

## ON THE NUMBER OF NON-EQUIVALENT ODD 1-FACTORS OF A COMPLETE GRAPH

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**Abstract.** For even  $n > 0$ , let  $K_n$  be the complete graph with vertices  $v_0, v_1, \dots, v_{n-1}$ . An edge  $v_i v_j$  is called odd or even accordingly as  $|i - j|$  is odd or even. An odd(even) 1-factor of  $K_n$  is a 1-factor of  $K_n$  whose edges are all odd(even). The Dihedral group  $D_n$  acts on  $K_n$  naturally, and this action induces an action of  $D_n$  on the family of all 1-factors of  $K_n$ . In this paper, by applying Burnside's lemma, we calculate the number of the equivalence classes of odd(even) 1-factors under the action of  $D_n$ .

### 1 Introduction

For even  $n > 0$ , let  $K_n$  be the complete graph with vertices  $v_0, v_1, \dots, v_{n-1}$ . An edge  $v_i v_j$  is called odd or even accordingly as  $|i - j|$  is odd or even. An odd(even) 1-factor of  $K_n$  is a 1-factor of  $K_n$  whose edges are all odd (even). Let  $X_n$  be the set of the odd 1-factors of  $K_n$  and let  $Y_n$  be the set of the even 1-factors of  $K_n$ .

The action of the Dihedral group  $D_n = \{\rho_0, \rho_1, \dots, \rho_{n-1}, \sigma_0, \sigma_1, \dots, \sigma_{n-1}\}$  on  $K_n$  is defined by

$$\rho_i(v_k) = v_{(k+i) \pmod n} \text{ for } 0 \leq i \leq n-1, 0 \leq k \leq n-1$$

$$\sigma_i(v_k) = v_{(n+i-k) \pmod n} \text{ for } 0 \leq i \leq n-1, 0 \leq k \leq n-1$$

Then this action induces the action of  $D_n$  on  $X_n$  and  $Y_n$ .

The equivalence classes of  $X_8$  are given with the next figure.



The equivalence classes of  $X_{10}$  are given with the next figure.



By applying Burnside's lemma, we calculate the number of the equivalence classes of  $X_n$  and  $Y_n$  under this group action. This problem was presented by Dr. Shun-ichiro Koh who is a physicist of Kochi University. This problem is related to Feynman diagram in quantum mechanical many body problem.

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**Notation 1.** For each integer  $i$  such that  $0 \leq i \leq n-1$ , let  $d = (n, i)$  and  $R_i^n$  be defined by the following formula:

$$R_i^n = \begin{cases} 0 & \text{if } d \text{ is odd and } n/2 \equiv 0 \pmod{2} \\ \sum_{\substack{d=2s+t \\ s \geq 0, t \geq 0}} \frac{d!}{2^s s! t!} \left(\frac{n}{2d}\right)^s & \text{if } d \text{ is odd and } n/2 \equiv 1 \pmod{2} \\ \left(\frac{d}{2}\right)! \times \left(\frac{n}{d}\right)^{d/2} & \text{if } d \text{ is even} \end{cases}$$

**Remark 1.** It is easily checked that  $R_0^n$  is equal to  $(n/2)!$ .

**Notation 2.** Let  $S_0^n$  and  $S_1^n$  be defined by the following formula:

$$S_0^n = \begin{cases} 0 & \text{if } n/2 \equiv 0 \pmod{2} \\ \left(\frac{n-2}{4}\right)! \times 2^{\frac{n-2}{4}} & \text{if } n/2 \equiv 1 \pmod{2} \end{cases}$$

$$S_1^n = \sum_{\substack{\frac{n}{2}=2s+t \\ s \geq 0, t \geq 0}} \frac{\left(\frac{n}{2}\right)!}{2^s s! t!}$$

