RATIONAL DECOMPOSITIONS OF P-ADIC MEROMORPHIC FUNCTIONS

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Abstract. Let K be a non archimedean algebraically closed field of characteristic π, complete for its ultrametric absolute value. In a recent paper by Escassut and Yang ([6]) polynomial decompositions $P(f) = Q(g)$ for meromorphic functions f, g on K (resp. in a disk $d(0, r^-) \subset K$) have been considered, and for a class of polynomials P, Q, estimates for the Nevanlinna function $T(\rho, f)$ have been derived.
In the present paper we consider as a generalization rational decompositions of meromorphic functions, i.e., we discuss properties of solutions f, g of the functional equation $P(f) = Q(g)$, where P, Q are in $K(x)$ and satisfy a certain condition (M). We infer that in the case, where f, g are analytic functions, the Second Nevanlinna Theorem yields an analogue result as in the mentioned paper [6]. However, if they are meromorphic, non trivial estimates for $T(\rho, f)$ are more sophisticated.

1 Introduction Throughout this paper, K denotes an algebraically closed ultrametric field of characteristic π, complete with respect to the topology induced by its non-archimedean valuation, $K^* = K \setminus \{0\}$. Let $\mathcal{A}(K)$ denote the ring of entire functions on K, and $\mathcal{M}(K)$ the field of meromorphic functions in K, i.e., the field of fractions of $\mathcal{A}(K)$.
Moreover, for any real number $r > 0$, $d(a, r^-) = \{x : |x - a| < r\}$, i.e. the open ball with radius $r > 0$ and center $a \in K$; then similarly as above, $\mathcal{A}(d(a, r^-))$ is the ring of analytic functions on $d(a, r^-)$, i.e.: the ring of analytic functions with radius of convergence $\rho \geq r$. The ring of meromorphic functions on $d(a, r^-)$ is denoted by $\mathcal{M}(d(a, r^-))$.
We denote by $\mathcal{A}_b(d(a, r^-))$ the K-subalgebra of analytic functions with bounded norm, furthermore $\mathcal{A}_u(d(a, r^-)) = \mathcal{A}(d(a, r^-)) \setminus \mathcal{A}_b(d(a, r^-))$. Similarly, by $\mathcal{M}_b(d(a, r^-))$ we denote the field of fractions of $\mathcal{A}_b(d(a, r^-))$, and $\mathcal{M}_u(d(a, r^-)) = \mathcal{M}(d(a, r^-)) \setminus \mathcal{M}_b(d(a, r^-))$. For $R > 0$ we denote the interval I as the set $I = [\rho, R[$, where $0 < \rho < R$, and for some $\rho > 0$ we write $J = [\rho, \infty[$.

Notation in Nevanlinna Theory Let $R \in]0, \infty[$, $f \in \mathcal{M}(d(0, r^-))$ (resp. $\mathcal{M}(K)$) such that 0 is neither a zero nor a pole of f. Let $w_\alpha(f) = n$ (resp. $w_\alpha(f) = -n$), if f has a zero (resp. a pole) of order n at α. Then the functions Z and N are defined as

$$Z(\rho, f) := \sum_{w_\alpha > 0, |\alpha| \leq \rho} w_\alpha(f) \log \frac{\rho}{|\alpha|}$$

and $N(\rho, f) := Z(\rho, 1/f)$, moreover the Nevanlinna function is given by

$$T(\rho, f) := \max\{Z(\rho, f), N(\rho, f)\}$$

In addition we use similar functions not respecting multiplicities of zeros (resp. poles):

$$\widetilde{Z}(\rho, f) := \sum_{w_\alpha > 0, |\alpha| \leq \rho} \log \frac{\rho}{|\alpha|}$$

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and similarly $\widetilde{N}(\rho, f) := \widetilde{Z}(\rho, 1/f)$.

Notation in positive characteristic. If $\pi \neq 0$ we define the characteristic exponent $\chi := \pi$, otherwise we set $\chi := 1$. Due to [6] we call the ramification index of h in $\mathcal{M}(d(0, R^-))$(resp. $\mathcal{M}(K)$) the unique integer t such that $\sqrt[\chi^t]{h}$ belongs to $\mathcal{M}(d(0, R^-))$(resp. $\mathcal{M}(K)$). If $\pi = 0$, every function h in $\mathcal{M}(d(0, R^-))$(resp. $\mathcal{M}(K)$) has ramification index equals 0.
We note that in nonzero characteristic, the counting functions $\widetilde{N}$ (resp. $\widetilde{Z}$) of poles (resp. zeros) are defined slightly differently. For more information, we refer to [4].
In the present paper we apply the Second Nevanlinna Theorem due to Boutabaa and Escassut ([4], Theorem 2):

Theorem N. *Let $\alpha_1, \ldots, \alpha_n \in K$, with $n \geq 2$, and let $f \in \mathcal{M}(d(0, R^-))$ (resp. $f \in \mathcal{M}(K)$) of ramification index s, have no zero and no pole at 0. Let $S := \{ \sqrt[\chi^s]{\alpha_1}, \ldots, \sqrt[\chi^s]{\alpha_s}\}$. Assume that f, $\sqrt[\chi^s]{f}$, $f - \alpha_j$ have no zero and no pole at 0 $(1 \leq j \leq n)$. Then we have:*

$$\frac{(n-1)T(r, f)}{\chi^s} \leq \sum_{i=1}^{n} \widetilde{Z}(r, f - \alpha_i) + \widetilde{N}(r, f) - \log r + O(1), \quad r \in I \quad (r \in J)$$

Many Applications of the Nevanlinna Theory to Functional and Differential Equations have been worked out in the last years, and the Theory has only recently been generalized to fields of characteristic π (see [4] which contains the Theorem from above, resp. [8]). One of the most famous examples, where the archimedean Theorem is due to F. Gross [7] (generalizations were firstly made by N. Toda [9], and the non-archimedean work is due to A. Boutabaa [1]) is the equation

$$f^n + g^m = 1$$

A recent p-adic article on this topic deals with unbounded meromorphic solutions in a ball ([4]). In characteristic zero, the most comprehensive work on this class of functional equations can be found in [2].
Here we discuss properties of analytic or meromorphic solutions f, g of the functional equation

$$P(f) = Q(g) \tag{1}$$

where P and Q are certain rational functions on K.

