A NOTE ON THE PEARCE-PEČARIĆ INEQUALITIES

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Abstract. We discuss two inequalities appeared in the paper by Pearce and Pečarić in 1997. These inequalities are related to Hua's one. We first generalize one of them in the normed space setting. We next point out that the other is false and correct it in two directions.

0 Introduction. In [3], C. Pearce and J. Pečarić present two inequalities as generalizations of Hua's inequaliy:

Theorem A ([3, Theorem 2.1]). *Let f be a nondecreasing convex function on $[0,\infty)$. If $\alpha > 0$ and δ, $z_1, \dots, z_n$, $w_1, \dots, w_n \in \mathbb{C}$, then*

$$f\left(\left|\delta - \sum_{i=1}^n z_i w_i\right|\right) + \frac{1}{\alpha}\sum_{i=1}^n |w_i| f(\alpha|z_i|) \geq \frac{\alpha + \sum_{i=1}^n |w_i|}{\alpha} f\left(\frac{\alpha|\delta|}{\alpha + \sum_{i=1}^n |w_i|}\right). \tag{1}$$

When f is strictly convex and $w_i \neq 0$ for $i = 1, \dots, n$, the equality holds in (1) *if and only if*

$$z_j = \frac{\delta \overline{w}_j}{(\alpha + \sum_{i=1}^n |w_i|)|w_j|} \quad (j = 1, \dots, n).$$

Theorem B ([3, Theorem 2.2]). *If $\alpha > 0$, $a_1, \dots, a_n \in \mathbb{R}$ and δ, $z_1, \dots, z_n \in \mathbb{C}$, then*

$$\left|\delta - \sum_{i=1}^n a_i z_i\right|^2 + \frac{\alpha}{2}\left(\sum_{i=1}^n |z_i|^2 + \left|\sum_{i=1}^n z_i\right|^2\right) \geq \frac{\alpha|\delta|^2}{\alpha + \sum_{i=1}^n a_i^2}. \tag{2}$$

In (2)*, the equality holds if and only if $a_j = \mathrm{Re}(\lambda z_j)$ for $j = 1, \dots, n$, where λ is a complex number, $\sum_{i=1}^n \lambda^2 z_i^2$ is real and nonnegative and*

$$\sum_{i=1}^n a_i z_i = \frac{\sum_{i=1}^n a_i^2 \delta}{\alpha + \sum_{i=1}^n a_i^2}.$$

In this note, we give these theorems careful thought. In Section 1, we describe general results based on a formulation in [4]. Using this formulation, we show a general form of Theorem A in Section 2. Moreover, we deduce Dragomir's inequality ([1]) and compare it with (2). As a consequence, we see that Theorem B is false. In Sections 3, we show the following two improvements of Theorem B:

Theorem 1. *If $\alpha > 0$, δ, $a_1, \dots, a_n$, $z_1, \dots, z_n \in \mathbb{C}$ and $(a_1, \dots, a_n) \neq (0, \dots, 0)$, then*

$$\left|\delta - \sum_{i=1}^n a_i z_i\right|^2 + \frac{\alpha}{2}\left(\kappa \sum_{i=1}^n |z_i|^2 + \left|\sum_{i=1}^n z_i\right|^2\right) \geq \frac{\alpha|\delta|^2}{\alpha + 2(1+\kappa)}, \tag{3}$$

where $\kappa = \sqrt{\sum_{i=1}^n |a_i - 1|^2}$. In (3)*, the equality holds if and only if*

$$\sum_{i=1}^n a_i z_i = \frac{2(1+\kappa)\delta}{\alpha + 2(1+\kappa)}, \quad \sum_{i=1}^n z_i = \frac{2\delta}{\alpha + 2(1+\kappa)} \quad \text{and} \quad \left|\sum_{i=1}^n z_i\right|^2 = \sum_{i=1}^n |z_i|^2.$$

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Theorem 2. *If $\alpha > 0$, δ, $a_1, \dots, a_n$, $z_1, \dots, z_n \in \mathbb{C}$ $(n \geq 2)$ and $(a_1, \dots, a_n) \neq (0, \dots, 0)$, then*

$$\Big|\delta - \sum_{i=1}^{n} a_i z_i\Big|^2 + \frac{\alpha}{2}\left(\sum_{i=1}^{n} |z_i|^2 + \Big|\sum_{i=1}^{n} z_i\Big|^2\right) \geq \frac{\alpha|\delta|^2}{2\alpha + \left(\sqrt{2\sum_{i=1}^{n}|a_i|^2} + \sqrt{\alpha n}\right)^2}. \tag{4}$$

When two vectors $(a_1, \dots, a_n)$, $(1, \dots, 1)$ in $\mathbb{C}^n$ are linearly independent, the equality holds in (4) if and only if $z_1 = \cdots = z_n = \delta = 0$.

