A}_L$ has the same $\langle\backslash, 0\rangle$-subreducts as $\mathbf{A}$. It follows that $\mathsf{Q}(\mathbf{B}_2)$ is the class of all $\langle\backslash, 0\rangle$-subreducts of SBA. ∎

The proof of the following proposition may be understood as a simplification of an argument due to Blok and Raftery [8].

Proposition 3.5. [8, Proposition 6] $\mathbf{Q}(\mathbf{B}_2)$ *is not a variety.*

Proof. Let $\mathbf{B} := \mathbf{3}_L \times \mathbf{2}$ and let $\mathbf{A}$ be the implicative BCS-algebra reduct of $\mathbf{B}$. By Theorem 3.3, $\mathbf{A} \in \mathbf{Q}(\mathbf{B}_2)$. Let $\Theta := \omega_A \cup \{(\langle 1,0\rangle, \langle 2,0\rangle), (\langle 2,0\rangle, \langle 1,0\rangle)\}$. Then it is easily checked that Θ is a congruence on $\mathbf{A}$ such that $\mathbf{A}/\Theta$ is isomorphic to the five-element subdirectly irreducible implicative BCS-algebra of Example 2.1. But no subdirectly irreducible member of iBCS with more than three elements can be a member of $\mathbf{Q}(\mathbf{B}_2)$. Hence $\mathbf{Q}(\mathbf{B}_2)$ is not closed under homomorphic images, and so is not a variety. ∎

A *pseudo-complemented semilattice* is an algebra $\langle A; \wedge, {}^*, 0\rangle$ of type $\langle 2,1,0\rangle$ such that: (i) the reduct $\langle A; \wedge, 0\rangle$ is a meet semilattice with zero; and (ii) for all $a \in A$, the greatest element of A disjoint from a exists and is a^*. It is well known that the class PCSL of all pseudo-complemented semilattices is a variety. By Jones [15, Theorem 11.1] PCSL is generated (as a variety) by the three-element bounded chain $\mathbf{3}$ (considered as a pseudo-complemented semilattice). Since, for any bounded chain $\mathbf{A}$ (considered as a pseudo-complemented semilattice) and $a \in A$, $a^* = 0$ if $a \neq 0$, while 0^* is the maximal element of the chain, PCSL is a binary discriminator variety with binary discriminator term $x \backslash y := x \wedge y^*$. From this observation it follows immediately that any pseudo-complemented semilattice $\mathbf{A}$ has a canonical term definable implicative BCS-algebra reduct $\langle A; \backslash, 0\rangle$. In the statement of the following theorem and in the sequel, we always denote this reduct by $\mathbf{A}_{\mathrm{I}}$.

Theorem 3.6. iBCS *is the class of all* $\langle \backslash, 0\rangle$*-reducts of* PCSL*. Hence an algebra* $\langle A; \backslash, 0\rangle$ *of type* $\langle 2,0\rangle$ *is an implicative BCS-algebra if and only if it is isomorphic to* $\mathbf{A}_{\mathrm{I}}$ *for some pseudo-complemented semilattice* $\mathbf{A}$*.*

Proof. It is sufficient to show that for any implicative BCS-algebra $\mathbf{B}'$ there is a pseudo-complemented semilattice $\mathbf{B}$ such that $\mathbf{B}'$ is isomorphic to $\mathbf{B}_{\mathrm{I}}$.

Let $\mathbf{A}$ be any subdirectly irreducible implicative BCS-algebra. By the remarks concluding Section 2, $\mathbf{A}$ is order isomorphic to a Boolean lattice with its unit element replaced by a two-element maximal clique. Let m_1 and m_2 be the two elements making up this maximal clique of A. In view of the description of the subdirectly irreducible pseudo-complemented semilattices given in Jones [15, Theorem 7.2], we can construct a subdirectly irreducible pseudo-complemented semilattice $\mathbf{A}^*$ from $\mathbf{A}$ by extending the partial order on A_0 to include the pair $\langle m_1, m_2\rangle$. The pseudo-complemented semilattice operations in this case are given by taking $a \wedge b$ to be the meet of a and b under this extended partial order and by defining $0^* = m_2$, $m_1^* = m_2^* = 0$, while a^* is defined to be a', the complement of a in the underlying Boolean lattice, when a is not 0, m_1 or m_2. It is easy to see that $\mathbf{A}$ is the $\langle \backslash, 0\rangle$-reduct of $\mathbf{A}^*$.