**Theorem 1.** For even  $n > 0$ , the number of non-equivalent odd 1-factors of  $K_n$  under the action of the Dihedral group  $D_n$  is

$$\frac{1}{2n} \left\{ \sum_{i=0}^{n-1} R_i^n + \frac{n}{2} (S_0^n + S_1^n) \right\}$$

**Remark 2.** We calculated the non-equivalent odd 1-factors of  $K_n$ ,  $n \leq 16$ , under the action of the Dihedral group  $D_n$  by a computer. The numbers agreed with the numbers that are given by Theorem 1. The results is as follows:

n=2	1
n=4	1
n=6	3
n=8	5
n=10	17
n=12	53
n=14	260
n=16	1466

**Notation 3.** Let  $n \equiv 0 \pmod{4}$ . For each integer  $i$  such that  $0 \leq i \leq n-1$ , let  $d = (n, i)$  and  $P_i^n$  be defined by the following formula:

$$P_i^n = \begin{cases} \sum_{\substack{d/2=2s_1+t_1 \\ d/2=2s_2+t_2 \\ s_1, t_1, s_2, t_2 \geq 0}} \frac{(d/2)! \times (d/2)!}{2^{s_1+s_2} s_1! s_2! t_1! t_2!} \left(\frac{n}{d}\right)^{s_1+s_2} & \text{if } d \text{ is even and } n/d \equiv 0 \pmod{2} \\ \left(\frac{d-2}{2}\right)!! \times \left(\frac{d-2}{2}\right)!! \times \left(\frac{n}{d}\right)^{\frac{d}{2}} & \text{if } d \text{ is even and } n/d \equiv 1 \pmod{2} \\ \sum_{\substack{d=2s+t \\ s \geq 0, t \geq 0}} \frac{d!}{2^s s! t!} \left(\frac{n}{2d}\right)^s & \text{if } d \text{ is odd} \end{cases}$$

**Remark 3.** It is easily checked that if  $n \equiv 0 \pmod{4}$  then  $P_0^n$  is equal to  $(n/2 - 1)!! \times (n/2 - 1)!!$ .

**Notation 4.** Let  $n \equiv 0 \pmod{4}$ . Let  $Q_0^n$  and  $Q_1^n$  be defined by the following formula:

$$Q_0^n = \sum_{\substack{\frac{n}{4} = 2s_1 + t_1 \\ \frac{n}{4} = 2s_2 + t_2 \\ s_1, s_2, t_1, t_2 \geq 0}} \frac{\left(\frac{n}{4}\right)! \left(\frac{n-4}{4}\right)!}{2^{s_1+s_2} s_1! s_2! t_1! t_2!} \times 2^{s_1+s_2}$$

$$Q_1^n = \left(\frac{n-2}{2}\right)!!.$$

**Theorem 2.** If  $n \equiv 2 \pmod{4}$  then  $K_n$  has no even 1-factors. If  $n \equiv 0 \pmod{4}$  then the number of the non-equivalent even 1-factors of  $K_n$  under the action of the Dihedral group  $D_n$  is

$$\frac{1}{2n} \left\{ \sum_{i=0}^{n-1} P_i^n + \frac{n}{2} (Q_0^n + Q_1^n) \right\}.$$

**Remark 4.** We calculated the non-equivalent even 1-factors of  $K_n$ ,  $n \leq 16$ , under the action of the Dihedral group  $D_n$  by a computer. The numbers agreed with the numbers that are given by Theorem 2. The results is as follows:

n=4	1
n=8	3
n=12	22
n=16	436

These computations can be done by applying Burnside's lemma.

**Theorem 3.** (Burnside's lemma) Let  $G$  be a group of permutations acting on a set  $S$ . Then the number of orbits induced on  $S$  is given by

$$\frac{1}{|G|} \sum_{\pi \in G} |fix(\pi)|$$

where  $fix(\pi) = \{x \in S | \pi(x) = x\}$ .