1 General Theory. We begin with a formulation in [4], which is the essence of many genaralizations of Hua's inequality.

Theorem C ([4, Corollary 2]). *Let $(G, +)$ be a semigroup, and let φ and ψ be nonnegative functions on G. Suppose φ is subadditive on G and there is a positive constant λ such that $\varphi(x) \leq \lambda\psi(x)$ for $x \in G$. Let f be a nondecreasing convex function on $[0, \infty)$. If $a, b \in G$, then*

$$f\big(\varphi(a)\big) + \lambda f\big(\psi(b)\big) \geq (1 + \lambda) f\Big(\frac{\varphi(a + b)}{1 + \lambda}\Big). \tag{5}$$

When f is strictly convex, the equality holds in (5) if and only if

$$\varphi(a + b) = \varphi(a) + \varphi(b), \quad \varphi(b) = \lambda\psi(b) \quad \textit{and} \quad \varphi(a) = \psi(b). \tag{6}$$

The proof is very simple and will be important in the later discussion, so we state it here.

Proof. By the hypothesis on φ and ψ, we have

$$\varphi(a + b) \leq \varphi(a) + \varphi(b) \leq \varphi(a) + \lambda\psi(b).$$

Since f is nondecreasing and convex, we obtain

$$f\Big(\frac{\varphi(a + b)}{1 + \lambda}\Big) \leq f\Big(\frac{\varphi(a) + \lambda\psi(b)}{1 + \lambda}\Big) \leq \frac{f(\varphi(a)) + \lambda f(\psi(b))}{1 + \lambda},$$

which implies (5). The condition (6) for equality is easily obtained by considering the case that all the above inequalities become equalities. □

Corollary 1. *Let $(G, +)$ be a semigroup, and let φ and ψ be nonnegative functions on G. Suppose φ is subadditive on G and there is a positive constant λ such that $\varphi(x) \leq \lambda\psi(x)$ for $x \in G$. Suppose $p, q > 1$ and $1/p + 1/q = 1$. If $a, b \in G$, then*

$$\varphi(a)^p + \psi(b)^p \geq \frac{\varphi(a + b)^p}{(1 + \lambda^q)^{p-1}}. \tag{7}$$

In (7), the equality holds if and only if

$$\varphi(a + b) = \varphi(a) + \varphi(b), \quad \varphi(b) = \lambda\psi(b) \quad \textit{and} \quad \lambda^q\varphi(a)^p = \psi(b)^p. \tag{8}$$

Proof. Since $\varphi(x) \leq \lambda^q\big(\lambda^{1-q}\psi(x)\big)$ for $x \in G$, we can replace ψ and λ in Theorem C by $\lambda^{1-q}\psi$ and λ^q, respectively. Moreover, we consider the case $f(t) = t^p$, where f is surely nondecreasing and strictly convex on $[0, \infty)$. Then (5) becomes

$$\varphi(a)^p + \lambda^q\big(\lambda^{1-q}\psi(b)\big)^p \geq \big(1 + \lambda^q\big)\left(\frac{\varphi(a + b)}{1 + \lambda^q}\right)^p,$$

which is equivalent to (7). Also, (6) becomes (8). □

Corollary 2. *Let* $(G,+)$ *be a semigroup, and let* φ_1*,* φ_2*,* ψ *be nonnegative functions on* G*. Suppose* φ_1*,* φ_2 *are subadditive on* G *and there are positive constants* λ_1*,* λ_2 *such that* $\varphi_k(x) \le \lambda_k \psi(x)$ *for* $x \in G$ *and* $k = 1, 2$*. Suppose* $p, q > 1$ *and* $1/p + 1/q = 1$*. If* $a, b \in G$*, then*

$$\varphi_1(a)^p + \varphi_2(a)^p + \psi(b)^p \ge \frac{\big(\varphi_1(a+b) + \varphi_2(a+b)\big)^p}{\big(2 + (\lambda_1+\lambda_2)^q\big)^{p-1}}. \tag{9}$$