We are starting with analytic functions in section 2 and receive similar conclusions as in a recent paper due to Escassut and Yang ([6]), where P, Q are elements in $K[x]$. However, the meromorphic case turns out to be more sophisticated than the analytic one, i.e., it is more complicated to derive non trivial estimations for the Nevanlinna function $T(\rho, f)$. This case is worked out in section 3.

When not explicitly stated, we write $P = R/S$, $Q = V/W$, where $R, S, V, W \in K[x]$, $(R, S) := \gcd(R, S) = 1$, $(V, W) = 1$, and the degrees of P (resp. Q) are defined by $p := \deg P = \max\{\deg R, \deg S\}$ (resp. $q := \deg Q = \max\{\deg V, \deg W\}$).

Preparatory statements

Theorem 0.1. *Let $f \in \mathcal{A}(K)$ such that $f(K) \subset K^*$ (i.e. f has no zero in K). Then f is a* *constant.*

Proof. Let $f(x) = \sum_{j\geq 0} a_j x^j$. It is well known that the number of zeros of f in any disk $d(0,r) := \{x \in K : |x| \leq r\}$ is equal to the largest integer k such that $|a_k|r^k = \sup_{i\geq 0} |a_i|r^i$ (see Theorem 23.5 in [5]). Hence, if f has no zero in K, obviously for all $n > 0 :\ a_n = 0$.□

Corollary 0.2. *If $f,\ g \in \mathcal{A}(K)$ have the same zeros respecting multiplicity, then $\frac{f}{g}$ is a constant.* □

Theorem 0.3. ([3]) *Let f be in $\mathcal{M}(d(0,r^-))$ with $f(0) \neq 0, \infty$. Then f belongs to $\mathcal{M}_b(d(0,r^-))$ if and only if $T(\rho, f)$ is bounded in $[0, r[$.*

Theorem 0.4. ([3]) *Let f b in $\mathcal{M}(d(0,r^-))$ and let $P \in K[x]$ be of degree n. Then $T(\rho, P(f)) = nT(\rho, f) + O(1)$.*

Remark 0.5. ([4], [6]) There is a well defined mapping

$$\sqrt[\pi]{\ }: \qquad x \mapsto \sqrt[\pi]{x}$$

which is a homomorphism on K and can be extended to a homomorphism on $K[x]$ (or even to one on $K(x)$) in the following way: Let $R = \sum_{i=0}^{n} a_i x^i = \mu(x-\alpha_1)\dots(x-\alpha_n)$ in $K[x]$, then we define

$$\sqrt[\pi]{\ }: \qquad \sqrt[\pi]{P}(x) := \sum_{i=0}^{n} \sqrt[\pi]{a_i} x^i = \sqrt[\pi]{\mu}(x - \sqrt[\pi]{\alpha_1})\dots(x - \sqrt[\pi]{\alpha_n})$$

Lemma 0.6. ([2], [4]) *Suppose $\pi \neq 0$. Let $r > 0$ and $f \in \mathcal{M}(d(0,r^-))$(resp. $f \in \mathcal{M}(K)$). Then $\sqrt[\pi]{f}$ belongs to $\mathcal{M}(d(0,r^-))$(resp. $\mathcal{M}(K)$) if and only if $f' \equiv 0$. Moreover, there exists a unique $t \in \mathbb{N}$ such that $\sqrt[\pi^t]{f}$ in $\mathcal{M}(d(0,r^-))$(resp. $\mathcal{M}(K)$) and $(\sqrt[\pi^t]{f})' \neq 0$.*

2 Decompositions of analytic functions First, we apply Theorem 2.9 ([6]) to the question (1) for entire f, g. In section 2.2 we prove a somewhat similar result with appropriate conditions, particularly tailored to our "rational" problem. Indeed this does not only yield a quite more general result (see examples 2.2.6, 2.2.8), but also an analogue result for elements f, g in $\mathcal{A}(d(0,r^-))$.

2.1 An Application of a previous paper ([6])

Remark 2.1.1. Let f, g be in $A(K)$ (resp $A(d(0,r^-))$), satisfying (1). Any pole b of $P(f)$ is a zero of $S(f)$, hence a zero of $W(g)$ of the same order. In the meromorphic case, this conclusion is wrong.

Theorem 2.1.2. *Let $f,\ g \in \mathcal{A}(K)$ solve (1), then there exists a constant $\lambda \in K^*$ such that*

$$S(f) = \lambda W(g)$$

Proof. Since f, g are analytic functions and $S(f)$ and $W(g)$ have the same zeros of the same order, $S(f)/W(g)$ is a constant by Corollary 0.2. □

Thus our problems reads

$$R(f) = \lambda S(g),\ \lambda \in K^* \tag{2}$$

Although λ is an undetermined constant, we are able to apply the following Theorem (Theorem 2.9 in [6]) to it, worked out for decompositions $A(f) = B(g)$ for f, g, where A, B are polynomials (in order to avoid confusion, we write A, B instead of P, Q used in [6]):

Theorem 2.1.3. ([6]) *Let A, B be in $K[x]$ with $A'B'$ not identically zero, such that $2 \leq \min\{\deg A, \deg B\}$. Assume that there exist k distinct zeros $c_1, \dots, c_k$ of A' such that $A(c_i) \neq A(c_j) \forall i \neq j$ and $A(c_i) \neq B(d)$ for every zero d of $B' (i = 1, \dots, k)$. Assume that there exist two nonconstant functions f, $g \in \mathcal{M}(K)$ such that $A(f) = B(g)$, and let $t = \nu(f)$. Then $q \leq p$ and f satisfies*

$$\widetilde{N}(\rho, f) \geq \frac{k \deg B - \deg A}{\chi^t \deg B} T(\rho, f) + \log \rho + O(1)$$

Moreover, if $\frac{p}{2} < q$, then $k = 1$ and c_1 is a simple root of A'.

