# A NOTE ON MOORE-SMITH CONVERGENCE, SUMMABLE FAMILIES AND CONTINUOUS LINEAR OPERATORS BETWEEN LOCALLY CONVEX SPACES

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Abstract. We obtain a counterexample about the hypothesis of a linear operator on a locally convex space to a Hilbert space mapping bounded convergent nets to convergent nets, implying that the operator is continuous. Accordingly, a linear operator defined on a locally convex space and, taking values in a Hilbert space, might map summable families to summable families without being continuous. We may take the weak dual of a Hilbert space for the domain space.

## 1. Introduction

We had conjectured about a linear operator between (separated) locally convex spaces mapping bounded convergent nets to convergent nets or, summable families to summable families, these would be sufficient conditions for the operator to be continuous. In the course of the research work, we found the counterexample $Id : (W^{1,p}(\Omega), \sigma(W^{1,p}(\Omega))') \to L^p(\Omega)$ $(1 \prec p \prec \infty)$, $\Omega$ a bounded open subset of $\mathbf{R}^N$ of class $C^1$ in the sense of [Bré] (IX.2., Definición pp. 157). In paragraph 2, we obtain Theorem 1 and a counterexample in Theorem 4. We state the well known Theorem 2 and Theorem 3 and, we conclude that $Id : (W^{1,p}(\Omega), \sigma) \to L^p(\Omega)$ maps bounded convergent nets to convergent nets. However, this operator is not continuous, as follows from Theorem 3. In Theorem 4 we obtain the counterexample concerning the convergence of bounded nets and of summable families, for the case where the domain space is the weak dual of a Hilbert space and, the range is a Hilbert space.

## 2. Obtaining a counterexample

In what follows, if $E$ is a (real or complex) vector space, $p$ a seminorm on $E$, we let $U_p = \{x \in E : p(x) \preceq 1\}$ stand for the closed unit semiball of $E$. $f$ being a linear functional on $E$, we denote by $\mid f \mid$ the seminorm on $E$ defined through $\mid f \mid (x) = \mid \prec f, x \succ \mid$ $(x \in E)$, $\prec .,. \succ$ for the duality. If $(E, \|.\|)$ is a normed space, we let $B_E$ stand for the closed unit ball of $E$. We consider the dual $E'$ of $E$ and, we denote by $(E, \sigma(E, E'))$ the space $E$, endowed with the weak topology $\sigma(E, E')$.

**Theorem 1.** *Let $(E, \|.\|)$ be a normed space with a separable dual $E'$, $F$ a normed space and, let $T : E \to F$ be a linear operator. If $Tx_n \to 0$ in $F$ for each sequence $(x_n)$ in $E$ such that, $\|x_n\| \preceq 1$ for all $n$ and, $x_n \to 0$ in $(E, \sigma(E, E'))$, it follows that $T : B_E \to F$ is continuous where, $B_E$ is the closed unit ball of $E$ endowed with the weak topology.*

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*Proof.* Let $A = \{f_m : m \in \mathbf{N}\}$ be a strongly dense subset of $E'$, and suppose that $T : (B_E, \sigma(E, E')) \to (F, \|.\|)$ is not continuous where, we denote the norms in $E, F$ by the same symbol $\|.\|$. Then there exists some $\varepsilon \succ 0$ such that, for each $m \in \mathbf{N}$ there exists some $x_m, \|x_m\| \leq 1, x_m \in 2^{-m} \bigcap\{U_{|f_n|} : 1 \leq n \leq m\}$ with $\|Tx_m\| \geq \varepsilon$. We find: if $m \geq p$, it follows that $x_m \in 2^{-m} U_{|f_p|}$ whence $|\prec f_p, x_m \succ| \leq 2^{-m}$ and $\prec f_p, x_m \succ \to 0$ as $m \to \infty$. Since the closure of the linear span of $A$ is all of $E'$, it follows ([Tay, Lay],Theorem 10.4 in Ch. III) that $x_m \to 0$ in $(B_E, \sigma(E, E'))$. Nevertheles the sequence $Tx_m$ does not converge to 0 in $F$ and the theorem is proved. ∎

**Theorem 2.** *Let $E, F$ be normed spaces, $T$ a compact operator on $E$ to $F$. Then $Tx_n$ converges to zero for each sequence $x_n \to 0$ in $(E, \sigma(E, E'))$.*