## 2 Odd 1-factors

We prove Theorem 1. We must determine the numbers of the fixed points of each permutation  $\rho_i$  and  $\sigma_i$  to prove the Theorem by applying Burnside's Lemma.

**Lemma 1.** The number of the odd 1-factors of  $K_n$  is  $(n/2)!$ . This is the number of the fixed points of  $\rho_0$ .

**Proof.**  $v_0$  is able to join any vertex of  $\{v_1, v_3, \dots, v_{n-1}\}$ . Then the number of ways of joining is  $n/2$ . Since the suffix of one vertex of the first edge is even and the suffix of the other is odd, the number of ways of choosing of the second edge is  $(n-2)/2$  even if we choose any vertex as one end vertex of the edge. When we choose  $j$  edges, we use  $j$  vertices which have even suffix and  $j$  vertices which have odd suffix. Then the number of ways of choosing of the  $j+1$ th edge is  $(n-2j)/2$  even if we choose any vertex as one end vertex of the edge. Therefore, the number of the odd 1-factors of  $K_n$  is  $(n/2)!$ .  $\square$

**Lemma 2.** If  $(n,i)=1$  then the number of the fixed points of  $\rho_i$  is one if  $n \equiv 2 \pmod{4}$  and is zero if  $n \equiv 0 \pmod{4}$ .

*Proof.* Let  $M_n$  be the 1-factor  $\{v_\alpha v_{n/2+\alpha} | 0 \leq \alpha \leq n/2 - 1\}$  of  $K_n$ . If  $n \equiv 2 \pmod{4}$  then  $n/2$  is odd and  $M_n$  is the odd 1-factor of  $K_n$  and  $\rho_i(M_n) = M_n$ . Conversely, let  $H$  be a odd 1-factor of  $K_n$  which is fixed by  $\rho_i$  and let  $v_0 v_m$  be an edge of  $H$ . Since  $(n,i)=1$ , there is an integer  $\alpha$  such that  $\alpha i \equiv m \pmod{n}$ . Then  $\rho_i^\alpha(v_0) = v_m$  and  $\rho_i^\alpha(v_m) = v_{(m+i\alpha) \pmod{n}}$ . Since  $\rho_i(H) = H$ , we have  $v_0 v_m = v_m v_{(m+i\alpha) \pmod{n}}$ . Then we have  $m + i\alpha \equiv 0 \pmod{n}$  and  $2m \equiv 0 \pmod{n}$  and therefore  $m = n/2$  and  $v_0 v_{n/2} \in H$ . If  $n \equiv 0 \pmod{4}$  then  $n/2$  is even. This is contradiction. Then if  $n \equiv 0 \pmod{4}$  then the number of fixed points of  $\rho_i$  is zero. We assume that  $n \equiv 2 \pmod{4}$ . Since  $\{\rho_i^\alpha(0) | 0 \leq \alpha \leq n-1\} = \{0, 1, 2, \dots, n-1\}$ ,  $H$  is uniquely determined by  $v_0 v_{n/2}$  and  $H = \{v_\alpha v_{n/2+\alpha} | 0 \leq \alpha \leq n/2 - 1\}$ . Then the number of the fixed points of  $\rho_i$  is one.  $\square$

**Lemma 3.** Let  $(n, i) = d$  be greater than one. The number of the fixed points of  $\rho_i$  is given by the following formula:

1. In the case that  $d$  is odd:
  - (a) if  $n/2 \equiv 0 \pmod{2}$  then 0.
  - (b) if  $n/2 \equiv 1 \pmod{2}$  then

$$\sum_{\substack{d=2s+t \\ s \geq 0, t \geq 0}} \frac{d!}{2^s s! t!} \left(\frac{n}{2d}\right)^s.$$

2. If  $d$  is even then

$$\left(\frac{d}{2}\right)! \times \left(\frac{n}{d}\right)^{\frac{d}{2}}.$$

*Proof.* Let  $V_0 = \{v_0, v_d, v_{2d}, \dots, v_{n-d}\}$ ,  $V_1 = \{v_1, v_{d+1}, v_{2d+1}, \dots, v_{n-d+1}\}$ ,  $V_2 = \{v_2, v_{d+2}, v_{2d+2}, \dots, v_{n-d+2}\}$ ,  $\dots$ ,  $V_{d-1} = \{v_{d-1}, v_{2d-1}, v_{3d-1}, \dots, v_{n-1}\}$ .