Theorem 2.1.4. *Let $P = R/S$, $Q = V/W$, where R, S, V, W are in $K[x]$, $(R, S) = 1$, $(V, W) = 1$. Let $c_1, \dots, c_k$ be zeros of R' such that $R(c_i) \neq R(c_j) \forall i \neq j$. Moreover, let the degree of V satisfy $l = k - \deg V + 1 > 0$. Then, if $f, g \in \mathcal{A}(K) \setminus K$ solve $P(f) = Q(g)$, we have*

$$0 \geq \frac{l \deg R - \deg V}{\chi^t \deg V} T(\rho, f) + \log \rho + O(1)$$

i.e.: $\deg V (\deg R + 1) > (k + 1) \deg R$.

Proof. Due to Theorem 2.1.2 we have $R(f) = \lambda V(g)$ for some $\lambda \in K^*$; set $A := R,\ B := \lambda V$. Now, obviously there exist at least l distinct roots $c_{j_1}, \dots, c_{j_l}$ of A', such that $A(c_{j_r}) \neq B(d)$ for any zero d of B ($r = 1, \dots, l$ with $l = k - \deg V + 1 > 0$), and $\forall i \neq j, (i, j \in \{j_1, \dots, j_l\})$: $A(c_i) \neq A(c_j)$. Trivially, $\widetilde{N}(\rho, f)$ is identically zero for non constant analytic f. □

Remark 2.1.5. Unfortunately the condition of Theorem 2.1.4 implies $\deg V \leq \deg R - 1$, which follows from $\deg V \leq k < \deg R$, since $k \leq \deg R'$. This inequality together with the statement of Theorem 2.1.4 tells us therefore:

$$\frac{(k + 1) \deg R}{\deg R + 1} < \deg V < \deg R$$

In the next section, however, we present conditions for P, Q, where not necessarily $\deg V < \deg R$, such that (1) has only non constant entire solutions. Also, we consider the case where f, g are unbounded analytic functions inside a disk $d(0, r^-)$.

Corollary 2.1.6. *Let P, Q be in $K(x)$ with $P'Q'$ not identically zero and let $p = \deg(R)$, $q = \deg(V)$ with $2 \leq \min(p, q)$ and $\frac{p}{2} < q$. Assume that there exist q distinct zeros c_i, $(1 \leq i \leq q)$ of R' such that $R(c_i) \neq R(c_j)\ \forall i \neq j$. If two functions f, $g \in \mathcal{A}(K)$ satisfy $P(f) = Q(g)$, then f and g are constants.*

Proof. Assume that two functions f, $g \in \mathcal{A}(K)$ satisfy $P(f) = Q(g)$. By Theorem 2.1.2 there exists $\lambda \in K$ such that $R(f) = \lambda V(g)$. Let $d_1, \dots, d_n$ be the distinct zeros of V'. We notice that $n \leq q - 1$. In order to apply Theorem 2.10 in [6], we only have to check that there exists a zero c_k of R' satisfying $R(c_k) \neq \lambda V(d_j)$ for every $j = 1, \dots, n$. Suppose it is not true. Then, up to a reordering, we can assume that $R(c_1) = \lambda V(d_1), \dots, R(c_1) = \lambda V(d_n)$. Since $q > n$ and since $R(c_i) \neq R(c_j)\ \forall i \neq j$, we then have $R(c_q) \neq \lambda V(d_j)\ \forall j = 1, \dots, n$. Thus, we can apply Corollary 2.10 in [6] to the polynomial $A := R$ and $B := \lambda V$. □

2.2 Generalizations of [6]

Remark 2.2.1. We further distinguish some cases: Suppose f, g are non constant entire solutions (resp. unbounded analytic solutions in a disk $d(a, r^-)$) of (1), then by using growth at infinity (resp. growth, when $\rho \to r-$) we have

- Case 1: $\deg V > \deg W$, then obviously $\deg R > \deg S$
- Case 2: $\deg V < \deg W$, then obviously $\deg R < \deg S$,
- Case 3: $\deg V = \deg W$, then obviously $\deg R = \deg S$,

furthermore we obtain in any case, when $\rho \to \infty$ (resp. when $\rho \to r-$)

$$pT(\rho, f) = qT(\rho, g) + O(1) \tag{3}$$

which follows from the functional equation (1):
$T(\rho, P(f)) = \max\{Z(\rho, P(f)),\ N(\rho, P(f))\}$, and since $Z(\rho, P(f)) = Z(\rho, R(f))$, $N(\rho, P(f)) = Z(\rho, S(f))$, we have by Theorem 0.4, $T(\rho, P(f)) = \max\{r, s\}T(\rho, f) + O(1) = pT(\rho, f) + O(1)$.

