Scientiae Mathematicae Japonicae Online, **Vol. 9,** (2003), 159–165

B-ALGEBRAS AND GROUPS

P. J. Allen, J. Neggers and Hee Sik Kim

Received February 18, 2003

Abstract. In this note, we give another proof of the close relationship of B-algebras with groups using the observation that the zero adjoint mapping is surjective. Moreover, we find a condition for an algebra defined on the real numbers to be a B-algebra using the analytic method. In addition we note certain other facts about commutative B-algebras.

1. Introduction.

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([9]) introduced the notion of d-algebras, i.e., (I) $x * x = 0$; (V) $0 * x = 0$; (VI) $x * y = 0$ and $y * x = 0$ imply $x = y$, which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([6]) introduced a new notion, called an BH-algebra, i.e., (I), (II) $x*0 = x$ and (VI), which is a generalization of $BCH/BCI/BCK$-algebras, and defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. Recently J. Neggers and H. S. Kim ([11]) introduced a new notion which appears to be of some interest, i.e., that of a B-algebra, and studied some of its properties. M. Kondo and Y. B. Jun ([7]) proved that the class of B-algebras is equivalent in one sense to the class of groups by using the property: $x = 0 * (0 * x)$, for all $x \in X$. J. Neggers and H. S. Kim ([11]) argued slightly differently in taking their position. In this note, we give another proof using that the zero adjoint mapping is surjective. Moreover, we find a condition for an algebra defined on the real numbers to be a B-algebra using the analytic method. In addition we note certain other facts about commutative B-algebras.

2. Preliminaries.

A *B-algebra* ([11]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $(x * y) * z = x * (z * (0 * y))$
for all x, y, z in X.

2000 *Mathematics Subject Classification.* 06F35, 20N02.
Keywords and phrases. B-algebra, group, commutative, center.

If we let $y := x$ in (III), then we have

(a) $$(x * x) * z = x * (z * (0 * x)).$$

If we let $z := x$ in (a), then we obtain also

(b) $$0 * x = x * (x * (0 * x)).$$

Using (I) and (a), it follows that

(c) $$0 = x * (0 * (0 * x)).$$

We have already discussed that the three axioms (I), (II) and (III) are independent (see [11]).

Example 2.1. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X : *, 0)$ is a B-algebra.

Example 2.2 ([11])**.** Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x * y := \frac{n(x - y)}{n + y}.$$

Then $(X; *, 0)$ is a B-algebra.

Lemma 2.3 ([11])**.** *If $(X; *, 0)$ is a B-algebra, then $y * z = y * (0 * (0 * z))$ for all $y, z \in X$.*

If we take $y := 0$ in Lemma 2.3, we obtain a useful formula

(d) $$0 * z = 0 * (0 * (0 * z)).$$

Proposition 2.4 ([11])**.** *If $(X; *, 0)$ is a B-algebra, then*

(IV) $$x * (y * z) = (x * (0 * z)) * y$$

for any $x, y, z \in X$.

3. B-algebras and groups.

Proposition 3.1. *Let* $(X; \circ, 0)$ *be a group. If we define* $x * y := x \circ y^{-1}$*, then* $(X; *, 0)$ *is a* B*-algebra.*

Proof. We know that $x * x = x \circ x^{-1} = 0$ and $x * 0 = x \circ 0^{-1} = x \circ 0 = x$. For any x, y, z in X, we see that $(x*y)*z = (x \circ y^{-1}) \circ z^{-1} = x \circ (z \circ y)^{-1} = x*(z*y^{-1}) = x*(z*(0*y))$. □

¿From the above Proposition 3.1 we can see that every group $(X; \circ, 0)$ determines a B-algebra $(X; *, 0)$, called a *group-derived* B-algebra. It is then a question of interest to determine whether or not all B-algebras are so group-derived. We claim that this is not the case in general, and thus that this class of algebras contains the class of groups indirectly via the group-derived principle we have explained in Proposition 3.1.

Proposition 3.2. *Let* $(X; *, 0)$ *be a group-derived* B*-algebra. Define a map* $\varphi : X \to X$ *by* $\varphi(x) := 0 * x$*, then* φ *is a surjection.*

Proof. If $g \in X$, then $\varphi(g^{-1}) = 0 * g^{-1} = 0 \circ (g^{-1})^{-1} = g$. □

The mapping φ discussed in Proposition 3.2 is called a *zero adjoint* mapping. The Proposition 3.2 means that if φ is not surjective, then the algebra $(X; *, 0)$ cannot be a group-derived B-algebra. Hence the condition that $\varphi : X \to X$ be a surjection is certainly necessary for the B-algebra to be group derived.