In (9)*, the equality holds if and only if*

$$\begin{gathered} \varphi_k(a+b) = \varphi_k(a) + \varphi_k(b), \quad \varphi_k(b) = \lambda_k \psi(b) \quad (k = 1, 2) \\ \text{and} \quad \varphi_1(a) = \varphi_2(a) = \frac{\psi(b)}{(\lambda_1 + \lambda_2)^{q-1}}. \end{gathered} \tag{10}$$

Proof. Put $\mu = (\lambda_1+\lambda_2)^{q-1}$. Then $\varphi_k(x) \le \mu\lambda_k\big(\psi(x)/\mu\big)$ for $x \in G$ and $k = 1, 2$. We apply Theorem C, replacing φ, ψ, λ by φ_k, ψ/μ, $\mu\lambda_k$, respectively, and putting $f(t) = t^p$. Then we have

$$\varphi_k(a)^p + \mu\lambda_k\Big(\frac{\psi(b)}{\mu}\Big)^p \ge (1 + \mu\lambda_k)\left(\frac{\varphi_k(a+b)}{1 + \mu\lambda_k}\right)^p. \tag{11}$$

Summing these inequalities as $k = 1, 2$ yields

$$\varphi_1(a)^p + \varphi_2(a)^p + \mu^{1-p}(\lambda_1+\lambda_2)\,\psi(b)^p \ge (1+\mu\lambda_1)\left(\frac{\varphi_1(a+b)}{1 + \mu\lambda_1}\right)^p + (1+\mu\lambda_2)\left(\frac{\varphi_2(a+b)}{1 + \mu\lambda_2}\right)^p. \tag{12}$$

By the convexity of $f(t) = t^p$,

$$\frac{(1+\mu\lambda_1)\left(\dfrac{\varphi_1(a+b)}{1 + \mu\lambda_1}\right)^p + (1+\mu\lambda_2)\left(\dfrac{\varphi_2(a+b)}{1 + \mu\lambda_2}\right)^p}{2 + \mu\lambda_1 + \mu\lambda_2} \ge \left(\frac{\varphi_1(a+b) + \varphi_2(a+b)}{2+\mu\lambda_1+\mu\lambda_2}\right)^p, \tag{13}$$

and so the right side of (12) is greater than or equal to that of (9). While the left side of (12) is equal to that of (9). Thus we proved (9).

By the above argument, the equality holds in (9) precisely when the inequality signs in (11) and (13) become equality signs, namely when

$$\begin{gathered} \varphi_k(a+b) = \varphi_k(a) + \varphi_k(b), \quad \varphi_k(b) = \mu\lambda_k\frac{\psi(b)}{\mu}, \quad \varphi_k(a) = \frac{\psi(b)}{\mu} \quad (k = 1, 2) \\ \text{and} \quad \frac{\varphi_1(a+b)}{1 + \mu\lambda_1} = \frac{\varphi_2(a+b)}{1 + \mu\lambda_2}. \end{gathered}$$

It is easy to see that these equations are equivalent to (10). □

2 Pearce-Pečarić Theorems.

We first generalize Theorem A as follows:

Theorem 3. *Let* X *be a real or complex normed space with dual* X^* *and let* f *be a nondecreasing convex function on* $[0, \infty)$*. If* $\alpha > 0$*,* $\delta \in \mathbb{C}$*,* $x_1, \dots, x_n \in X$ *and* $h_1, \dots, h_n \in X^*$*, then*

$$f\left(\Big|\delta - \sum_{i=1}^n h_i(x_i)\Big|\right) + \frac{1}{\alpha}\sum_{i=1}^n \|h_i\| f\big(\alpha\|x_i\|\big) \ge \frac{\alpha+\sum_{i=1}^n\|h_i\|}{\alpha} f\Big(\frac{\alpha|\delta|}{\alpha+\sum_{i=1}^n\|h_i\|}\Big). \tag{14}$$

When f is strictly convex, the equality holds in (14) *if and only if*

$$h_j(x_j) = \frac{\delta\|h_j\|}{\alpha+\sum_{i=1}^n\|h_i\|} \quad \textit{and} \quad |h_j(x_j)| = \|h_j\|\,\|x_j\| \quad (j = 1, \dots, n). \tag{15}$$

We prove it by using the method in the proof of Theorem C.