Now let $\mathbf{B}$ be any implicative BCS-algebra. Without loss of generality, we can assume that $\mathbf{B}$ is the subdirect product $\prod\{\mathbf{B}_\gamma : \gamma \in \Gamma\}$ of a family $\{\mathbf{B}_\gamma\}$ of subdirectly irreducible implicative BCS-algebras. Thus each element $a \in B$ is a function mapping Γ onto the disjoint union of the sets B_γ such that each projection map $\prod_\gamma$ from $\mathbf{B}$ to $\mathbf{B}_\gamma$ is an epimorphism. For each $\mathbf{B}_\gamma$, construct the subdirectly irreducible pseudo-complemented semilattice $\mathbf{B}_\gamma^*$ as above. Define the operations $\wedge$ and * on the set B pointwise by $(a \wedge b)(\gamma) := a(\gamma) \wedge b(\gamma)$ for each $\gamma \in \Gamma$ and $a^*(\gamma) := a(\gamma)^*$. Then $\langle B; \wedge, {}^*, 0\rangle$ is a pseudo-complemented semilattice and it is clear that $\mathbf{B}$ is its $\langle \backslash, 0\rangle$-reduct. ∎

A *pseudo-complemented distributive lattice* is an algebra $\langle A; \wedge, \vee, {}^*, 0\rangle$ of type $\langle 2,2,1,0\rangle$ such that: (i) the reduct $\langle A; \wedge, \vee, 0\rangle$ is a distributive lattice with zero; and (ii) for all $a \in A$, the greatest element of A disjoint from a exists and is a^*. It is well known that the class PCDL of all pseudo-complemented distributive lattices is a variety. Because of the description of the subdirectly irreducible members of PCDL given by Lakser in [17], an obvious modification of the proof of Theorem 3.6 shows that iBCS is also the class of all $\langle \backslash, 0\rangle$-*reducts* of PCDL when $x\backslash y$ is the term $x \wedge y^*$. However, the results of this section notwithstanding, PCDL is *not* a binary discriminator variety, since it is not generated by any class of ideal simple algebras in the sense of Chajda *et al* [9]. For a discussion of this point, see [2].

4. The Quasivariety $\mathbf{Q}(\mathbf{B}_2)$

Let K be a quasivariety and let $\mathbf{A} \in \mathsf{K}$. Recall from universal algebraic logic that a congruence θ on $\mathbf{A}$ is called a K-*congruence* on $\mathbf{A}$ if $\mathbf{A}/\theta \in \mathsf{K}$. It is folklore that, when ordered by inclusion, the set $\mathrm{Con}_{\mathsf{K}}\, \mathbf{A}$ of all K-congruences on $\mathbf{A}$ gives rise to an algebraic lattice $\mathbf{Con}_{\mathsf{K}}\mathbf{A}$. In the following lemma and in the sequel, by a $\mathbf{Q}(\mathbf{B}_2)$-*algebra* we mean a member of the quasivariety $\mathbf{Q}(\mathbf{B}_2)$.