In the present paper our statements on rational decompositions of meromorphic functions always concern a specific class of rational functions P, Q, admitting certain decompositions themselves (see Lemma 2.2.2, below). They are described by the following condition to which we always refer:

Condition (M) Let $P, Q \in K(x)$ and denote the zeros of P' by $c_1, \dots c_k$.
P, Q are said to satisfy Condition (M), if

1. $P'Q' \neq 0$
2. $P = R/S$, $(R, S) = 1$, $Q = V/W$, V, W monic, $(V, W) = 1$ (R, S, V, $W \in K[x]$)
3. $k > 0$, and for any $i \in \{1, \dots, k\}$ we have
$$(Q(d) \neq P(c_i)) \quad \wedge \quad (W(d) \neq 0)\,,$$
for any zero d of $V' - W'P(c_i)$,
4. $P(c_i) \neq P(c_j)$ whenever $i \neq j$,
5. Finally, if $v = w$ we assume $\forall i \in \{1, \dots, k\} : P(c_i) \neq 1$.

Remark on Condition (M). Let f, g be non constant entire functions. If we set $S \in K^*$, it easily follows that also $W \in K^*$, moreover $P, Q \in K[x]$ and (like before, denote the zeros of P' by $c_1, \dots c_k$) satisfy:

1. $P'Q' \neq 0$
2. $k > 0$, and for any $i \in \{1, \dots, k\}$ we have $Q(d) \neq P(c_i)$ for any zero d of Q',
3. $P(c_i) \neq P(c_j)$ whenever $i \neq j$

On this condition Theorem 2.1 and 2.9 are based in [6]. In this sense, our paper can be considered as a generalization to part of [6].
Now we are ready to state the basic Lemma:

Lemma 2.2.2. *Let P, Q satisfy Condition (M). Then for any $i \in 1,\dots,k$, $P-P(c_i)$ resp. $(Q-P(c_i))W$ we have the following factorizations:*

$$P(x)-P(c_i)=(x-c_i)^{s_i}R_i(x), \quad s_i \geq 2, \quad R_i(c_i)\neq 0 \tag{4}$$

and

$$(Q(x)-P(c_i))W(x)=\prod_{j=1}^{q}(x-b_{i,j}) \tag{5}$$

Furthermore, the set $\{b_{i,j}\}$ consists of qk distinct elements.

Proof. Let $i\in\{1,\dots,k\}$ be arbitrary, but fixed. Since $P'(c_i)=0$ we can clearly write $P(x)-P(c_i)=(x-c_i)^{s_i}R_i(x)$, with $R_i(c_i)\neq 0$, and $s_i\geq 2$; indeed, suppose $s_i=1$, then for the derivative we have $P'(x)=(x-c_i)R_i'(x)+R_i(x)$, so that $P'(c_i)=R_i(c_i)$, which is a contradiction.
In any case, we have $\deg((Q-P(c_i))W)=\deg(Q)=\max\{\deg V,\deg W\}=q$: In Case 1 and Case 2 this is obvious, and in Case 3 we infer this by the additional condition $P(c_i)\neq 1$ and V, W being monic polynomials. Thus we can write $(Q(x)-P(c_i))W(x)=\prod_{j=1}^{q}(x-b_{i,j})$; furthermore for any fixed i, $b_{i,j}\neq b_{i,j'}$, since for any d with $W(d)=0$ or $Q(d)-P(c_i)=0$ we have $V'(d)-W'(d)P(c_i)\neq 0$.
Now, let $(i,j)\neq(i',j')$, then $\varphi(x):=(Q(x)-P(c_i))-(Q(x)-P(c_{i'}))=P(c_i)-P(c_{i'})\neq 0$ is a constant function different from zero; on the other hand, assume $b_{i,j}=b_{i',j'}$, then by the right side of decomposition (5) we infer $\varphi(x)=0$, which contradicts our assumption and thus $b_{i,j}\neq b_{i',j'}$. □

Let R_i $(i=1,\dots,k)$ be the rational functions due to the notation of Lemma 2.2.2. The following lemma presents upper bounds for their degrees $(i=1,\dots,k)$:

Lemma 2.2.3. *Let P, Q satisfy condition (M). Then for any $i\in\{1,\dots,k\}$, R_i can be written in the way $R_i=\frac{A_i}{B_i}$ with $(A_i,B_i)=1$, where $B_i=S$. Moreover we have $\deg A_i\leq\max\{\deg R,\deg S\}-s_i=p-s_i$.*

Proof. We may write $R_i=\frac{A_i}{B_i}$ and from (5) we get

$$(x-c_i)^{s_i}\frac{A_i(x)}{B_i(x)}=\frac{R(x)-P(c_i)S(x)}{S(x)}$$

$(R,S)=1$ clearly implies $(R(x)-P(c_i)S(x),S(x))=1$, thus $B_i=S$ and $\deg A_i=\deg(R(x)-P(c_i)S(x))-s_i$.

- Case 1: $\deg(R(x)-P(c_i)S(x))=r$, i.e. $\deg A_i=\deg(R(x)-P(c_i)S(x))-s_i=r-s_i=p-s_i$.
- Case 2: we get exactly in the same way as before, switching the roles of r and s: $\deg(R(x)-P(c_i)S(x))=s$, i.e. $\deg A_i=\deg(R(x)-P(c_i)S(x))-s_i=s-s_i=p-s_i$.
- Case 3: Obviously R might be not monic; we conclude $\deg A_i=\deg(R(x)-P(c_i)S(x))-s_i\leq\deg S-s_i=\deg R-s_i=\deg P-s_i$.

□

Particularly for the case where K has nonzero characteristic, we note two useful Lemmas:

Lemma Π_1. *If for P, Q in $K(x)$, $P'Q'\neq 0$ and f, g in $\in\mathcal{M}(K)\setminus K$ (resp. $\mathcal{M}_u(d(0,r^-))$)*

satisfy (1), then $f' \equiv 0 \Leftrightarrow g' \equiv 0$. *Obviously,* $t = \nu(f) = \nu(g)$.