Proof. By the triangle inequality and the well-known functional inequality, we have

$$\begin{aligned}|\delta| = \left|\Big(\delta - \sum_{i=1}^n h_i(x_i)\Big) + \sum_{i=1}^n h_i(x_i)\right| &\le \left|\delta - \sum_{i=1}^n h_i(x_i)\right| + \sum_{i=1}^n \big|h_i(x_i)\big| \\ &\le \left|\delta - \sum_{i=1}^n h_i(x_i)\right| + \sum_{i=1}^n \|h_i\|\,\|x_i\|.\end{aligned}$$

Since f is nondecreasing and convex, it follows that

$$\begin{aligned}f\left(\frac{\alpha|\delta|}{\alpha+\sum_{i=1}^n\|h_i\|}\right) &\le f\left(\frac{\alpha\,|\delta - \sum_{i=1}^n h_i(x_i)| + \sum_{i=1}^n\|h_i\|(\alpha\|x_i\|)}{\alpha+\sum_{i=1}^n\|h_i\|}\right) \\ &\le \frac{\alpha\, f\big(|\delta - \sum_{i=1}^n h_i(x_i)|\big) + \sum_{i=1}^n\|h_i\| f\big(\alpha\|x_i\|\big)}{\alpha+\sum_{i=1}^n\|h_i\|}.\end{aligned}$$

This proves (14).

Suppose f is strictly convex and the equality holds in (14). By the above argument, we see

$$\left|\Big(\delta - \sum_{i=1}^n h_i(x_i)\Big) + \sum_{i=1}^n h_i(x_i)\right| = \left|\delta - \sum_{i=1}^n h_i(x_i)\right| + \sum_{i=1}^n \big|h_i(x_i)\big|, \tag{16}$$

$$|h_j(x_j)| = \|h_j\|\,\|x_j\| \qquad (j = 1, \dots, n), \tag{17}$$

$$\left|\delta - \sum_{i=1}^n h_i(x_i)\right| = \alpha\|x_1\| = \cdots = \alpha\|x_n\|. \tag{18}$$

In (16), the triangle inequality becomes equality, and so there is a complex number ξ, $|\xi| = 1$, such that

$$\delta - \sum_{i=1}^n h_i(x_i) = \left|\delta - \sum_{i=1}^n h_i(x_i)\right|\xi \quad \text{and} \quad h_j(x_j) = |h_j(x_j)|\,\xi \quad (j = 1, \dots, n).$$

By (17) and (18),

$$\delta - \sum_{i=1}^n h_i(x_i) = \alpha\|x_1\|\,\xi \quad \text{and} \quad h_j(x_j) = \|h_j\|\,\|x_1\|\,\xi \quad (j = 1, \dots, n).$$

Hence $\delta = \alpha\|x_1\|\xi + \sum_{i=1}^n\|h_i\|\,\|x_1\|\,\xi = \big(\alpha + \sum_{i=1}^n\|h_i\|\big)\big(\|x_1\|\,\xi\big)$ and so

$$h_j(x_j) = \|h_j\|\big(\|x_1\|\xi\big) = \frac{\delta\|h_j\|}{\alpha+\sum_{i=1}^n\|h_i\|} \quad (j = 1, \dots, n).$$

Thus we obtain (15).

Conversely, suppose (15) holds. Then

$$f\left(\left|\delta - \sum_{j=1}^n h_j(x_j)\right|\right) = f\left(\left|\delta - \sum_{j=1}^n \frac{\delta\|h_j\|}{\alpha+\sum_{i=1}^n\|h_i\|}\right|\right) = f\left(\frac{\alpha|\delta|}{\alpha+\sum_{i=1}^n\|h_i\|}\right).$$

If $h_j \neq 0$, then

$$\|h_j\|\, f\big(\alpha\|x_j\|\big) = \|h_j\|\, f\left(\alpha\frac{|h_j(x_j)|}{\|h_j\|}\right) = \|h_j\|\, f\left(\frac{\alpha|\delta|}{\alpha+\sum_{i=1}^{n}\|h_i\|}\right),$$

while, if $h_j = 0$, the initial and terminal sides are clearly equal. These equations show the equality in (14). Thus Theorem 3 was proved. □

Remark. Theorem 3 can be proved by using Theorem C directly.