Lemma 4.1. *Let $\mathbf{A}$ be an implicative BCS-algebra and let $a \in A$ be fixed. The following assertions hold:*

1. *The maps $c \mapsto c \wedge a$ and $c \mapsto c\backslash a$ are epimorphisms from $\mathbf{A}$ onto $A \wedge a := \{b \wedge a : b \in A\}$ and $\mathrm{ann}(a) := \{b \in A : a \wedge b = 0\}$ respectively.*
2. *The relations Φ_a and Ψ_a, defined respectively for all $b, c \in A$ by:*

$$b \equiv c \pmod{\Phi_a} \quad \text{if and only if} \quad b \wedge a = c \wedge a$$
$$b \equiv c \pmod{\Psi_a} \quad \text{if and only if} \quad b\backslash a = c\backslash a$$

   are congruences on $\mathbf{A}$. Moreover, when $\mathbf{A} \in \mathbf{Q}(\mathbf{B}_2)$, both Φ_a and Ψ_a are $\mathbf{Q}(\mathbf{B}_2)$-congruences on $\mathbf{A}$.
3. *The sets $A \wedge a$ and $\mathrm{ann}(a)$ are (the base sets of) retracts of $\mathbf{A}$. Thus the map $\varphi_a : A \to (A \wedge a) \times \mathrm{ann}(a)$ defined for all $c \in A$ by:*

$$\varphi_a(c) := \langle c \wedge a, c\backslash a\rangle$$

   is an epimorphism.

Proof. (1) Identities (2.3) and (2.10) ensure that the two maps are endomorphisms. The map $c \mapsto c \wedge a$ is obviously a surjection, so it remains to show $c \mapsto c\backslash a$ is surjective. Let $c \in \mathrm{ann}(a)$. Then $a \wedge c = 0$, whence $c = c\backslash 0$ (by (2.5)) $= c\backslash(a \wedge c) = c\backslash\big(a\backslash(a\backslash c)\big) = c\backslash a$ (by (2.8)).

(2) It follows from (1) that the relations Φ_a and Ψ_a are congruences on $\mathbf{A}$. Also, $A \wedge a$ and $\mathrm{ann}(a)$ are both subalgebras of $\mathbf{A}$, because of the identities (2.6) and (2.9) and the proof of (1). Hence, when $\mathbf{A} \in \mathbf{Q}(\mathbf{B}_2)$, $\mathbf{A}/\Phi_a$ and $\mathbf{A}/\Psi_a$ are both $\mathbf{Q}(\mathbf{B}_2)$-algebras, and so Φ_a and Ψ_a are both $\mathbf{Q}(\mathbf{B}_2)$-congruences.

(3) This follows immediately from the proofs of (1) and (2). ∎

The assertions of Lemma 4.1 also hold for skew Boolean algebras. Actually, rather more is true, since a skew Boolean algebra $\mathbf{A}$ always decomposes as a direct product of $A \wedge a$ with $\mathrm{ann}(a)$, which means that the map φ_a must be one-to-one in this case. In view of Theorem 3.4, it follows that the map φ_a of Lemma 4.1 will be a bijection whenever $\mathbf{A}$ is a member of $\mathbf{Q}(\mathbf{B}_2)$. The condition that every map of the form φ_a be one-to-one is captured by the quasi-identity:

$$x \wedge z \approx y \wedge z \;\&\; x\backslash z \approx y\backslash z \supset x \approx y. \tag{4.1}$$

The above considerations suggest that (4.1) is a likely candidate for axiomatising $\mathbf{Q}(\mathbf{B}_2)$ relative to iBCS.

Lemma 4.2. *Suppose* $\mathbf{A}$ *is a subdirectly irreducible member of* iBCS *with more than three elements. Then* $\mathbf{A}$ *possesses a subalgebra isomorphic to the five-element subdirectly irreducible implicative BCS-algebra of Example 2.1.*

Proof. Denote by m_1 and m_2 the two elements in the maximal clique of A. Since $\mathbf{A}$ has more than three elements, there exists $c \in A$ such that $c \neq 0$ and c is not equal to either m_1 or m_2. Let c' be the complement of c in the principal subalgebra $(\boldsymbol{m_1}]$ generated by m_1. Then one easily checks that $c' = m_1 \backslash c = m_2 \backslash c$ and that $m_1 \wedge c = m_2 \wedge c = c$. A straightforward series of checks now confirms that $B := \{0, c, c', m_1, m_2\}$ is closed under the operation $\backslash$ and that $\langle B; \backslash, 0\rangle$ is isomorphic to the five-element subdirectly irreducible implicative BCS-algebra of Example 2.1. ■

Theorem 4.3. *The following are equivalent for* $\mathbf{A} \in \mathsf{iBCS}$.