Since  $(n, i) = d$ , the equation  $xi \equiv m \pmod{n}$  has a solution if and only if  $d$  divides  $m$ . Then we have  $\rho_i(V_k) = V_k$  for  $0 \leq k \leq d-1$ .

Let  $H$  be an odd 1-factor of  $K_n$  which is fixed by  $\rho_i$  and let  $v_\alpha v_\beta$  be an edge of  $H$ . If  $v_\alpha \in V_k$  and  $v_\beta \in V_k$  then the induced subgraph  $H|V_k$  is an odd 1-factor of  $K_{n/d}$  which is fixed by  $\rho_{i/d}$  and it is unique odd 1-factor  $M_{n/d}$  by Lemma 2. If  $v_\alpha \in V_{k_1}$  and  $v_\beta \in V_{k_2}$  then the induced subgraph  $H|V_{k_1} \cup V_{k_2}$  is an odd 1-factor of  $K_{2n/d}$  which is fixed by  $\rho_{i/d}$ .

We first consider the case that  $d$  is odd. Since  $n/d$  is even, the number of the vertices in  $V_k$  whose suffixes are odd is equal to the number of the vertices in  $V_k$  whose suffixes are even for any  $k$ .

If  $n/2$  is even then  $V_k$  is not able to make the odd 1-factor which is fixed by  $\rho_i$  alone, because  $\frac{n}{2d}$  is even. Furthermore, we can not partition  $\{V_0, V_1, V_2, \dots, V_{d-1}\}$  into  $d/2$  pairs, since  $d$  is odd. Therefore, if  $n/2$  is even then the number of the fixed points of  $\rho_i$  is zero.

On the other hand, if  $n/2$  is odd then  $v_0 v_{d(\frac{n}{2d})}$  is an odd edge. Then  $V_k$  is able to make the unique odd 1-factor which is fixed by  $\rho_i$  alone. Furthermore, any  $V_{k_1}$  and  $V_{k_2}$  are able to make  $\frac{n}{2d}$  odd 1-factors which are fixed by  $\rho_i$ . The number of ways to partition  $\{V_0, V_1, V_2, \dots, V_{d-1}\}$  into  $s$  pairs and  $t$  singletons is equal to

$$\frac{d!}{2^s s! t!}.$$

Then, the number of the odd 1-factors fixed by  $\rho_i$  is

$$\sum_{\substack{d=2s+t \\ s \geq 0, t \geq 0}} \frac{d!}{2^s s! t!} \left(\frac{n}{2d}\right)^s.$$

Next we consider the case that  $d$  is even.

$V_0, V_2, V_4, \dots, V_{d-2}$  are the set whose elements have even suffix and  $V_1, V_3, V_5, \dots, V_{d-1}$  are the set whose elements have odd suffix. Then  $V_k$  is not able to make odd 1-factor which is fixed by  $\rho_i$  alone. But  $V_k$ , for  $k$  is even, and  $V_l$ , for  $l$  is odd, are able to make  $\frac{n}{d}$  odd 1-factors fixed by  $\rho_i$ . The number of ways to partition  $\{V_0, V_1, V_2, \dots, V_{d-1}\}$  into  $d/2$  pairs of the type  $\{V_{even}, V_{odd}\}$  is equal to  $(\frac{d}{2})!$ . Then, the number of the odd 1-factors fixed by  $\rho_i$  is

$$\left(\frac{d}{2}\right)! \times \left(\frac{n}{d}\right)^{\frac{d}{2}}.$$

We have the results. □

**Lemma 4.** The number of the fixed points of  $\sigma_0$  is equal to the number of the fixed points of  $\sigma_{2d}$  for all  $1 \leq d \leq n/2 - 1$ .