Theorem 3.3. *Let* $(X; *, 0)$ *be a* B*-algebra. If the map* $\varphi : X \to X$ *by* $\varphi(x) := 0 * x$ *is a surjection, then the algebra* $(X; *, 0)$ *is group derived.*

Proof. Let $(X; *, 0)$ be a B-algebra. Assume the zero adjoint mapping $\varphi : X \to X$ is a surjection. If $x \in X$, then there is $y \in X$ such that $x = 0 * y$ and hence $0 * (0 * x) = 0 * (0 * (0 * y)) = (0 * y) * 0 = 0 * y = x$, i.e.,

$$0 * (0 * x) = x. \tag{e}$$

Define a binary operation " $\circ$ " on X by

$$x \circ y := x * (0 * y).$$

Then $(X; *, 0)$ is a group. In fact, it follows that $x \circ 0 = x * (0 * 0) = x * 0 = x$ and $0 \circ x = 0 * (0 * x) = x$. Therefore 0 acts like an identity element on X. Also, $x \circ (0 * x) = x * (0 * (0 * x)) = (x * x) * 0 = 0$ and $(0 * x) \circ x = (0 * x) * (0 * x) = 0$, i.e., $0 * x$ behaves like a multiplicative inverse for the element x with respect to the operation $\circ$. Finally, in order to establish the associative law, we obtain:

$$\begin{aligned}
x \circ (y \circ z) &= x * (0 * (y * (0 * z)) \\
&= x * ((0 * z) * y) && \text{[by (III)]} \\
&= x * ((0 * z) * (0 * (0 * y))) && \text{[by (e)]} \\
&= (x * (0 * y)) * (0 * z) && \text{[(III)]} \\
&= (x \circ y) \circ z.
\end{aligned}$$

Note that $x \circ y^{-1} = x * (0 * y^{-1}) = x * (0 * (0 * y)) = x * y$, whence $(X, ; *, 0)$ is also group derived from the group $(X; \circ, 0)$ as defined. This proves the theorem. □

Theorem 3.4. *Every B-algebra is group derived.*

Proof. Let $\varphi : X \to X$ be the zero adjoint mapping defined by $\varphi(x) := 0 * x$. Let $t \in X$, and let $x = \varphi(t) \in X$. Then we observe that

$$\begin{aligned} \varphi(x) &= 0 * x \\ &= (t * t) * x && \text{[by (I)]} \\ &= t * (x * (0 * t)) && \text{[by (III)]} \\ &= t * (x * x) && [x = \varphi(t) = 0 * x] \\ &= t. && \text{[by (I), (II)]} \end{aligned}$$

Conssequently, φ is a surjective. By applying Theorem 3.3 we conclude that every B-algebra is group derived. □

Remark. Let $(G; \circ, e)$ be an arbitrary group. If we define $x * y := yxy^{-2}$, then $x * x = e$ and $x * e = x$ and $e * x = x^{-1}$. Now consider the expressions $(x * y) * z = zyxy^{-2}z^{-2}$ and $x * (z * (e * y)) = x * (y^{-1}zy^2) = (y^{-1}zy^2)x(y^{-1}zy^2)^{-2}$. Thus, let us assume that is actually the case that $zyxy^{-2}z^{-2} = (y^{-1}zy^2)x(y^{-1}zy^2)^{-2} \cdots (*)$ in $(G; \circ, e)$. It follows that since $\varphi(x) = e * x = x^{-1}$ is a surjection, $(G; *, e)$ is group derived, i.e., there is an operation "$\circledast$" such that $x * y = x \circledast y^{(-1)}$, where $y^{(-1)} \circledast y = y \circledast y^{(-1)} = e = y * y$. But this means that $x^{-1} = e * x^{-1} = e \circledast x^{(-1)} = x^{(-1)}$, i.e., $x^{-1} = x^{(-1)}$, and hence that $x * y = x \circledast y^{-1}$. In fact, the condition leads to the conclusion that G is an abelian group, i.e., yxy^{-2} becomes xy^{-1}.