Let us consider the case $X = \mathbb{C}$. For each $i = 1, \ldots, n$, pick $w_i \in \mathbb{C}$ and define $h_i \in \mathbb{C}^*$ by $h_i(z) = w_i z$ for $z \in \mathbb{C}$. Then $\|h_i\| = |w_i|$. Applying Theorem 3 with $x_i = z_i$ $(i = 1, \ldots, n)$, we obtain Theorem A.

Next, we discuss Theorem B. The inequality in Theorem B is similar to Dragomir's one:

Theorem D ([1]). *If $\alpha > 0$ and $\delta, a_1, \ldots, a_n, z_1, \ldots, z_n \in \mathbb{C}$, then*

$$\left|\delta - \sum_{i=1}^{n} a_i z_i\right|^2 + \alpha \sum_{i=1}^{n} |z_i|^2 \geq \frac{\alpha|\delta|^2}{\alpha+\sum_{i=1}^{n}|a_i|^2}. \tag{19}$$

In (19), the equality holds if and only if $z_j = \delta\overline{a}_j/\big(\alpha+\sum_{i=1}^{n}|a_i|^2\big)$ for $j = 1, \ldots, n$.

For the sake of completeness, we prove it. Our proof depends on Theorem A.

Proof. We first consider the case that $a_i \neq 0$ for all $i = 1, \ldots, n$. In Theorem A, take $p = q = 2$, and repalce z_i, w_i by z_i/a_i, a_i^2, respectively. The theorem follows immediately. In general case, we discard the ith terms a_i, z_i such that $a_i = 0$. Since the remaining ones are applicable to the first case, it follows that

$$\left|\delta - \sum_{a_i\neq 0} a_i z_i\right|^2 + \alpha \sum_{a_i\neq 0} |z_i|^2 \geq \frac{\alpha|\delta|^2}{\alpha+\sum_{a_i\neq 0}|a_i|^2}. \tag{20}$$

Clearly, the left side of (20) is less than or equal to that of (19). While the right side of (20) is equal to that of (19). Thus (19) was proved. The condition for equality is easily obtained. □

Let us compare (2) and (19). Since (19) is true by the above proof, we doubt whether the case $n \geq 2$ of (2) is true. Indeed, take $\alpha = 1$, $\delta = 4$, $a_1 = 2$, $a_2 = \cdots = a_n = 0$ and $z_1 = 2$, $z_2 = -1$, $z_3 = \cdots = z_n = 0$. Then

$$\left|\delta - \sum_{i=1}^{n} a_i z_i\right|^2 + \frac{\alpha}{2}\left(\sum_{i=1}^{n}|z_i|^2 + \left|\sum_{i=1}^{n} z_i\right|^2\right) = 3 < \frac{16}{5} = \frac{\alpha|\delta|^2}{\alpha+\sum_{i=1}^{n} a_i^2}.$$

We conclude that Theorem B is false.

Remark. Suppose $a_1, \ldots, a_n \in \mathbb{C}$ and $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$. We can easily show that (2) holds for all $\alpha > 0$ and all $\delta, z_1, \ldots, z_n \in \mathbb{C}$ if and only if the inequality

$$\left|\sum_{i=1}^{n} a_i z_i\right|^2 \leq \frac{1}{2}\sum_{i=1}^{n}|a_i|^2\left(\sum_{i=1}^{n}|z_i|^2 + \left|\sum_{i=1}^{n} z_i\right|^2\right) \tag{21}$$

holds for all $z_1, \ldots, z_n \in \mathbb{C}$ (cf. [3, Proposition 2.1]). In case $n = 2$, a_1, a_2 satisfy the inequality (21), namely

$$\left|a_1 z_1 + a_2 z_2\right|^2 \leq \frac{1}{2}\big(|a_1|^2+|a_2|^2\big)\big(|z_1|^2+|z_2|^2+|z_1+z_2|^2\big)$$

for all $z_1, z_2 \in \mathbb{C}$, if and only if

$$|a_1|^2 + |a_2|^2 \le 4\,\mathrm{Re}(a_1\overline{a}_2). \tag{22}$$

Thus (22) is the necessary and sufficient condition that (2) with $n = 2$ holds for all $\alpha > 0$, $\delta, z_1, z_2 \in \mathbb{C}$. We note that there exist $a_1, a_2 \in \mathbb{C}$ which do not satisfy (22).