*Proof.* Let  $H$  be an odd 1-factor of  $K_n$  fixed by  $\sigma_0$ . Then it is easily verified that  $\rho_d(H)$  is an odd 1-factor of  $K_n$  fixed by  $\sigma_{2d}$ . Conversely, if  $H$  is an odd 1-factor of  $K_n$  fixed by  $\sigma_{2d}$  then  $\rho_d^{-1}(H)$  is an odd 1-factor of  $K_n$  fixed by  $\sigma_0$ . Then we have the results. □

Similarly, we have the next lemma.

**Lemma 5.** The number of the fixed points of  $\sigma_1$  is equal to the number of the fixed points of  $\sigma_{2d+1}$  for all  $1 \leq d \leq n/2 - 1$ .

**Lemma 6.** The number of the fixed points of  $\sigma_0$  is given by the following formula:

1. if  $n/2 \equiv 0 \pmod{2}$  then 0.
2. if  $n/2 \equiv 1 \pmod{2}$  then  $\left(\frac{n-2}{4}\right)! \times 2^{\frac{n-2}{4}}$ .

*Proof.* Since the axis of  $\sigma_0$  passes through  $v_0$  and  $v_{n/2}$ , the odd 1-factor of  $K_n$  fixed by  $\sigma_0$  must contain the edge  $v_0 v_{n/2}$ . If  $n/2 \equiv 0 \pmod{2}$  then the edge  $v_0 v_{n/2}$  is even and the number of the fixed points of  $\sigma_0$  must be zero. Therefore we assume that  $n/2 \equiv 1 \pmod{2}$ . Let  $V_1 = \{v_1, v_{n-1}\}, V_2 = \{v_2, v_{n-2}\}, V_3 = \{v_3, v_{n-3}\}, \dots, V_{n/2-1} = \{v_{n/2-1}, v_{n/2+1}\}$ . Then we have  $\sigma_0(V_k) = V_k$  for  $1 \leq k \leq n/2 - 1$ .  $V_1, V_3, V_5, \dots, V_{n/2-2}$  are the sets whose elements have odd suffixes and  $V_2, V_4, V_6, \dots, V_{n/2-1}$  are the sets whose elements have even suffixes.

Let  $H$  be an odd 1-factor of  $K_n$  fixed by  $\sigma_0$  and let  $v_\alpha v_\beta$  be an edge of  $H$ . If  $\alpha$  and  $\beta$  are not equal to zero then  $v_\alpha \in V_{2k}$  and  $v_\beta \in V_{2l+1}$  or  $v_\alpha \in V_{2k+1}$  and  $v_\beta \in V_{2l}$  for some  $k$  and  $l$ . The number of ways to partition  $\{V_1, V_2, \dots, V_{n/2-1}\}$  into  $(n-2)/4$  pairs of the type  $\{V_{even}, V_{odd}\}$  is equal to  $(\frac{n-2}{4})!$ . The number of ways to choosing the edge between  $V_{2k}$  and  $V_{2l+1}$  is two. Then the number of the fixed points of  $\sigma_0$  is

$$\left(\frac{n-2}{4}\right)! \times 2^{\frac{n-2}{4}}.$$

We have the results. □

**Lemma 7.** The number of the fixed points of  $\sigma_1$  is

$$\sum_{\substack{n/2=2s+t \\ s \geq 0, t \geq 0}} \frac{\left(\frac{n}{2}\right)!}{2^s s! t!}.$$