Proof. Say $f' = 0$, then by the derivative of (1) we can see that $g' = 0$: Firstly, $\mathcal{M}(K)$ (resp. $\mathcal{M}(d(0, r^-))$) is a field, secondly Q' is not identically zero by our assumption, i.e. it vanishes at finitely many points only; and since g takes infinitely many values, $Q'(g)$ is not identically zero. Conversely, by the same argument if $f' \neq 0$, then $g' \neq 0$. □

Lemma Π_2**.** *Let* P, Q *satisfy Condition (M) and* $f' \equiv 0$. *Then Condition (M) is satisfied by* $P_1 := \sqrt[\chi]{P}$, $Q_1 := \sqrt[\chi]{Q}$. *Moreover if* f, $g \in \mathcal{M}(K)$ *(resp.* $\mathcal{M}(d(0, r^-))$*) satisfy (1), then* $P_1(f_1) = Q_1(g_1)$, *where* $f_1 := \sqrt[\chi]{f}$, $g_1 := \sqrt[\chi]{g}$. *Thus by repeating the same process* t *times, where* t *is the unique integer from Lemma 0.6, we derive* $P_t(f_t) = Q_t(g_t)$ *and Condition (M) is satisfied by.* P_t, Q_t *(where we denote similarly* $f_t := \sqrt[\chi^t]{f}$, $g_t := \sqrt[\chi^t]{g}$*).*

Proof. Use Remark 0.5. □

Theorem 2.2.4. *Let* f, $g \in \mathcal{A}(K)$ *be non constant solutions of (1), where* P, Q *satisfy condition (M) and let* $t := \nu(f) = \nu(g)$. *Then*

$$0 \geq (\frac{kq - p}{\chi^t q})T(\rho, f) + \log \rho + O(1), \tag{6}$$

i.e., $qk - p < 0$.

Theorem 2.2.5. *Let* f, $g \in \mathcal{A}_u(d(0, r^-))$ *be solutions to (1), where* P, Q *satisfy condition (M) and let* $t := \nu(f) = \nu(g)$. *Then*

$$0 \geq (\frac{kq - p}{\chi^t q})T(\rho, f) + O(1), \tag{7}$$

i.e., $qk - p \leq 0$.

For the proof of these two theorems we refer to section 4, where a unified proof including analogue statements on meromorphic functions (see section 3) is given.

2.3 Examples Let $\deg R = \deg S = \deg V = \deg W = 2$, then due to Remark 2.1.5 Theorem 2.1.4 can not be applied. But 2.2.4 works: To demonstrate this explicitly, we consider the following example:

Example 2.2.6. Let $K = \mathbb{C}_p$, let $P(x) = \frac{R(x)}{x^2}$, $\deg R = 2$, $Q(x) = \frac{V(x)}{W(x)} = \frac{x^2}{x^2+x+1}$. Write $R(x) = ax^2 + bx + c$, $a \neq 0$, $b \neq 0$, $\frac{a-c}{b} = \sqrt{3}$. Then

$$P'(x) = \frac{-bx^2 + x(2a - 2c) + b}{(x^2 + 1)^2} = \frac{-bx^2 + 2(b\sqrt{3})x + b}{(x^2 + 1)^2} = -b\frac{x^2 - 2\sqrt{3}x - 1}{(x^2 + 1)^2}$$

and P' has two distinct roots

$$c_1 = \sqrt{3} + 2, \quad c_2 = \sqrt{3} - 2$$

which yields

$$P(c_1) = \frac{(4\sqrt{3} + 7)b + (4 + 2\sqrt{3})c}{4 + 2\sqrt{3}}, \quad P(c_2) = \frac{(4\sqrt{3} - 7)b + (4 - 2\sqrt{3})c}{4 - 2\sqrt{3}}$$

We may set $P(c_1) = \frac{2}{3}$, $P(c_2) = -\frac{4}{3}$. Thus, the only zero d of $V' - P(c_1)W' = x(2 - 2P(c_1)) - P(c_1)$ is $d = \frac{P(c_1)}{2(1-P(c_1))} = 1$, furthermore $W(d) = W(1) \neq 0$ and $1 \neq \frac{2}{3} = P(c_1) \neq Q(d) = \frac{1}{3}$;

similarly, the only zero d of $V' - P(c_2)W' = x(2 - 2P(c_2)) - P(c_2)$ is $d = \frac{P(c_2)}{2(1-P(c_2))} = -\frac{2}{7}$, furthermore $W(d) = W(-\frac{2}{7}) \neq 0$ and $1 \neq -\frac{4}{3} = P(c_2) \neq Q(d) = \frac{4}{39}$.
By applying Theorem 2.2.4, we conclude that there are no non constant, entire solutions f, g of the equation

$$\frac{(\frac{\sqrt{3}}{2} - \frac{1}{3})f^2 + f - (\frac{\sqrt{3}}{2} + \frac{1}{3})}{f^2} = \frac{g^2}{g^2 + g + 1}$$

Note that in the case $b = 0$ the problem turns out to be almost trivial (use Theorem 2.1.2). Theorem 2.2.5 can be applied, too, since $kq - p = 4 - 2 = 2$, and we get, that there are no unbounded elements f, g in $\mathcal{A}(d(0, r^-))$ having the above decomposition.