3 Proofs. We prove a general form of Theorem 1.

Theorem 4. *Let X be a real or complex normed space with dual X^*. Suppose $p, q > 1$ and $1/p + 1/q = 1$. If $\alpha > 0$, $\delta \in \mathbb{C}$, $x \in X$, $g, h \in X^*$ and $g \ne 0$, then*

$$\big|\delta - g(x)\big|^p + \alpha\big(\|g-h\|\,\|x\|^p + |h(x)|^p\big) \ge \frac{|\delta|^p}{\big(1 + \alpha^{1-q}(1+\|g-h\|)\big)^{p-1}}. \tag{23}$$

In (23)*, the equality holds if and only if*

$$g(x) = \frac{\delta\,(1+\|g-h\|)}{\alpha^{q-1}+1+\|g-h\|}, \quad h(x) = \frac{\delta}{\alpha^{q-1}+1+\|g-h\|} \quad \textit{and} \quad |h(x)| = \|x\|. \tag{24}$$

Proof. Take $\alpha > 0$, $\delta \in \mathbb{C}$, $x \in X$ and $g, h \in X^*$ $(g \ne 0)$. We first observe that

$$|g(z)| \le (1+\|g-h\|)^{1/q}\big(\|g-h\|\,\|z\|^p + |h(z)|^p\big)^{1/p} \tag{25}$$

for all $z \in X$. This follows from

$$|g(z)| \le |g(z) - h(z)| + |h(z)| \le \|g - h\|\,\|z\| + |h(z)|$$

and

$$\begin{aligned} |g(z)|^p &\le \big(1+\|g-h\|\big)^p \left(\frac{\|g-h\|\,\|z\| + |h(z)|}{1 + \|g - h\|}\right)^p \\ &\le \big(1+\|g-h\|\big)^{p-1}\big(\|g-h\|\,\|z\|^p + |h(z)|^p\big). \end{aligned}$$

Put $G = X$, and define nonnegative functions φ, ψ on G by

$$\varphi(z) = |g(z)|, \qquad \psi(z) = \alpha^{1/p}\big(\|g-h\|\,\|z\|^p + |h(z)|^p\big)^{1/p}$$

for $z \in G$. Clearly, φ is subadditive on G. If we set $\lambda = (1+\|g-h\|)^{1/q}/\alpha^{1/p}$, then (25) implies $\varphi(z) \le \lambda\psi(z)$ for $z \in G$. Choose $y \in X$ so that $g(y) = \delta$ (This is possible because $g \ne 0$). Now, apply Corollary 1 with $a = y - x$ and $b = x$. The result is

$$\big|g(y) - g(x)\big|^p + \alpha\big(\|g-h\|\,\|x\|^p + |h(x)|^p\big) \ge \frac{|g(y)|^p}{\big(1 + (1+\|g-h\|)/\alpha^{q/p}\big)^{p-1}}.$$

This proves (23).

Suppose that the equality holds in (23). By (8) in Corollary 1, we see

$$|\delta| = |\delta - g(x)| + |g(x)|, \tag{26}$$

$$|g(x)| = \frac{(1 + \|g-h\|)^{1/q}}{\alpha^{1/p}}\alpha^{1/p}\big(\|g-h\|\,\|x\|^p + |h(x)|^p\big)^{1/p}, \tag{27}$$

$$\frac{1 + \|g-h\|}{\alpha^{q/p}}|\delta - g(x)|^p = \alpha\,\big(\|g-h\|\,\|x\|^p + |h(x)|^p\big). \tag{28}$$

Here (27) says that the equality holds in (25), and so

$$|g(x)| = |g(x)-h(x)| + |h(x)|, \quad |g(x)-h(x)| = \|g-h\| \, \|x\|, \quad \|x\| = |h(x)|. \tag{29}$$