1. $\mathbf{A} \in \mathbf{Q}(\mathbf{B}_2)$.
2. $\mathbf{A} \models (4.1)$.
3. *For any* $a \in A$ *the map* φ_a *of Lemma 4.1*(3) *is an isomorphism.*
4. *For any* $a \in A$ *the relations* Φ_a *and* Ψ_a *of Lemma 4.1*(2) *are complementary factor congruences.*
5. $\mathbf{A}$ *is the* $\langle \backslash, 0\rangle$*-subreduct of a skew Boolean algebra.*

Proof. (1) ⇔ (2) It is easily checked that $\mathbf{B}_2 \models (4.1)$ and hence that $\mathbf{Q}(\mathbf{B}_2) \models (4.1)$. Conversely, suppose $\mathbf{A} \models (4.1)$. Without loss of generality, we may assume that $\mathbf{A}$ is the subdirect product of a family $\{\mathbf{A}_\gamma : \gamma \in \Gamma\}$ of subdirectly irreducible implicative BCS-algebras. Suppose that one of the $\mathbf{A}_\gamma$ has more than three elements. Then by Lemma 4.2, this $\mathbf{A}_\gamma$ has a subalgebra $\mathbf{B}$ that is isomorphic to the five-element subdirectly irreducible implicative BCS-algebra of Example 2.1. We denote the elements of $\mathbf{B}$ by $0, c, c', m_1, m_2$, as in this lemma. Since $m_1 \wedge c = m_2 \wedge c$ and $m_1 \backslash c = m_2 \backslash c$, but $m_1 \neq m_2$, it follows that $\mathbf{A}_\gamma$ does not satisfy (4.1). But this means that $\mathbf{A}$ can not satisfy (4.1) either. This contradiction implies that each $\mathbf{A}_\gamma$ has at most three elements. But then each $\mathbf{A}_\gamma$ must be isomorphic to either $\mathbf{B}_1$ or $\mathbf{B}_2$ and so $\mathbf{A} \in \mathbf{Q}(\mathbf{B}_2)$.

(2) ⇒ (3) Suppose $\mathbf{A} \models (4.1)$. Now if $b, c \in A$ are such that $\varphi_a(b) = \varphi_a(c)$ then we have $a \wedge b = a \wedge c$ and $a \backslash b = a \backslash c$; whence $b = c$. Thus φ_a is one-to-one and therefore is an isomorphism.

(3) ⇒ (4) Suppose that the map φ_a is an isomorphism for any $a \in A$. Let $b, c \in A$ be such that $b \equiv c \pmod{\Phi_a}$ and $b \equiv c \pmod{\Psi_a}$. Then $b \wedge a = c \wedge a$ and $b \backslash a = c \backslash a$, which implies that $b = c$ by Lemma 4.1(3), since φ_a is a bijection. Thus $\Phi_a \cap \Psi_a = \omega$. Also, $b = \varphi_a^{-1}\big(\langle b \wedge a, b \backslash a\rangle\big)\Phi_a \varphi_a^{-1}\big(\langle b \wedge a, c \backslash a\rangle\big)\Psi_a \varphi_a^{-1}\big(\langle c \wedge a, c \backslash a\rangle\big) = c$, for any $b, c \in A$. Thus $\Phi_a \circ \Psi_a = \iota$. Hence Φ_a and Ψ_a are complementary factor congruences.

(4) ⇒ (2) Suppose that Φ_a and Ψ_a are complementary factor congruences for any $a \in A$. Let $a, b, c \in A$ be such that $b \wedge a = c \wedge a$ and $b \backslash a = c \backslash a$. Then we have $b \equiv c \pmod{\Phi_a}$ and $b \equiv c \pmod{\Psi_a}$, which implies that $b = c$ since $\Phi_a \cap \Psi_a = \omega$. Hence $\mathbf{A} \models (4.1)$.

