Scientiae Mathematicae Japonicae Online, Vol.**9** (2003), 239–244

SAKAGUCHI-TYPE HARMONIC UNIVALENT FUNCTIONS

Om P. Ahuja and Jay M. Jahangiri

Received September 24, 2002; revised May 23, 2003

Abstract. We take the Sakaguchi class of analytic univalent functions which are starlike with respect to symmetric points in the open unit disc Δ and extend it to the complex-valued harmonic univalent functions in Δ. A necessary and sufficient convolution characterization for such harmonic functions is determined. Also, a sufficient coefficient bound for these functions is introduced which in turn proves that they are harmonic starlike of order $\alpha/2,\ 0 \le \alpha < 1$, in the open unit disc.

1. Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. Harmonic functions have been studied by differential geometers such as Choquet [1], Kneser [5], Lewy [6], and Radó [7]. Recent interest in harmonic complex functions has been triggered by geometric function theorists Clunie and Sheil-Small [2]. In [2] they developed the basic theory of complex harmonic univalent functions f defined on the open unit disk $\Delta = \{z : |z| < 1\}$ with the normalization $f(0) = 0$ and $f_z(0) = 1$. Such functions may be written as $f = h + \bar{g}$ where h and g are analytic in Δ. In this case, f is sense-preserving if $|g'| < |h'|$ in Δ, or equivalently, if the dilatation function $w = g'/h'$ satisfies $|w(z)| < 1$ for $z \in \Delta$. To this end, without loss of generality, for $f = h + \bar{g}$ we may write

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

On the other hand, Sakaguchi [8] introduced the class S of analytic univalent functions in Δ which are starlike with respect to symmetrical points. An analytic function $f(z)$ is said to be starlike with respect to symmetrical points if there exists an $\epsilon > 0$ sufficiently small such that, for every ρ in $(1-\epsilon, 1)$ and every ζ with $|\zeta| = \rho$, the angular velocity of $f(z)$ about the point $f(-\zeta)$ is positive at $z = \zeta$ as z traverses the circle $|z| = \rho$ in a positive direction. Thus, we have the inequality

$$\Re \frac{2\zeta f'(\zeta)}{f(\zeta) - f(-\zeta)} > 0 \tag{2}$$

for all ζ in some ring $1 - \epsilon < |\zeta| < 1$, where $\epsilon > 0$ is sufficiently small. Note that (e.g. see [3] Vol. I, p. 165) the inequality (2) in $r < |z| < 1$ does not in itself imply univalence.

2000 AMS Subject Classification: Primary 30C45; Secondary 30C50.
Key words and phrases. Harmonic, Univalent, Starlike.

Extending the definition (2) to include the harmonic functions, for $0 \leq \alpha < 1$ we let $SH(\alpha)$ denote the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1) which satisfy the condition

$$\Im\left(\frac{2\,\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta})-f(-re^{i\theta})}\right) \geq \alpha \tag{3}$$

where $z = re^{i\theta}$, $0 \leq r < 1$ and $0 \leq \theta < 2\pi$.

In this paper we determine a convolution characterization for functions in $SH(\alpha)$. We then introduce a sufficient coefficient condition for harmonic functions to be in $SH(\alpha)$. It is also shown that such functions in $SH(\alpha)$ are also starlike of order α.

2. Main Results

To prove our results in this section, we shall need the following lemma which is due to the second author [4].

2.1. Lemma. Let $f = h + \bar{g}$ be of the form (1) and suppose that the coefficients of h and g satisfy the condition

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 2,\ a_1 = 1,\ 0 \leq \alpha < 1. \tag{4}$$

Then f is sense-preserving, harmonic univalent, and starlike of order α in Δ.

The condition (4) for $\alpha = 0$ was obtained by Silverman and Silvia [10].

A function f is said to be starlike of order α in Δ (e.g. see [9] p. 244) if

$$\frac{\partial}{\partial\theta}(arg\ f(re^{i\theta})) \geq \alpha, \quad |z| = r < 1. \tag{5}$$

We also define the convolution or Hadamard product of two power series $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $F(z) = \sum_{n=1}^{\infty} A_n z^n$ by

$$(f * F)(z) = f(z) * F(z) = \sum_{n=1}^{\infty} a_n A_n z^n.$$

2.2. Theorem. Let α be a constant such that $0 \leq \alpha < 1$. Then a harmonic function $f = h + \bar{g}$ is in $SH(\alpha)$ if and only if

$$h(z)\ *\ \frac{(1-\alpha)z + (\alpha+\xi)z^2}{(1-z)^2(1+z)} \ -\ \overline{g(z)}\ *\ \frac{(\alpha+\xi)\bar{z} + (1-\alpha)\bar{z}^2}{(1-\bar{z})^2(1+\bar{z})} \ \neq 0$$

where $|\xi| = 1$, $\xi \neq -1$ and $0 < |z| < 1$.