Combining (28) and the third equation in (29), we easily see that

$$|\delta - g(x)| = \alpha^{q-1}|h(x)|. \tag{30}$$

Moreover, by (26) and the first equation in (29), we find complex numbers ξ, η, $|\xi| = |\eta| = 1$, such that

$$\delta - g(x) = |\delta - g(x)| \, \xi, \quad g(x) = |g(x)| \, \xi, \tag{31}$$

$$g(x) - h(x) = |g(x) - h(x)| \, \eta, \quad h(x) = |h(x)| \, \eta. \tag{32}$$

We now assume $g(x) \neq 0$. Then $\xi = \eta$, because (32) implies $g(x) = |g(x)| \, \eta$. Hence, by (30), (31) and (32),

$$\delta - g(x) = \alpha^{q-1}|h(x)| \, \xi = \alpha^{q-1}|h(x)| \, \eta = \alpha^{q-1}h(x). \tag{33}$$

While, by (29) and (32),

$$g(x) - h(x) = \|g-h\| \, \|x\| \, \eta = \|g-h\| \, |h(x)| \, \eta = \|g-h\| \, h(x). \tag{34}$$

From (33) and (34), we easily get the first and second equations in (24). The third one in (24) has been obtained in (29). Thus we proved (24) in case $g(x) \neq 0$. If $g(x) = 0$, then $h(x) = 0$ by (32), $\|x\| = 0$ by the third equation of (29) and $\delta = 0$ by (30). Thus (24) also holds in case $g(x) = 0$.

It is a routine work to see that (24) implies the equality in (23). $\square$

Pick $\alpha > 0$, $\delta, a_1, \dots, a_n, z_1, \dots, z_n \in \mathbb{C}$ as in Theorem 1. In Theorem 4, take X to be the n-dimensional complex Euclidean space $\mathbb{C}^n$, set $p = q = 2$, replace α by $\alpha/2$, put $x = (z_1, \dots, z_n) \in \mathbb{C}^n = X$ and define $g, h \in (\mathbb{C}^n)^* = X^*$ by

$$g(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i, \qquad h(x_1, \dots, x_n) = \sum_{i=1}^n x_i,$$

for all $(x_1, \dots, x_n) \in \mathbb{C}^n = X$. Noting $\|g-h\| = \sqrt{\sum_{i=1}^n |a_i - 1|^2} = \kappa$, we arrive at Theorem 1.

Next, we prove Theorem 2.

Proof of Theorem 2. Take $\alpha > 0$ and $\delta, a_1, \dots, a_n, z_1, \dots, z_n \in \mathbb{C}$ $(n \geq 2)$. We first consider the case that two vectors $(a_1, \dots, a_n)$, $(1, \dots, 1)$ in $\mathbb{C}^n$ are linearly independent. Put $G = \mathbb{C}^n$, and define nonnegative functions φ_1, φ_2, ψ on G by

$$\varphi_1(x_1, \dots, x_n) = \Big|\sum_{i=1}^n a_i x_i\Big|, \quad \varphi_2(x_1, \dots, x_n) = \sqrt{\frac{\alpha}{2}} \Big|\sum_{i=1}^n x_i\Big|, \quad \psi(x_1, \dots, x_n) = \sqrt{\frac{\alpha}{2} \sum_{i=1}^n |x_i|^2}$$

for $(x_1, \dots, x_n) \in \mathbb{C}^n = G$. It is clear that φ_1 and φ_2 are subadditive on G. If we set

$$\lambda_1 = \sqrt{\frac{2}{\alpha} \sum_{i=1}^n |a_i|^2}, \qquad \lambda_2 = \sqrt{n},$$

then the Cauchy-Schwarz inequality shows

$$\varphi_1(x_1,\ldots,x_n) \le \sum_{i=1}^n |a_i||x_i| \le \sqrt{\sum_{i=1}^n |a_i|^2}\sqrt{\sum_{i=1}^n |x_i|^2} = \lambda_1\psi(x_1,\ldots,x_n),$$
$$\varphi_2(x_1,\ldots,x_n) \le \sqrt{\frac{\alpha}{2}}\sum_{i=1}^n |x_i| \le \sqrt{\frac{\alpha}{2}}\sqrt{n}\sqrt{\sum_{i=1}^n |x_i|^2} = \lambda_2\psi(x_1,\ldots,x_n)$$

for $(x_1,\ldots,x_n) \in G$. We use the linear independence of $(a_1,\ldots,a_n)$ and $(1,\ldots,1)$ to find a vector $(w_1,\ldots,w_n) \in \mathbb{C}^n$ such that

$$\sum_{i=1}^n a_i w_i = \delta \quad \text{and} \quad \sum_{i=1}^n w_i = 0. \tag{35}$$