Example 2.2.5 shows that in the following case condition (M) does not yield an empty set of rational functions P, Q:

Corollary 2.2.7. *Let $P = R/S$, $Q = V/W$, where R, S, V, W are polynomials over K, V, W, monic, (R,S)= 1, (V, W)= 1, each of which having degree two. Let P' have two distinct zeros c_i $(i = 1, 2)$ such that $P(c_1) \neq P(c_2)$ and $P(c_i) \neq 1$ $(i = 1, 2)$. Assume that for any i, $\deg(V' - P(c_i)W') > 0$ and let d_i be its unique zero. Also, suppose $P(c_i) \neq Q(d_i)$ and $W(d_i) \neq 0$ $(i = 1, 2)$. Then for a pair $(f, g) \in \mathcal{A}(K) \times \mathcal{A}(K)$ having the decomposition $P(f) = Q(g)$ it follows that $(f, g) \in K^2$.* □

Similarly to Example 2.2.6 we show now, that there exist rational functions P, Q, $\deg R = \deg V = 2, \deg S = 2$ which satisfy Condition (M).

Example 2.2.8. Let $K = \mathbb{C}_p$. For f, $g \in \mathcal{A}(K)$(resp. $\mathcal{A}(d(0, r^-))$), we consider the functional equation

$$\frac{af^2 + bf + c}{f^3} = \frac{g^2}{g^3 - 6g^2 + 11g + 6}$$

where a, b, c in K are chosen in a way, that Condition (M) is satisfied: We write $R = ax^2 + bx + c$, $S(x) = x^3$, $V(x) = x^2$, $W(x) = (x - 1)(x - 2)(x - 3)$. Clearly we have to choose $c \neq 0$, such that $(R(x), x^3) = 1$. Whenever $a \neq 0$, the derivative P' is

$$P'(x) = \frac{(-a(x^2 + 2b/ax + 3c/a))}{x^4}$$

For $c = \frac{b^2}{3a}$, P' has a single zero of multiplicity two only: $c_1 = -\frac{b}{a}$; furthermore $P(c_1) = -\frac{a^2}{3b}$. We set $t := -P(c_1) = \frac{a^2}{3b}$. Now the reader can easily verify that we may choose t in such a way that

1. $V' - P(c_1)W' = 3tx^2 + (2 - 12t)x + 11t = 0$ has one single solution of multiplicity two, $d = \frac{6t-1}{3t}$,
2. $d \notin \{1, 2, 3\}$ (i.e., $W(d) \neq 0$)
3. $P(c_1) \neq Q(d)$.

Since $p = q = 3,\ k = 1$, Theorem 2.2.4 assures us that there are no non constant entire functions f, g satisfying the functional equation from above. However, elements in $\mathcal{A}_u(d(0, r^-))$ with this specific decomposition might exist.

Corollary 2.2.9. *Let $P = R/S$, $Q = V/W$, where R, S, V, W are polynomials over K, (V, W, monic, (R,S)= 1, (V, W)= 1), R, V having degree 2, $\deg W = 3$. Let P' have a zero c of multiplicity 2, and let $V' - P(c)W'$ have a zero of multiplicity 2. If $P(c) \neq Q(d)$, then (1) has no non constant entire solutions.* □

3 Decompositions of meromorphic functions Let us have a look at the functional equation (1) again, f and g now being meromorphic functions in all of K (resp. in $M_u(d(0,r^-)))$ and P, Q in $K(x)$.
What is new and has to be precisely considered, is that f, g might have poles in K. This is the reason for some differences to the preceding case. We note:

1. There are not such cases 1, 2, 3, as in Remark 2.2.1 (for the "analytic case"): the fact, that f, g might have poles yields more "degrees of freedom" for the decomposition (1), i.e.: $\deg R > \deg S \Rightarrow \deg V > \deg W$
2. If $\deg V = \deg W$, then if g has a pole at $b \in K$, $Q(g)$ has no pole at b; this means, in this case we cannot get an estimation of $\widetilde{N}(\rho, g)$ by calculating $\widetilde{N}(\rho, Q(g))$.
3. Finally, Remark 2.1.1 tells us, that estimation (14) in the proof of Theorem 2.2.4-2.2.5 turns worse, compared with the analytic case.

The aim of this section is to establish statements along the lines of Theorem 2.2.4 and 2.2.5 for rational decompositions (1) of two distinct meromorphic functions f, g. To begin with, we repeat a statement of [8], presenting a precise asymptotic formula for the Nevanlinna function of a rational function composed with a meromorphic one (this is a generalization of Theorem 0.4 and the analogue formula in Remark 2.2.1):

Proposition 3.1.1. *Let $f \in \mathcal{M}(K) \setminus K$ (resp. $f \in \mathcal{M}_u(d(0,r^-)))$), $L \in K(x)$, where $L = A/B$, and A, B in $K[x]$, $(A,B) = 1$ and $\deg A = k, \deg B = q$. Then we have*

$$T(\rho, L(f)) = \max\{k, q\}T(\rho, f) + O(1) \tag{8}$$

For the proof in the case $f \in \mathcal{M}(K) \setminus K$ we refer to [8], where a little more general statement is shown. For f being meromorphic in a disk, the proof is analogue, since the only non elementary facts used are the Jensen's Formula and the analogue statement Theorem 0.4 for $L \in K[x]$.

