ORDER AMONG POWER MEANS OF POSITIVE OPERATORS

Jadranka Mićić and Josip Pečarić

Received September 16, 2003; revised June 2, 2004

Abstract. Let $M_k^{[r]}(\mathbf{A},\Phi,\omega) := (\sum_{j=1}^k \omega_j \ \Phi_j(A_j^r))^{1/r}$ ($r \in \mathbf{R}\backslash\{0\}$) be a weighted power mean of positive operators A_j with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $0 < m < M$, where Φ_j are normalized positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$ and $\omega_j \in \mathbf{R}_+$ such that $\sum_{j=1}^k \omega_j = 1$ ($j = 1,\dots,k$). As a continuation of our paper [Sci. Math. Japon., **57** (2003), 139–148] we determined real constants α_1, α_2, β_1 and β_2 such that

$$\alpha_2 M_k^{[s]}(\mathbf{A},\Phi,\omega) \le M_k^{[r]}(\mathbf{A},\Phi,\omega) \le \alpha_1 M_k^{[s]}(\mathbf{A},\Phi,\omega)$$

and

$$\beta_2 I \le M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) \le \beta_1 I$$

hold if $r \le s$, $r,s \ne 0$. As applications, these inequalities for some special maps were given.

1 Introduction.

Let $\mathcal{B}(H)$ be the C*-algebra of all bounded linear operators on a Hilbert space H, $\mathcal{B}_+(H)$ be the set of all positive operators of $\mathcal{B}(H)$ and $P_N[\mathcal{B}(H),\mathcal{B}(K)]$ be the set of all normalized positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$. We denote by $\mathsf{Sp}(A)$ the spectrum of the operator A.

We consider the following weighted power means of positive operators. Let $\Phi_j \in P_N[\mathcal{B}(H),\mathcal{B}(K)]$, $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $0 < m < M$ and $\omega_j \in \mathbf{R}_+$ such that $\sum_{j=1}^k \omega_j = 1$ ($j = 1,\dots,k$). We define

$$M_k^{[r]}(\mathbf{A},\Phi,\omega) := (\sum_{j=1}^k \omega_j \ \Phi_j(A_j^r))^{1/r} \qquad \text{if} \quad r \in \mathbf{R}\backslash\{0\}. \tag{1}$$

Mond and Pečarić proved in [6] that

$$M_k^{[r]}(\mathbf{A},\Phi,\omega) \le M_k^{[s]}(\mathbf{A},\Phi,\omega) \tag{2}$$

holds if $r \le s$, $s \notin \langle -1,1\rangle$, $r \notin \langle -1,1\rangle$ or $1/2 \le r \le 1 \le s$ or $r \le -1 \le s \le -1/2$.

We considered in [7] the power means when $\Phi_j = I$ ($j = 1,\dots,k$), i.e. $M_k^{[r]}(\mathbf{A},\omega) := (\sum_{j=1}^k \omega_j A_j^r)^{1/r}$ if $r \in \mathbf{R}\backslash\{0\}$ and proved that

$$\tilde{\Delta}^{-1} M_k^{[s]}(\mathbf{A},\omega) \le M_k^{[r]}(\mathbf{A},\omega) \le M_k^{[s]}(\mathbf{A},\omega)$$

holds if $r \le s$, $s \notin \langle -1,1\rangle$, $r \notin \langle -1,1\rangle$ or $1/2 \le r \le 1 \le s$ or $r \le -1 \le s \le -1/2$ and

$$\tilde{\Delta}^{-1} M_k^{[s]}(\mathbf{A},\omega) \le M_k^{[r]}(\mathbf{A},\omega) \le \tilde{\Delta} M_k^{[s]}(\mathbf{A},\omega)$$

2000 *Mathematics Subject Classification.* Primary 47A63.

Key words and phrases. Operator order, Mond-Pečarić method, weighted power mean.

holds if $s \geq 1$, $-1 < r < 1/2$, $r \neq 0$ or $r \leq -1$, $-1/2 < s < 1$, $s \neq 0$, where

$$\tilde{\Delta} = \left\{ \frac{r(\kappa^s - \kappa^r)}{(s-r)(\kappa^r - 1)} \right\}^{\frac{1}{s}} \left\{ \frac{s(\kappa^r - \kappa^s)}{(r-s)(\kappa^s - 1)} \right\}^{-\frac{1}{r}}, \qquad \kappa = \frac{M}{m}.$$