*Proof.* This is essentially Theorem 1 in [Lus; Sobo], Chapter IV. ∎

**Theorem 3.** *If $(E, \|.\|)$, $(F, \|.\|)$ are normed spaces, $T : (E, \sigma(E, E')) \to (F, \|.\|)$ is a continuous linear operator, then $T$ has finite rank.*

*Proof.* This follows from I. 30, Livre III (pp. 432) in [Garni; De Wi; Schme], since that $T$ is completely bounded in the sense of I.21 (pp. 425) due of $E$ being a normed space and, clearly $E'$ is separating in the sense of pp. 151 that is, if $\prec f, x \succ = 0$ for all $f \in E'$ it follows that $x = 0$. ∎

Supposing now that: $(P)$ each linear operator $T$ on a locally convex space $L$ to a normed space $N$, carrying bounded convergent nets in $L$ to convergent nets in $N$ is continuous, (See [Kan, Aki] for this concept) we conclude an absurd by taking $T = I_d : (W^{1,p}(\Omega), \sigma) \to L^p$ where, $\Omega$ is a bounded open subset of $\mathbf{R}^N$ of class $C^1$, $\sigma$ stands for the weak topology of the Sobolev space $W^{1,p}(\Omega)$, $1 \prec p \prec \infty$ and $Idx = x$ is the identity operator. In fact, $Id$ is compact ([Bré], remark (4) to Teorema IX.16 (Rellich-Kondrachov)) and, $W^{1,p}(\Omega)$ has a separable dual due of being reflexive, separable ([Bré], Corolario III.24 and Proposición IX.1). According to theorem 2, $Id$ maps weakly convergent sequences to zero in the closed unit ball $B$ of $W^{1,p}(\Omega)$ to convergent sequences to zero in $L^p(\Omega)$ so that, by theorem 1, $Id$ is continuous from $B$ to $L^p(\Omega)$, $B$ endowed with the weak topology $\sigma$ of $W^{1,p}(\Omega)$. This implies that $Id$ maps weakly convergent nets to 0 in $B$ to convergent nets to 0 in $L^p(\Omega)$; thus if $(x_i)$ is a bounded net in $W^{1,p}(\Omega)$ weakly convergent to 0 it follows that, for some constant $C \succ 0$ it holds that $(x_i/C)$ is a weakly convergent net to 0 in $B$ thence $Tx_i/C$ converges to 0 in $L^p(\Omega)$ that is, $Tx_i$ converges to zero in $L^p(\Omega)$. If $(P)$ were true, it follows from theorem 3 that $W^{1,p}(\Omega)$ would be finite dimensional, which is an absurd. We obtain:

**Theorem 4.** *The conditions that a linear operator $T$ from a separated locally convex space to a Hilbert space to map bounded convergent nets to convergent nets and summable families to summable families, are not sufficient for continuity of $T$. We may take for the domain the weak dual of a Hilbert space.*

*Proof.* This follows from above, taking $p = 2$, if we prove that, the sum $s$ of a summable family (See [Choq]) is the limit of a bounded net that we may take for the summable family. We have: the family $(a_i)$ $(i \in I)$ in the locally convex space $E$ is summable with sum $s$ if and only if the net $(\sum_A a_i)$ where, $\sum_A a_i$ stands for the (finite) sum of the $a_i \in A$, $A \in \mathcal{F}(I) = \{A \subset I\colon \phi \neq A, A \subset I,\ A\ finite\}$ converges to $s$ in the sense that, for each neighborhood $s + U$ of $s$ in $E$, $U$ a neighborhood of 0, there exists some finite set $I_U \subset I$ such that, $\sum_A a_i - s \in U$ for each $A \in \mathcal{F}(I)$, $A \supset I_U$. We prove that the set of the sums $S_A = \sum_A a_i$ is bounded in $E$. Let $U$ be a convex neighborhood of 0. According to [Bourba], TG III.38 (2., Le critère de Cauchy), if $(a_i)$ is summable, then there exists a finite set $D(U) \subset I$ such that, for every finite set $C \subset I \setminus D(U)$ it holds that $S_C \in U$. Therefore, if $B \in \mathcal{F}(I)$, it follows that $S_B = S_{B \cap D(U)} + S_{B \setminus D(U)} \subset (\lambda + 1)U$ where, $\lambda \succ 0$ is such that $S_{D(U)} \subset \lambda U$ and the theorem follows. ∎

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