(1) ⇔ (5) This is immediate from Theorem 3.4. ■

Corollary 4.4. *A quasi-equational base for* $\mathbf{Q}(\mathbf{B}_2)$ *is given by the implicative BCS-algebra identities* (2.1) *to* (2.4) *of Section 2 together with the quasi-identity* (4.1) *above.* ■

A skew Boolean algebra $\langle A; \wedge, \vee, \backslash, 0\rangle$ is said to be *flat* (also *smooth* or *primitive* in the skew Boolean algebra literature) if its implicative BCS difference operation $\backslash$ is the binary discriminator on A. In other words, a skew Boolean algebra is flat if and only if it is a *binary discriminator algebra* in the sense of Chajda *et al* [9]. Example 3.2 shows that such algebras play an important role in the theory of (left handed) skew Boolean algebras. This observation suggests attention be devoted to the study of *flat* implicative BCS-algebras, namely those implicative BCS-algebras $\langle A; \backslash, 0\rangle$ for which (by analogy with the theory of skew Boolean algebras) the fundamental operation $\backslash$ is the binary discriminator on A. It is easy to see that every flat implicative BCS-algebra is the $\langle \backslash, 0\rangle$-*reduct* of a flat skew Boolean algebra and hence is a member of $\mathbf{Q}(\mathbf{B}_2)$. The following useful technical lemma, which clarifies the relationship between flat implicative BCS-algebras and flat skew Boolean algebras, shows that even more is true.

Lemma 4.5.

1. *Let* $\mathbf{A}$ *be a flat left handed skew Boolean algebra. Then the* $\langle \backslash, 0\rangle$*-reduct of* $\mathbf{A}$ *is flat. Moreover,* $\operatorname{Con}\langle A; \backslash, 0\rangle = \operatorname{Con}\mathbf{A}$.
2. *Let* $\mathbf{A} := \langle A; \backslash, 0\rangle$ *be a flat implicative BCS-algebra. For all* $a, b \in A$*, let:*

$$a \wedge b := a\backslash(a\backslash b) = \begin{cases} a & \text{if } b \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a \vee b := \begin{cases} b & \text{if } b \neq 0 \\ a & \text{otherwise.} \end{cases}$$

   Then the induced structure $\mathbf{A}' := \langle A; \wedge, \vee, \backslash, 0\rangle$ *is a flat left handed skew Boolean algebra. Moreover,* $\mathbf{A}$ *is the* $\langle \backslash, 0\rangle$*-reduct of* $\mathbf{A}'$ *and* $\operatorname{Con}\mathbf{A} = \operatorname{Con}\mathbf{A}'$.
3. *Any flat implicative BCS-algebra is a* $\mathbf{Q}(\mathbf{B}_2)$*-algebra.*
4. *Every congruence on a flat* $\mathbf{Q}(\mathbf{B}_2)$*-algebra is a* $\mathbf{Q}(\mathbf{B}_2)$*-congruence.*

Proof. (1) The first assertion is clear, while the second may be deduced by use of (2).

(2) Let $\mathbf{A}$ be a flat implicative BCS-algebra. Easy but tedious case-splitting arguments show that the induced structure $\mathbf{A}' := \langle A; \wedge, \vee, \backslash, 0\rangle$ satisfies all the identities defining left handed skew Boolean algebras (see [10, Section 2] or [19, Theorem 1.8]) and hence is a left handed skew Boolean algebra. Clearly $\mathbf{A}$ is the $\langle \backslash, 0\rangle$-reduct of $\mathbf{A}'$ and $\operatorname{Con}\mathbf{A}' \subseteq \operatorname{Con}\mathbf{A}$. Now by [10, Lemma 4.8] any partitition of the non-zero elements of A, together with the singleton $\{0\}$, is the set of congruence classes of some congruence on $\mathbf{A}'$. To see $\operatorname{Con}\mathbf{A} \subseteq \operatorname{Con}\mathbf{A}'$ it therefore suffices to show $\Theta^{\mathbf{A}}(0, b) = \iota$ for any $0 \neq b \in A$. So let $a, b \in A$ with $0 \neq b$. Then $a = a\backslash 0 \; \Theta^{\mathbf{A}}(0, b) \; a\backslash b = 0$. Hence $\Theta^{\mathbf{A}}(0, b) = \iota$ as required.