*Proof.* Let  $V_1 = \{v_1, v_0\}, V_2 = \{v_2, v_{n-1}\}, V_3 = \{v_3, v_{n-2}\}, \dots, V_{n/2} = \{v_{n/2}, v_{n/2+1}\}$ . Then we have  $\sigma_1(V_k) = V_k$  for  $1 \leq k \leq n/2$ . Each  $V_k$  contains one vertex whose suffix is even and one vertex whose suffix is odd. Let  $H$  be an odd 1-factor of  $K_n$  fixed by  $\sigma_1$  and let  $v_\alpha v_\beta$  be an edge of  $H$ . Then  $v_\alpha$  and  $v_\beta$  are contained in same  $V_k$  or  $v_\alpha \in V_{k_1}$  and  $v_\beta \in V_{k_2}$  for some  $k_1$  and  $k_2$ . The number of ways to partition  $\{V_1, V_2, V_3, \dots, V_{n/2}\}$  into  $s$  pairs and  $t$  singletons is equal to

$$\frac{\left(\frac{n}{2}\right)!}{2^s s! t!}.$$

Then the number of the fixed points of  $\sigma_1$  is

$$\sum_{\substack{n/2=2s+t \\ s \geq 0, t \geq 0}} \frac{\left(\frac{n}{2}\right)!}{2^s s! t!}.$$

We have the results. □

Then we completely proved Theorem 1.

### 3 Even 1-factors

Next we prove Theorem 2. We must determine the numbers of the fixed points of each permutation  $\rho_i$  and  $\sigma_i$  to prove the Theorem by applying Burnside's Lemma.

**Lemma 8.** The number of the even 1-factors of  $K_n$  is zero if  $n \equiv 2 \pmod{4}$  and is  $(n/2 - 1)!! \times (n/2 - 1)!!$  if  $n \equiv 0 \pmod{4}$ . This is the number of the fixed points of  $\rho_0$ .

*Proof.* In order to partition  $\{v_0, v_2, v_4, \dots, v_{n-2}\}$  into  $n/4$  subsets which consist two elements  $n/2$  must be even. Then, if  $n \equiv 2 \pmod{4}$  then the number of the even 1-factors of  $K_n$  is zero. Therefore we assume that  $n \equiv 0 \pmod{4}$ . The number of ways to partition  $\{v_0, v_2, v_4, \dots, v_{n-2}\}$  into  $n/4$  pairs is equal to  $(n/2 - 1)!!$  and the number of ways to partition  $\{v_1, v_3, v_5, \dots, v_{n-1}\}$  into  $n/4$  pairs is equal to  $(n/2 - 1)!!$ . Therefore if  $n \equiv 0 \pmod{4}$  then the number of the even 1-factors of  $K_n$  is  $(n/2 - 1)!! \times (n/2 - 1)!!$ . We have the results. □

By Lemma 8 we have that if  $n \equiv 2 \pmod{4}$  then the number of the non-equivalent even 1-factors of  $K_n$  is zero.

Therefore we assume from now on that  $n \equiv 0 \pmod{4}$ .

**Lemma 9.** If  $(n, i) = 1$  then the number of the fixed points of  $\rho_i$  is one.

*Proof.* Let  $M_n$  be the 1-factor  $\{v_\alpha v_{n/2+\alpha} | 0 \leq \alpha \leq n/2 - 1\}$  of  $K_n$ . Since  $n/2$  is even,  $M_n$  is the even 1-factor of  $K_n$  and  $\rho_i(M_n) = M_n$ . Conversely, let  $H$  be an even 1-factor of  $K_n$  which is fixed by  $\rho_i$  and let  $v_0 v_m$  be an edge of  $H$ . By the essentially the same methods, we have  $H = \{v_\alpha v_{n/2+\alpha} | 0 \leq \alpha \leq n/2 - 1\}$ . Then the number of the fixed points of  $\rho_i$  is one. □

**Lemma 10.** Let  $(n, i) = d$  be greater than one. The number of the fixed points of  $\rho_i$  is given by the following formula:

1. If  $d$  is odd then

$$\sum_{\substack{d=2s+t \\ s, t \geq 0}} \frac{d!}{2^s s! t!} \left(\frac{n}{2d}\right)^s.$$

2. In the case that  $d$  is even:

(a) If  $n/d \equiv 0 \pmod{2}$  then

$$\sum_{\substack{d/2=2s_1+t_1 \\ d/2=2s_2+t_2 \\ s_1, s_2, t_1, t_2 \geq 0}} \frac{(d/2)! \times (d/2)!}{2^{s_1+s_2} s_1! s_2! t_1! t_2!} \left(\frac{n}{d}\right)^{s_1+s_2}.$$

(b) If  $n/d \equiv 1 \pmod{2}$  then

$$\left(\frac{d-2}{2}\right)!! \times \left(\frac{d-2}{2}\right)!! \times \left(\frac{n}{d}\right)^{\frac{d}{2}}.$$

*Proof.* Let  $V_0 = \{v_0, v_d, v_{2d}, \dots, v_{n-d}\}$ ,  $V_1 = \{v_1, v_{d+1}, v_{2d+1}, \dots, v_{n-d+1}\}$ ,  $V_2 = \{v_2, v_{d+2}, v_{2d+2}, \dots, v_{n-d+2}\}$ ,  $\dots$ ,  $V_{d-1} = \{v_{d-1}, v_{2d-1}, v_{3d-1}, \dots, v_{n-1}\}$ . Then we have  $\rho_i(V_k) = V_k$  for  $0 \leq k \leq d-1$ .

We first consider the case that  $d$  is odd. Each  $V_k$ , for  $0 \leq k \leq d-1$ , contains  $\frac{n}{2d}$  vertices whose suffix is even and  $\frac{n}{2d}$  vertices whose suffixes are odd. Since  $\frac{n}{2d}$  is even, each  $V_k$  is able to make unique even 1-factor fixed by  $\rho_i$  alone. Furthermore any  $V_k$  and  $V_l$  are able to make  $\frac{n}{2d}$  even 1-factors fixed by  $\rho_i$ . Then the number of the fixed points of  $\rho_i$  is

$$\sum_{\substack{d=2s+t \\ s, t \geq 0}} \frac{d!}{2^s s! t!} \left(\frac{n}{2d}\right)^s.$$

Next we consider the case that  $d$  is even. We first consider the case that  $n/d$  is even.  $V_0, V_2, V_4, \dots, V_{d-2}$  contain  $n/d$  vertices whose suffixes are even and  $V_1, V_3, V_5, \dots, V_{d-1}$  contain  $n/d$  vertices whose suffixes are odd. Since  $n/d$  is even, each  $V_k$  is able to make unique even 1-factor fixed by  $\rho_i$  alone. Furthermore any two elements of  $\{V_0, V_2, V_4, \dots, V_{d-2}\}$  and any two elements of  $\{V_1, V_3, V_5, \dots, V_{d-1}\}$  are able to make  $\frac{n}{d}$  even 1-factors fixed by  $\rho_i$ , respectively. Then the number of the fixed points of  $\rho_i$  is

$$\sum_{\substack{d/2=2s_1+t_1 \\ d/2=2s_2+t_2 \\ s_1, s_2, t_1, t_2 \geq 0}} \frac{(d/2)! \times (d/2)!}{2^{s_1+s_2} s_1! s_2! t_1! t_2!} \left(\frac{n}{d}\right)^{s_1+s_2}.$$

Next we consider the case that  $n/d$  is odd. Since  $n/d$  is odd,  $V_k$  is not able to make even 1-factor fixed by  $\rho_i$  alone. But any two elements of  $\{V_0, V_2, V_4, \dots, V_{d-2}\}$  and any two elements of  $\{V_1, V_3, V_5, \dots, V_{d-1}\}$  are able to make  $\frac{n}{d}$  even 1-factors fixed by  $\rho_i$ , respectively. Since  $n \equiv 0 \pmod{4}$  and  $n/d$  is odd,  $d \equiv 0 \pmod{4}$ . Then the number of the fixed points of  $\rho_i$  is

$$\left(\frac{d-2}{2}\right)!! \times \left(\frac{d-2}{2}\right)!! \times \left(\frac{n}{d}\right)^{\frac{d}{2}}.$$

We have the results. □

The next two Lemmas are proved by the essentially the same methods as Lemma 4.