Now, apply Corollary 2 with $p = q = 2$ and $a = (w_1 - z_1,\ldots,w_n - z_n)$, $b = (z_1,\ldots,z_n)$. The result is

$$\Big|\sum_{i=1}^n a_i w_i - \sum_{i=1}^n a_i z_i\Big|^2 + \frac{\alpha}{2}\Big|\sum_{i=1}^n w_i - \sum_{i=1}^n z_i\Big|^2 + \frac{\alpha}{2}\sum_{i=1}^n |z_i|^2$$
$$\ge \frac{\left(|\sum_{i=1}^n a_i w_i| + \sqrt{\alpha/2}\,|\sum_{i=1}^n w_i|\right)^2}{2 + \left(\sqrt{(2/\alpha)\sum_{i=1}^n |a_i|^2} + \sqrt{n}\right)^2}.$$

This is (4) by (35).

Suppose that two vectors $(a_1,\ldots,a_n)$, $(1,\ldots,1)$ are linearly independent and the equality holds in (4). By (10) in Corollary 2, we see

$$|\delta| = \Big|\delta - \sum_{i=1}^n a_i z_i\Big| + \Big|\sum_{i=1}^n a_i z_i\Big|, \quad \Big|\sum_{i=1}^n a_i z_i\Big| = \sqrt{\frac{2}{\alpha}\sum_{i=1}^n |a_i|^2}\sqrt{\frac{\alpha}{2}\sum_{i=1}^n |z_i|^2},$$
$$0 = \sqrt{\frac{\alpha}{2}}\Big|-\sum_{i=1}^n z_i\Big| + \sqrt{\frac{\alpha}{2}}\Big|\sum_{i=1}^n z_i\Big|, \quad \sqrt{\frac{\alpha}{2}}\Big|\sum_{i=1}^n z_i\Big| = \sqrt{n}\sqrt{\frac{\alpha}{2}\sum_{i=1}^n |z_i|^2},$$
$$\Big|\delta - \sum_{i=1}^n a_i z_i\Big| = \sqrt{\frac{\alpha}{2}}\Big|-\sum_{i=1}^n z_i\Big| = \sqrt{\frac{\alpha}{2}\sum_{i=1}^n |z_i|^2}\Big/\left(\sqrt{\frac{2}{\alpha}\sum_{i=1}^n |a_i|^2} + \sqrt{n}\right).$$

The third and fourth equations imply $z_1 = \cdots = z_n = 0$. Hence the fifth equation yields $\delta = 0$. Conversely, if $z_1 = \cdots = z_n = \delta = 0$, then both sides of (4) are equally zero. Thus the statement on equality was proved.

Finally we must consider the case that vectors $(a_1,\ldots,a_n)$, $(1,\ldots,1)$ are linearly dependent. In this case, we can select a sequence $\{(a_{k,1},\ldots,a_{k,n})\}_{k=1,2,\ldots}$ of vectors in $\mathbb{C}^n$ such that

$$a_{k,1} \to a_1, \quad \ldots, \quad a_{k,n} \to a_n \quad (k \to \infty)$$

and for each k, two vectors $(a_{k,1},\ldots,a_{k,n})$, $(1,\ldots,1)$ are linearly indepnedent. Then the preceeding argument shows that for each k,

$$\Big|\delta - \sum_{i=1}^n a_{k,i} z_i\Big|^2 + \frac{\alpha}{2}\left(\sum_{i=1}^n |z_i|^2 + \Big|\sum_{i=1}^n z_i\Big|^2\right) \ge \frac{\alpha|\delta|^2}{2\alpha + \left(\sqrt{2\sum_{i=1}^n |a_{k,i}|^2} + \sqrt{\alpha n}\right)^2}.$$

Letting $k \to \infty$, we obtain (4). The proof is finished. □

Remark. In [2], we may find another development of Hua-type inequalities.

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