Thus we infer the same asymptotic formula for $T(\rho, g)$ as in the analytic case (Remark 2.2.1):

Proposition 3.1.2. *If $f \in \mathcal{M}(K) \setminus K$ (resp. $f \in \mathcal{M}_u(d(0,r^-)))$, and f, g satisfy (1), then*

$$qT(\rho, g) = pT(\rho, f) + O(1)$$

□

Definitions and Notation 3.1.3. In this section we distinguish following cases with respect to the degrees of R, S, V and W and assign to each of them a certain rational number $\Lambda(P, Q, f, g)$:

1. Case. $v = w$: $\Lambda(P, Q, f, g) := \frac{p}{q}$
2. Case. $v < w$, $r \geq s$: $\Lambda(P, Q, f, g) := \min\{\ \gamma(R),\ \frac{p}{q}\}$
3. Case. $v > w$, $r \leq s$: $\Lambda(P, Q, f, g) := \min\{\ \gamma(S),\ \frac{p}{q}\}$
4. Case. $v > w$, $r > s$: $\Lambda(P, Q, f, g) := \min\{\ \gamma(S) + 1,\ \frac{p}{q}\}$
5. Case. $v < w$, $r < s$: $\Lambda(P, Q, f, g) := \min\{\ \gamma(R) + 1,\ \frac{p}{q}\}$

where for $L \in K[x]$, $\gamma(L)$ denotes the number of distinct zeros of L in K.
Λ arises in following estimation of $\widetilde{N}(\rho, g)$ by $T(\rho, f)$:

Proposition 3.1.4. *If $f,\ g \in \mathcal{M}(K)$ (resp. $\mathcal{M}(d(0, r^-))$) satisfy (1), then we have*

$$\widetilde{N}(\rho, g) \leq \Lambda(P, Q, f, g) T(\rho, f) + O(1) \tag{9}$$

The proof is given in section 4.
Similarly to Theorem 2.2.4 and 2.2.5 we state now

Theorem 3.1.5. *Let $f,\ g \in \mathcal{M}(K) \setminus K$, let $P,\ Q \in K(x)$ satisfy condition (M), and let $\Theta(P) := \sum_{i=1}^{k}(s_i - 2) > 0$, i.e., at least one zero c_j of $P - P(c_j)$ has multiplicity greater than 2. If f and g are solutions to equation (1), then,*

$$\widetilde{N}(\rho, g) \geq \left(\frac{q\Theta(P) - p(k\gamma(W) + 1)}{\chi^t q} \right) T(\rho, f) + \log \rho + O(1)$$

Theorem 3.1.6. *Let $f,\ g\ \mathcal{M}_u(d(0, r^-))$, let $P,\ Q \in K(x)$ satisfy condition (M), and let $\Theta(P) := \sum_{i=1}^{k}(s_i - 2) > 0$, i.e., at least one zero c_j of $P - P(c_j)$ has multiplicity greater than 2. If f and g are solutions to equation (1), then,*

$$\widetilde{N}(\rho, g) \geq \left(\frac{q\Theta(P) - p(k\gamma(W) + 1)}{\chi^t q} \right) T(\rho, f) + O(1)$$

The proofs can be found in section 4.

Corollary 3.1.7. *Let $f,\ g \in \mathcal{M}(K) \setminus K$, let $P,\ Q \in K(x)$ satisfy condition (M). If f and g are solutions to equation (1), then, we have*

$$q\Theta(P) < p(k\gamma(W) + 1) + q\Lambda(P, Q, f, g)$$

Corollary 3.1.8. *Let $f,\ g\ \mathcal{M}_u(d(0, r^-))$, let $P,\ Q \in K(x)$ satisfy condition (M). If f and g are solutions to equation (1), then, we have*

$$q\Theta(P) \leq p(k\gamma(W) + 1) + q\Lambda(P, Q, f, g)$$

Proof of the Corollaries 3.1.6-3.1.7. Both follow from Theorem 3.1.5 resp. Theorem 3.1.6 by Proposition 3.1.4 (i.e. the asymptotic formula for $\widetilde{N}(\rho, g)$) and the growth of the $\log \rho$-term. □

4 The Proofs In this section, we give a unified proof of Theorems 2.2.4, 2.2.5, 3.1.5 and 3.1.6. At first we show that (9) holds true:

Proof of Proposition 3.1.4. In any case we have $\widetilde{N}(\rho, g) \leq \frac{p}{q} T(\rho, f) + O(1)$ which immediately follows from Proposition 3.1.2. In certain cases, basic considerations improve this asymptotic formula:

1. Case. As mentioned in the beginning of section 3, in this case poles of g are cancelling in $Q(g)$, so no better result for $T(\rho, f)$ than the one from above can be achieved.

2. Case. Taking the reciprocal value of (1) we see that

$$\widetilde{N}(\rho,g)+\widetilde{Z}(\rho,V(g))=\widetilde{Z}(\rho,R(f)),$$

because $w>v$ means that $1/Q(g)$ has a pole if and only if g has a pole or $V(g)$ has a zero. Likewise we have $1/P(f)$ has a pole if and only if $S(f)$ has a zero, since a pole of f implied a zero of $1/P(f)$. Using Proposition 3.1.2 we derive

$$\widetilde{N}(\rho,g)\leq\widetilde{Z}(\rho,R(f))\leq\gamma(R)T(\rho,f)+O(1)$$

3. Case. Can be worked out like the 2. Case. Indeed, by taking the reciprocal value of (1), R and S merely change their roles.
4. Case. Obviously any pole of $P(f)$ either is a pole of f or a zero of $S(f)$, similarly any pole of $Q(g)$ either is a pole of g or a zero of $W(g)$, thus we infer

$$\widetilde{N}(\rho,g)+\widetilde{Z}(\rho,W(g))=\widetilde{N}(\rho,f)+\widetilde{Z}(\rho,S(f)),$$

which means

$$\widetilde{N}(\rho,g)\leq(1+\gamma(S))T(\rho,f)+O(1)$$

5. Case. Taking the reciprocal value of (1) we conclude as in the preceding case, with the roles of R, S exchanged.

□

Proof of Theorems 2.2.4, 2.2.5, 3.1.5 and 3.1.6. First, let us suppose f and g to be in $\mathcal{A}(K)\setminus K$ (resp. $\mathcal{A}_u(d(0,r^-))$) and assume $\pi=0$. By Lemma 2.2.2 we have certain decompositions (4) and (5) for any fixed i, thus by inserting f and g therein we derive by means of (1)

$$P(f)-P(c_i)=(f-c_i)^{s_i}R_i(f)=\frac{1}{W(g)}\prod_{j=1}^{q}(g-b_{i,j})=Q(g)-P(c_i) \tag{10}$$