Recently, J. Neggers and H. S. Kim ([10]) investigated analytic T-algebras and obtained useful formulas for finding some examples for various BCK-related algebras. We apply the same method discussed there to the class of B-algebras. Suppose that we set $x * y := x - \varphi(x, y)$ where $\varphi : R^2 \to R$ is an arbitrary function of two variables on the real numbers R. If $x * x = x - \varphi(x, x) = 0$, then $\varphi(x, x) = x$, while if $x * 0 = x - \varphi(x, 0) = x$, then $\varphi(x, 0) = 0$. If the condition (III) holds, then

$$\begin{aligned} (x * y) * z &= x * y - \varphi(x * y, z) \\ &= x - \varphi(x, y) - \varphi(x * y, z) \\ &= x - \varphi(x, y) - \varphi(x - \varphi(x, y), z) \end{aligned}$$

and

$$\begin{aligned} x * (z * (0 * y)) &= x - \varphi(x, z * (0 * y)) \\ &= x - \varphi(x, z - \varphi(z, 0 * y)) \\ &= x - \varphi(x, z - \varphi(z, -\varphi(0, y))). \end{aligned}$$

It follows that

(f) $$x - \varphi(x, y) - \varphi(x - \varphi(x, y), z) = x - \varphi(x, z - \varphi(z, -\varphi(0, y)))$$

If φ satisfies the condition (i), then $(R; *, 0)$ is a B-algebra. We summarize:

Proposition 3.5. *Let $\varphi : R^2 \to R$ be an arbitrary function of two variables on the real numbers R satisfying $\varphi(x, x) = x$ and $\varphi(x, 0) = 0$. If the mapping φ satisfies the condition (f), then $(R; *, 0)$ is a B-algebra.*

4. Commutativity and center.

A B-algebra $(X;*,0)$ is said to be *commutative* ([11]) if $a*(0*b)=b*(0*a)$ for any $a,b\in X$.

Proposition 4.1. ([11]) *If $(X;*,0)$ is a commutative B-algebra, then*

$$x*y=(0*y)*(0*x). \tag{g}$$

for any $x,y\in X$.

Proposition 4.2. ([1]) *If $(X;*,0)$ is a B-algebra, then $0*(0*x)=x$ for any $x\in X$.*

Proposition 4.3. *If $(X;*,0)$ is a B-algebra with the condition (g), then X is commutative.*

Proof. By applying Proposition 4.2 we obtain:

$$\begin{aligned} x*(0*y) &= (0*(0*y))*(0*x) \\ &= y*(0*x) \end{aligned}$$

for any $x,y\in X$. □

Theorem 4.4. *Let $(X;*,0)$ be a B-algebra derived from a group $(X;\circ,0)$. Then $(X;*,0)$ is commutative if and only if $(X;\circ,0)$ is commutative.*

Proof. Since $x*y=x\circ y^{-1}$, we have

$$\begin{aligned} x*(0*y) &= x*(0\circ y^{-1}) \\ &= x*y^{-1} \\ &= x\circ y \end{aligned}$$

and $x*(0*y)=y*(0*x)$ reduces to the condition $x\circ y=y\circ x$, i.e., x and y commute in the group $(X;\circ,0)$.

Since $x\circ y=x*(0*y)$, $x\circ y=y\circ x$ leads to $x*(0*y)=y*(0*x)$, i.e., $(X;*,0)$ is commutative. □

Let $(X;*,0)$ be a B-algebra and let $g\in X$. Define $g^n:=g^{n-1}*(0*g)$ $(n\geq 1)$ and $g^0:=0$. Note that $g^1=g^0*(0*g)=0*(0*g)=g$.

Proposition 4.5. *If $(X;*,0)$ is a B-algebra, then for any $x,y\in X$*
(i). $(x*y)*y=x*y^2$;
(ii). $(x*y)*(0*y)=x$.

Proof. (i). Refer to [1].
(ii). It follows from (III) and (I) that $(x*y)*(0*y)=x*((0*y)*(0*y))=x*0=x$. □

Corollary 4.6. *If $(X;*,0)$ is a B-algebra then the right cancellation law holds, i.e., $y*x = y'*x$ implies $y = y'$.*

Proof. Suppose that $y*x = y'*x$. Then

$$\begin{aligned} y &= (y*x)*(0*x) && \text{[by Proposition 4.5-(ii)]} \\ &= (y'*x)*(0*x) \\ &= y'*((0*x)*(0*x)) && \text{[by (III)]} \\ &= y'*0. \\ &= y. \end{aligned}$$