Proof. For $0 \leq \alpha < 1$, a harmonic function $f = h + \bar{g}$ is in $SH(\alpha)$ if and only if the condition (3) holds. Differentiating $f(re^{i\theta})$ with respect to θ and substituting in (3) we obtain

$$\Re\left[\frac{2zh'(z) - 2\overline{zg'(z)} - \alpha[f(z) - f(-z)]}{(1-\alpha)[f(z) - f(-z)]}\right] \geq 0.$$

Or equivalently,

$$\frac{2zh'(z) - 2\overline{zg'(z)} - \alpha[h(z) + \overline{g(z)} - h(-z) - \overline{g(-z)}]}{(1-\alpha)[h(z) + \overline{g(z)} - h(-z) - \overline{g(-z)}]} \neq \frac{\xi - 1}{\xi + 1}, \tag{6}$$

where $|\xi| = 1$, $\xi \neq -1$ and $0 < |z| < 1$.

Simplifying (6) we obtain the equivalent condition

$$2(1+\xi)zh'(z)+(1-2\alpha-\xi)[h(z)-h(-z)]-2(1+\xi)\overline{zg'(z)}+(1-2\alpha-\xi)[\overline{g(z)-g(-z)}] \neq 0. \tag{7}$$

Upon noting that $zh'(z) = h(z) * (z/(1-z)^2)$, $zg'(z) = g(z) * (z/(1-z)^2)$, $h(z) - h(-z) = 2h(z) * (z/(1-z^2))$ and $g(z) - g(-z) = 2g(z) * (z/(1-z^2))$, the condition (7) yields the necessary and sufficient condition required by Theorem 2.2.

Next we give a sufficient coefficient condition for harmonic functions in $SH(\alpha)$.

2.3. Theorem. For h and g as in (1), let the harmonic function $f = h + \bar{g}$ satisfy

$$\sum_{n=1}^{\infty} \left\{ \frac{2(n-1)}{1-\alpha}(|a_{2n-2}| + |b_{2n-2}|) + \frac{2n-1-\alpha}{1-\alpha}|a_{2n-1}| + \frac{2n-1+\alpha}{1-\alpha}|b_{2n-1}| \right\} \leq 2 \tag{8}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$. Then f is sense-preserving harmonic univalent in Δ and $f \in SH(\alpha)$.

Proof. Since

$\sum_{n=1}^{\infty} n(|a_n| + |b_n|)$

$= \sum_{n=1}^{\infty} 2(n-1)|a_{2n-2}| + \sum_{n=1}^{\infty}(2n-1)|a_{2n-1}| + \sum_{n=1}^{\infty} 2(n-1)|b_{2n-2}| + \sum_{n=1}^{\infty}(2n-1)|b_{2n-1}|$

$\leq \sum_{n=1}^{\infty} \left\{ \frac{2(n-1)}{1-\alpha}(|a_{2n-2}| + |b_{2n-2}|) + \frac{2n-1-\alpha}{1-\alpha}|a_{2n-1}| + \frac{2n-1+\alpha}{1-\alpha}|b_{2n-1}| \right\} \leq 2,$

by Lemma 2.1, we conclude that f is sense-preserving , harmonic, univalent and starlike in Δ. To prove $f \in SH(\alpha)$, according to the condition (3), we need to show that

$$\Im\left(\frac{2\,\frac{\partial}{\partial\theta} f(re^{i\theta})}{f(re^{i\theta}) - f(-re^{i\theta})}\right) = \Re\left(\frac{-2i\frac{\partial}{\partial\theta} f(re^{i\theta})}{f(re^{i\theta}) - f(-re^{i\theta})}\right) = \Re\frac{A(z)}{B(z)} \geq \alpha,$$