Applying the second Nevanlinna Theorem N to g we derive

$$(qk-1)T(\rho,g)\leq\sum_{i=1}^{k}\sum_{j=1}^{q}\widetilde{Z}(\rho,g-b_{i,j})+\widetilde{N}(\rho,g)-\log\rho+O(1)\qquad(\rho\to\infty) \tag{11}$$

wherein of course $\widetilde{N}(\rho,g)=0$; for g we may assume $g(0)\neq b_{i,j}$ whenever $(i,j)\in\{1,\dots k\}\times\{1,\dots,q\}$.
By means of (5) we easily obtain for any fixed i

$$\widetilde{Z}(\rho,(Q(g)-P(c_i))W(g))=\sum_{j=1}^{q}\widetilde{Z}(\rho,g-b_{i,j}) \tag{12}$$

Inserting g in (12) and in (11) yields

$$(qk-1)T(\rho,g)\leq\sum_{i=1}^{k}\widetilde{Z}(\rho,(Q(g)-P(c_i))W(g))-\log\rho+O(1) \tag{13}$$

By (3) we know that $T(\rho, g) = \frac{p}{q}T(\rho, f) + O(1)$, furthermore by Lemma 2.2.3 we can write

$$R_i(x) = \frac{A_i(x)}{B_i(x)}, \quad \deg(A_i) \le p - s_i$$

and obviously $\widetilde{N}(\rho, B_i(f)) = \widetilde{N}(\rho, f) = 0$, thus by Lemma 2.2.3,

$$\widetilde{Z}(\rho, R_i(f)) = \widetilde{Z}(\rho, A_i(f)) + \widetilde{N}(\rho, B_i(f)) \le (p - s_i)T(\rho, f) + O(1)$$

For any i, any zero of $W(g)$ is a pole of $R_i(f)$ of same order: this follows from equation (10) and the fact that R, S have no common zeros. Therefore $\widetilde{Z}(\rho, R_i(f)W(g)) = \widetilde{Z}(\rho, R_i(f))$. Thus we can finally estimate any term of the sum on the right side of (13) by

$$\begin{aligned} &\widetilde{Z}(\rho, (Q - P(c_i))W(g)) = \widetilde{Z}(\rho, (f - c_i)^{s_i} R_i(f)W(g)) \le \\ \le\ &\widetilde{Z}(\rho, (f - c_i)) + \widetilde{Z}(\rho, R_i(f)) \le \\ \le\ &T(\rho, f) + (p - s_i)T(\rho, f) + O(1) \end{aligned} \tag{14}$$

and (6), (7) easily follow.
It immediately follows (consider the growth of the $\log\rho$ term) that for $f \in \mathcal{A}(K) \setminus K$ we have $qk < p$ in Theorem 2.2.4. Moreover, by Theorem 0.3 we receive $qk \le p$ in Theorem 2.2.5.
Let now $\pi \ne 0$: By Lemma Π_1 and Π_2 we see that we may apply the Nevanlinna Theorem N in the same way for g_t having ramification index 0. So we may write in the same way as we derived for characteristic 0:

$$0 \ge (\frac{kq - p}{\chi^t q})T(\rho, f_t) + \log\rho + O(1),$$

since the numbers k, p, q are the same for P_t, Q_t. Due to Lemma Π_2 the ramification index of f and g are equal and we immediately get $T(\rho, f_t) = \frac{T(\rho, f)}{\chi^t}$ which finishes the proof of the Theorems 2.2.4, 2.2.5.
The proof of the Theorems 3.1.5, 3.1.6 is similar to the one of Theorem 2.2.4 and Theorem 2.2.5:
Suppose $\pi = 0$. Obviously, formulas (10), (11), (12) for f and g hold true. Since g might have poles, instead of (11) we must write now

$$(qk - 1)T(\rho, g) \le \sum_{i=1}^{k} \widetilde{Z}(\rho, (Q(g) - P(c_i))W(g)) + \widetilde{N}(\rho, g) - \log\rho + O(1) \tag{15}$$

Now we need to estimate (12): For any $i = 1, \ldots, k$, we receive

$$\widetilde{Z}(\rho, R_i(f)) \le \widetilde{Z}(\rho, A_i(f)) + \widetilde{N}(\rho, f) \le (p - s_i + 1)T(\rho, f) + O(1),$$

thus,

$$\begin{aligned} &\widetilde{Z}(\rho, (Q(g) - P(c_i))W(g)) = \widetilde{Z}(\rho, (f - c_i)^{s_i} R_i(f)W(g)) \le \\ \le\ &\widetilde{Z}(\rho, (f - c_i)) + \widetilde{Z}(\rho, R_i(f)) + \widetilde{Z}(\rho, W(g)) \le \\ \le\ &T(\rho, f) + (p - s_i + 1)T(\rho, f) + \gamma(W)T(\rho, g) + O(1) = \\ =\ &\left(p - (s_i - 2) + \frac{p\gamma(W)}{q}\right) T(\rho, f) + O(1) \end{aligned} \tag{16}$$

Summing up over i we derive

$$\sum_{i=1}^{k} \widetilde{Z}(\rho, (Q(g) - P(c_i))W(g)) \leq (kp - \Theta(P) + \frac{kp\gamma(W)}{q})T(\rho, f) + O(1)$$

And this estimation put into (15) yields the maintained result.
If K has characteristic $\pi \neq 0$, the proof is similar to the one of the Theorems 2.2.4-2.2.5 in the same situation. Note that $\widetilde{N}(\rho, g) = \widetilde{N}(\rho, g_t)$. □

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