(3) This is immediate from (2) and Theorem 3.3.

(4) This follows from (2) and (3), since any non-trivial homomorphic image of a flat skew Boolean algebra is itself flat. ∎

Corollary 4.6. *(cf.* [8, Proposition 3]*)*

1. *The class of congruence lattices of all (flat)* $\mathbf{Q}(\mathbf{B}_2)$*-algebras does not satisfy any particular lattice identity.*
2. *The class of* $\mathbf{Q}(\mathbf{B}_2)$*-congruence lattices of all (flat)* $\mathbf{Q}(\mathbf{B}_2)$*-algebras does not satisfy any particular lattice identity.*
3. *The class of congruence lattices of all implicative BCS-algebras does not satisfy any particular lattice identity.*

Proof. The corollary is immediate in view of [10, Corollary 4.9]. ∎

Let K be a quasivariety. An algebra $\mathbf{A} \in \mathsf{K}$ is said to be K-*subdirectly irreducible* if $\mathbf{A}$ has a smallest non-identity K-congruence. We denote the class of all K-subdirectly irreducible members of K by $\mathsf{K}_{\mathrm{RSI}}$. By a result due to Mal'cev [21], every member $\mathbf{A}$ of K is isomorphic to a subdirect product of K-subdirectly irreducible members of K (that are homomorphic images of $\mathbf{A}$). Thus, $\mathsf{K} = \mathbf{IPs}(\mathsf{K}_{\mathrm{RSI}})$.

In [10] Cornish characterised the subdirectly irreducible left handed skew Boolean algebras by exploiting the complementary factor congruences of Lemma 4.1(2) in conjunction with his description [10, Lemma 4.8] of the congruence structure of flat skew Boolean algebras. Because the congruences on any flat implicative BCS-algebra $\mathbf{A}$ coincide with the congruences on any flat left handed skew Boolean algebra that has $\mathbf{A}$ as its $\langle \backslash, 0\rangle$-reduct, Cornish's result yields the following characterisation of the $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible $\mathbf{Q}(\mathbf{B}_2)$-algebras.

Theorem 4.7. *(cf.* [10, Theorem 4.10]*) To within isomorphism, the only $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible members of $\mathbf{Q}(\mathbf{B}_2)$ are the three-element and two-element flat implicative BCS-algebras $\mathbf{B}_2$ and $\mathbf{B}_1$ of Section 2.*

Proof. Suppose $\mathbf{A}$ is a $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible $\mathbf{Q}(\mathbf{B}_2)$-algebra. Let Φ_a and Ψ_a be the $\mathbf{Q}(\mathbf{B}_2)$-congruences of Lemma 4.1(2) and let $b \in A$ be such that $b \neq 0$. As $b\backslash b = 0$ (by (2.1)) $= 0\backslash b$ (by (2.6)), $0 \equiv b \pmod{\Psi_b}$. Hence $\Psi_b \neq \omega$. Because Φ_b, Ψ_b are complementary factor congruences, $\Phi_b = \omega$. Now for any $a \in A$, $(a \wedge b) \wedge b = a \wedge b$, so $a \wedge b \equiv a \pmod{\Phi_b}$. Consequently, $a \wedge b = a$. But then $a\backslash b = (a \wedge b)\backslash b = \big(a\backslash(a\backslash b)\big)\backslash b = 0$ (by (2.7)). When $b = 0$, $a\backslash b = a$ by (2.5). Hence $\mathbf{A}$ is flat. Now by Lemma 4.5, $\mathbf{A}$ will be subdirectly irreducible, and hence $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible, if and only if its skew Boolean algebra extension $\mathbf{A}'$ of Lemma 4.5(2) is subdirectly irreducible. Since, to within isomorphism, the only subdirectly irreducible left handed skew Boolean algebras are (by Example 3.2) $\mathbf{3}_L$ and $\mathbf{2}$, it follows that, to within isomorphism, the only $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible members of $\mathbf{Q}(\mathbf{B}_2)$ are the $\langle \backslash, 0\rangle$-reducts of $\mathbf{3}_L$ and $\mathbf{2}$, *viz.*, the flat implicative BCS-algebras $\mathbf{B}_2$ and $\mathbf{B}_1$. ∎