where $z = re^{i\theta} \in \Delta$, $0 \leq \alpha < 1$,

$$\begin{aligned} A(z) &= -2i\frac{\partial}{\partial\theta} f(re^{i\theta}) \\ &= 2re^{i\theta} + 2\sum_{n=2}^{\infty} na_n r^n e^{ni\theta} - 2\sum_{n=1}^{\infty} n\bar{b}_n r^n e^{-ni\theta} \\ &= 2re^{i\theta} + 2\sum_{n=2}^{\infty}\left[2(n-1)a_{2n-2}r^{2n-2}e^{(2n-2)i\theta} + (2n-1)a_{2n-1}r^{2n-1}e^{(2n-1)i\theta}\right] \\ &- 2\sum_{n=1}^{\infty}\left[2(n-1)\bar{b}_{2n-2}r^{2n-2}e^{-(2n-2)i\theta} + (2n-1)\bar{b}_{2n-1}r^{2n-1}e^{-(2n-1)i\theta}\right], \end{aligned} \tag{9}$$

and

$$\begin{aligned} B(z) &= f(re^{i\theta}) - f(-re^{i\theta}) \\ &= 2\left[re^{i\theta} + \sum_{n=2}^{\infty} a_{2n-1}r^{2n-1}e^{(2n-1)i\theta} + \sum_{n=1}^{\infty} \bar{b}_{2n-1}r^{2n-1}e^{-(2n-1)i\theta}\right]. \end{aligned} \tag{10}$$

Using the fact that $\Re(\omega) \geq \alpha$ if and only if $|1-\alpha+\omega| \geq |1+\alpha-\omega|$, it suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0. \tag{11}$$

On the other hand, for $A(z)$ and $B(z)$ as given by (9) and (10) we have

$$\begin{aligned} &|A(z) + (1-\alpha)B(z)| \\ &= 2r|2-\alpha + \sum_{n=2}^{\infty}\left\{2(n-1)a_{2n-2}r^{2n-3}e^{(2n-3)i\theta} + (2n-\alpha)a_{2n-1}r^{2n-2}e^{(2n-2)i\theta}\right\} \\ &\qquad - \sum_{n=1}^{\infty}\left\{2(n-1)\bar{b}_{2n-2}r^{2n-3}e^{-(2n-1)i\theta} + (2n-2+\alpha)\bar{b}_{2n-1}r^{2n-2}e^{-2ni\theta}\right\}| \\ &\geq 2r[(2-\alpha) - \sum_{n=2}^{\infty} 2(n-1)|a_{2n-2}| - \sum_{n=2}^{\infty}(2n-\alpha)|a_{2n-1}| \\ &\qquad - \sum_{n=1}^{\infty} 2(n-1)|b_{2n-2}| - \sum_{n=1}^{\infty}(2n-2+\alpha)|b_{2n-1}|], \end{aligned} \tag{12}$$

and

$$\begin{aligned} &|A(z) - (1+\alpha)B(z)| \\ &= 2r| - \alpha + \sum_{n=2}^{\infty}\left\{2(n-1)a_{2n-2}r^{2n-3}e^{(2n-3)i\theta} + (2n-2-\alpha)a_{2n-1}r^{2n-2}e^{(2n-2)i\theta}\right\} \\ &\qquad - \sum_{n=1}^{\infty}\left\{2(n-1)\bar{b}_{2n-2}r^{2n-3}e^{-(2n-1)i\theta} + (2n+\alpha)\bar{b}_{2n-1}r^{2n-2}e^{-2ni\theta}\right\}| \\ &\leq 2r[\alpha + \sum_{n=2}^{\infty} 2(n-1)|a_{2n-2}| + \sum_{n=2}^{\infty}(2n-2-\alpha)|a_{2n-1}| \\ &\qquad + \sum_{n=1}^{\infty} 2(n-1)|b_{2n-2}| + \sum_{n=1}^{\infty}(2n+\alpha)|b_{2n-1}|]. \end{aligned} \tag{13}$$