**Lemma 11.** The number of the fixed points of  $\sigma_0$  is equal to the number of the fixed points of  $\sigma_{2d}$  for all  $1 \leq d \leq n/2 - 1$ .

**Lemma 12.** The number of the fixed points of  $\sigma_1$  is equal to the number of the fixed points of  $\sigma_{2d+1}$  for all  $1 \leq d \leq n/2 - 1$ .

**Lemma 13.** The number of the fixed points of  $\sigma_0$  is

$$\sum_{\substack{\frac{n}{4}=2s_1+t_1 \\ \frac{n-4}{4}=2s_2+t_2 \\ s_1, s_2, t_1, t_2 \geq 0}} \frac{\left(\frac{n}{4}\right)! \times \left(\frac{n-4}{4}\right)!}{2^{s_1+s_2} s_1! t_1! s_2! t_2!} \times 2^{s_1+s_2}.$$

**Proof.** Let  $V_1 = \{v_1, v_{n-1}\}, V_2 = \{v_2, v_{n-2}\}, V_3 = \{v_3, v_{n-3}\}, \dots, V_{n/2-1} = \{v_{n/2-1}, v_{n/2+1}\}$ . Then we have  $\sigma_0(V_k) = V_k$  for  $1 \leq k \leq n/2 - 1$ . Each of  $V_1, V_3, V_5, \dots, V_{n/2-1}$  contains two vertices whose suffixes are odd and each of  $V_2, V_4, V_6, \dots, V_{n/2-2}$  contains two vertices whose suffixes are even. Then each  $V_k$  is able to make unique even 1-factor fixed by  $\sigma_0$  alone. Furthermore any two elements of  $\{V_2, V_4, \dots, V_{n/2-2}\}$  and any two elements of  $\{V_1, V_3, V_5, \dots, V_{n/2-1}\}$  are able to make two even 1-factors fixed by  $\sigma_0$ , respectively. Then the number of the fixed points of  $\rho_i$  is

$$\sum_{\substack{\frac{n}{4}=2s_1+t_1 \\ \frac{n-4}{4}=2s_2+t_2 \\ s_1, s_2, t_1, t_2 \geq 0}} \frac{\left(\frac{n}{4}\right)! \times \left(\frac{n-4}{4}\right)!}{2^{s_1+s_2} s_1! t_1! s_2! t_2!} \times 2^{s_1+s_2}.$$

□

**Lemma 14.** The number of the fixed points of  $\sigma_1$  is  $(n/2 - 1)!!$ .

**Proof.** Let  $V_1 = \{v_1, v_0\}, V_2 = \{v_2, v_{n-1}\}, V_3 = \{v_3, v_{n-2}\}, \dots, V_{n/2} = \{v_{n/2}, v_{n/2+1}\}$ . Then we have  $\sigma_1(V_k) = V_k$  for  $1 \leq k \leq n/2$ . Each  $V_k$  contains one vertex whose suffix is even and one vertex whose suffix is odd. Then  $V_k$  is not able to make even 1-factor fixed by  $\sigma_1$  alone. But any two elements of  $\{V_1, V_2, V_3, \dots, V_{n/2}\}$  are able to make unique even 1-factor fixed by  $\sigma_1$ . Then the number of the fixed points of  $\sigma_1$  is  $(n/2 - 1)!!$ .

We have the results.

□

Then we completely proved Theorem 2.

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