□

Proposition 4.6. *If $(X;*,0)$ is a commutative B-algebra, then $(0*x)*(x*y) = y*x^2$ for any $x, y \in X$.*

Proof. If X is a commutative B-algebra then

$$\begin{aligned} (0*x)*(x*y) &= ((0*x)*(0*y))*x && \text{[by (IV)]} \\ &= (y*x)*x && \text{[Proposition 4.1]} \\ &= y*x^2. && \text{[by Proposition 4.5-(i)]} \end{aligned}$$

□

Let $(X;*,0)$ be a B-algebra. Define $Z(X) := \{x \in X \mid x*(0*y) = y*(0*x), \forall y \in X\}$, and we call it the *center* of X. Note that $0 \in Z(X)$. In fact, for any $x \in X$, $x = x*0 = x*(0*0)$. By applying Proposition 4.2 $0 \in Z(X)$.

Let $(X;*,0)$ be a B-algebra. A non-empty subset N of X is said to be a *subalgebra* ([12]) if $x*y \in N$ for any $x, y \in N$.

Theorem 4.7. *If $(X;*,0)$ is a B-algebra, then the center $Z(X)$ is a subalgebra of X.*

Proof. For any $x, y \in X$, by (IV) and Proposition 4.2 we obtain $0*(x*y) = (0*(0*y))*x = y*x$. If $\alpha, \beta \in Z(X)$, then

$$\begin{aligned} (\alpha*\beta)*(0*x) &= \alpha*((0*x)*(0*\beta)) && \text{[by (III)]} \\ &= \alpha*(\beta*(0*(0*x)) && [\beta \in Z(X)] \\ &= \alpha*(\beta*x) && \text{[by Proposition 4.2]} \\ &= (\alpha*(0*x))*\beta && \text{[by (IV)]} \\ &= (x*(0*\alpha))*\beta && [\alpha \in Z(X)] \\ &= x*(\beta*(0*(0*\alpha))) && \text{[by (III)]} \\ &= x*(\beta*\alpha) && \text{[by Proposition 4.2]} \\ &= x*(0*(\alpha*\beta)) \end{aligned}$$

for any $x \in X$. Hence $Z(X)$ is a subalgebra of X. □

J. Neggers and H. S. Kim ([12]) introduced the notion of a normal subalgebra, i.e., a non-empty subset N of X is normal if and only if $(x*a)*(y*b) \in N$ for any $x*y, a*b \in N$. It is not known that the notion of a normal subalgebra is equivalent to the normal subgroup of the derived group. It is also interesting to prove or disprove that the center $Z(X)$ of X is a normal subalgebra of X.

References

1. Jung R. Cho and H. S. Kim, *On B-algebras and quasigroups*, Quasigroups and related systems **7** (2001), 1-6.
2. Qing Ping Hu and Xin Li, *On BCH-algebras*, Math. Seminar Notes **11** (1983), 313-320.
3. Qing Ping Hu and Xin Li, *On proper BCH-algebras*, Math. Japonica **30** (1985), 659-661.
4. K. Iséki and S. Tanaka, *An introduction to theory of BCK-algebras*, Math. Japonica **23** (1978), 1-26.
5. K. Iséki, *On BCI-algebras*, Math. Seminar Notes **8** (1980), 125-130.
6. Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Sci. Mathematicae **1** (1998), 347-354.
7. M. Kondo and Y. B. Jun, *The class of B-algebras coincides with the class of groups*, Sci. Math. Japo. Online **7** (2002), 175-177.
8. J. Meng and Y. B. Jun, BCK-algebras, Kyung Moon Sa Co., Seoul (1994).
9. J. Neggers and H. S. Kim, *On d-algebras*, Math. Slovaca **49** (1999), 19-26.
10. J. Neggers and H. S. Kim, *On analytic T-algebras*, Sci. Math. Japonica **53** (2001), 25-31.
11. J. Neggers and H. S. Kim, *On B-algebras*, Mate. Vesnik **54** (2002), 21-29.
12. J. Neggers and H. S. Kim, *A fundamental theorem of B-homomorphism for B-algebras*, Intern. Math. J. **2** (2002), 207-214.

P. J. Allen and J. Neggers
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487-0350
U. S. A.
{pallen}{jneggers}@gp.as.ua.edu

Hee Sik Kim
Department of Mathematics
Hanyang University
Seoul 133-791, Korea
heekim@hanyang.ac.kr