Now, by substituting for (12) and (13) in (11), we obtain

$$\begin{aligned} &|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \\ &\geq 4r[2(1-\alpha) - \sum_{n=1}^{\infty} 2(n-1)|a_{2n-2}| - \sum_{n=1}^{\infty}(2n-1-\alpha)|a_{2n-1}| \\ &\qquad - \sum_{n=1}^{\infty} 2(n-1)|b_{2n-2}| - \sum_{n=1}^{\infty}(2n-1+\alpha)|b_{2n-1}|] \end{aligned}$$

$$\geq 4r(1-\alpha)[2 - \sum_{n=1}^{\infty} \frac{2(n-1)}{1-\alpha}|a_{2n-2}| - \sum_{n=1}^{\infty} \frac{2n-1-\alpha}{1-\alpha}|a_{2n-1}|$$

$$- \sum_{n=1}^{\infty} \frac{2(n-1)}{1-\alpha}|b_{2n-2}| - \sum_{n=1}^{\infty} \frac{2n-1+\alpha}{1-\alpha}|b_{2n-1}|] \geq 0.$$

2.4. Corollary. Let f be as in Theorem 2.3. Then f is starlike of order $\alpha/2$ for $0 \leq \alpha < 1$.

Proof. By the sufficient condition (4), we conclude that f is starlike of order $\alpha/2$; $0 \leq \alpha < 1$ if

$$\sum_{n=1}^{\infty} \left(\frac{n - \frac{\alpha}{2}}{1 - \frac{\alpha}{2}}|a_n| + \frac{n + \frac{\alpha}{2}}{1 - \frac{\alpha}{2}}|b_n| \right) \leq 2. \qquad (14)$$

We will show that the coefficient condition (8) required for $f \in SH(\alpha)$, $0 \leq \alpha < 1$, implies the sufficient coefficient condition (14), which in turn implies that f is starlike of order $\alpha/2$; $0 \leq \alpha < 1$. By a simple algebraic manipulation, we see that this is the case, since

$$\sum_{n=1}^{\infty} \left(\frac{n-\frac{\alpha}{2}}{1-\frac{\alpha}{2}}|a_n| + \frac{n+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}|b_n| \right)$$

$$= \sum_{n=2}^{\infty} \frac{2n-2-\frac{\alpha}{2}}{1-\frac{\alpha}{2}}|a_{2n-2}| + \sum_{n=1}^{\infty} \frac{2n-1-\frac{\alpha}{2}}{1-\frac{\alpha}{2}}|a_{2n-1}|$$

$$+ \sum_{n=2}^{\infty} \frac{2n-2+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}|b_{2n-2}| + \sum_{n=1}^{\infty} \frac{2n-1+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}|b_{2n-1}|$$

$$\leq \sum_{n=1}^{\infty} \frac{2(n-1)}{1-\alpha}|a_{2n-2}| + \sum_{n=1}^{\infty} \frac{2n-1-\alpha}{1-\alpha}|a_{2n-1}|$$

$$+ \sum_{n=1}^{\infty} \frac{2(n-1)}{1-\alpha}|b_{2n-2}| + \sum_{n=1}^{\infty} \frac{2n-1+\alpha}{1-\alpha}|b_{2n-1}| \leq 2.$$

We remark that for f as in Theorem 2.3, the function $(f(z) - f(-z))/2$ is also starlike of order $\alpha/2$ in Δ.

Acknowledgment. The authors would like to express their appreciation for the referee's careful review of this article.

References

1. G. Choquet, *Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques*, Bull. Sci. Math. **69(2)** (1945), 156-165.
2. J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 3-25.
3. A. W. Goodman, *Univalent Functions Vols. I & II*, Mariner Publ. Co., Tampa, Florida. (1983).
4. J. M. Jahangiri, *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl. **235** (1999), 470-477.
5. H. Kneser, *Lösung der Aufgabe 41*, Jahresber. Deutsch. Math.-Verein. **35** (1926), 123-124.
6. H. Lewy, *On the non-vanishing of the jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), 689-692.
7. T. Radó, *Aufgabe 41*, Jahresber. Deutsch. Math.-Verein. **35** (1926), 49.
8. K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **2(1)** (1959), 72-75.
9. T. Sheil-Small, *Constants for planar harmonic mappings*, J. London Math. Soc. **2(42)** (1990), 237-248.

10. H. Silverman and E.M. Silvia, *Subclasses of harmonic univalent functions*, New Zealand J. Math. **28** (1999), 275-284.

Department of Mathematical Sciences, Kant State University, 14111 Claridon Troy Road, Burton, Ohio 44021-9500, U.S.A.
e-mail: ahuja@geauga.kent.edu
e-mail: jay@geauga.kent.edu