¿From the description of the subdirectly irreducible left handed skew Boolean algebras given in Example 3.2, it is clear (see [10, Corollary 4.11]) that the lattice of subvarieties of LSBA is a three-element chain. It is well known (see for instance [25, p. 6]) that the algebra $\mathbf{B}_1$ generates the class of implicative BCK-algebras as a quasivariety, and hence that every subquasivariety of iBCK is a variety. These remarks, in conjunction with Theorem 4.7 above and Blok and Raftery [8, Corollary 10], yield the following characterisation of the lattice of subquasivarieties of $\mathbf{Q}(\mathbf{B}_2)$-algebras. In the statement of the corollary, for any $\langle \backslash, \mathbf{0}\rangle$-terms $u_1(\vec{x}), \ldots, u_n(\vec{x})$ in the variables $\vec{x}$, we denote by $x\backslash \prod_{i=1}^{n} u_i(\vec{x})$ the term $\big(\cdots(x\backslash u_1(\vec{x}))\backslash \cdots\big)\backslash u_n(\vec{x})$, $n \in \omega$.

Corollary 4.8. *The lattice of subquasivarieties of $\mathbf{Q}(\mathbf{B}_2)$-algebras is a three-element chain; the only non-trivial proper subquasivariety of $\mathbf{Q}(\mathbf{B}_2)$ is the variety of implicative BCK-algebras. A subquasivariety K of $\mathbf{Q}(\mathbf{B}_2)$ is proper if and only if it satisfies an identity of the form:*

$$x\backslash \prod_{i=1}^{n} u_i(x,y) \approx y\backslash \prod_{j=1}^{m} v_j(x,y) \tag{4.2}$$

where $n, m \in \omega$ and $u_1, \ldots, u_n, v_1, \ldots, v_m$ are $\langle \backslash, \mathbf{0}\rangle$-terms such that BCK satisfies:

$$u_i(x,x) \approx \mathbf{0} \approx v_j(x,x) \qquad (i = 1, \ldots, n; j = 1, \ldots, m).$$

Proof. The first assertion of the corollary is clear. If K is a proper subquasivariety of $\mathbf{Q}(\mathbf{B}_2)$, then $\mathsf{K} \subseteq \mathsf{iBCK}$ and hence satisfies (2.11), which is an identity of the form of (4.2). Conversely, suppose K satisfies an identity of the form of (4.2). By Blok and Raftery [8, Corollary 10], $\mathbf{H}(\mathsf{K}) \subseteq \mathsf{BCK}$, so K is not $\mathbf{Q}(\mathbf{B}_2)$. Hence K is proper. ∎

The results of this section may be understood as generalisations to $\mathsf{Q}(\mathbf{B}_2)$ and skew Boolean algebras of some well known theorems relating iBCK to GBA. For details of these latter, see in particular Kalman [16]. The theory of $\mathsf{Q}(\mathbf{B}_2)$-algebras [resp. skew Boolean algebras] may itself be seen as an amalgamation of the theory of implicative BCK-algebras [resp. generalised Boolean algebras] with that of left normal bands [resp. normal bands]. For a discussion and references, see in particular Leech [19, Section 1.19].

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