Mićić, Pečarić, Seo and Tominaga in [5] considered the power mean $\Phi(A^r)^{1/r}$ if $r \in \mathbf{R}\backslash\{0\}$, where A is a Hermitian matrix with $\mathsf{Sp}(A) \subseteq [m, M]$ for some scalars $0 < m < M$ and Φ is a normalized positive linear map. They proved that

$$0 \leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \Delta I$$

holds if $r \leq s$, $s \notin \langle -1, 1 \rangle$, $r \notin \langle -1, 1 \rangle$ or $1/2 \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -1/2$ and

$$\Delta^* I \leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \Delta I$$

holds if $s \geq 1$, $-1 < r < 1/2$, $r \neq 0$ or $r \leq -1$, $-1/2 < s < 1$, $s \neq 0$, where

$$\begin{aligned} \Delta &= \max_{\theta \in [0,1]} \left\{ [\theta M^s + (1-\theta) m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta) m^r]^{\frac{1}{r}} \right\}, \\ \Delta^* &= \min_{\theta \in [0,1] \cup [\frac{d}{M^r - m^r}, \frac{d}{M^r - m^r} + 1]} \left\{ [\theta M^s + (1-\theta) m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta) m^r - d]^{\frac{1}{r}} \right\}, \\ d &= \frac{M^s m^r - M^r m^s}{M^s - m^s} - \left(1 - \frac{r}{s}\right) \left(\frac{s}{r} \frac{M^r - m^r}{M^s - m^s} \right)^{\frac{r}{r-s}}. \end{aligned}$$

Of course, they proved that the lower and upper estimates for the ratio of this means are equal as in our results [7].

The aim of this paper is to generalize the above results as follows. We determine real constants α_1, α_2, β_1 and β_2 such that

$$\alpha_2 M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq \alpha_1 M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

and

$$\beta_2 I \leq M_k^{[s]}(\mathbf{A}, \Phi, \omega) - M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq \beta_1 I$$

hold if $r \leq s$, $r, s \neq 0$. As applications, these inequalities for some special maps were given.

2 Previous results.

The Löwner-Heinz theorem asserts that the function $f(t) = t^p$ is operator monotone only for $1 \geq p \geq 0$ though it is monotone increasing for $p > 0$. In our papers [4] and [8] we proved the following theorems for the operator order and the reversing operator order.

Theorem A *Let $A, B \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(B) \subseteq [m, M]$ for some scalars $0 < m < M$. If $A \geq B > 0$, then for a given $\alpha > 0$*

$$\alpha A^p + \beta I \geq B^p \qquad \textit{for all} \quad p > 1,$$

where

$$\beta = \begin{cases} \alpha(p-1) \left(\frac{1}{\alpha p} \frac{M^p - m^p}{M - m} \right)^{\frac{p}{p-1}} + \frac{M m^p - m M^p}{M - m} & \textit{if} \quad p m^{p-1} \leq \frac{M^p - m^p}{\alpha(M-m)} \leq p M^{p-1} \\ \max\{M^p - \alpha M^p, m^p - \alpha m^p\} & \textit{otherwise.} \end{cases}$$

Theorem B *Let* $A, B \in \mathcal{B}_+(H)$ *with* $\mathsf{Sp}(A) \subseteq [m, M]$ *for some scalars* $0 < m < M$. *If* $A \geq B > 0$, *then for a given* $\alpha > 0$

$$\alpha B^p + \beta I \geq A^p \qquad \text{for all} \quad p < -1,$$

where β *is defined in Theorem A.*

If we put $\alpha = 1$ in Theorem A and Theorem B, then we obtain the following corollary (see also [9]).

Corollary 1. *If* $A, B \in \mathcal{B}_+(H)$, $A \geq B > 0$ *and* $\mathsf{Sp}(B) \subseteq [m, M]$ *for some scalars* $0 < m < M$, *then*

$$A^p + C(m, M, p)I \geq B^p \qquad \text{for all} \quad p > 1.$$

But, if $A \geq B > 0$ *and* $\mathsf{Sp}(A) \subseteq [m, M]$, $0 < m < M$, *then*

$$B^p + C(m, M, p)I \geq A^p \qquad \text{for all} \quad p < -1,$$

where

$$C(m, M, p) = (p - 1)\left(\frac{1}{p}\frac{M^p - m^p}{M - m}\right)^{\frac{p}{p-1}} + \frac{Mm^p - mM^p}{M - m} \qquad \text{for } p \in \mathbf{R}. \tag{3}$$

Remark 2. *Notice that*

$$A \geq B > 0 \quad \text{implies} \quad A^p \geq B^p > 0 \quad \text{for all} \quad 0 < p \leq 1$$

and

$$A \geq B > 0 \quad \text{implies} \quad B^p \geq A^p > 0 \quad \text{for all} \quad -1 \leq p < 0$$

by the Löwner-Heinz theorem and by the statement: $A \geq B > 0$ *implies* $B^{-1} \geq A^{-1} > 0$.

If we choose α such that $\beta = 0$ in Theorem A and Theorem B, then we obtain the following corollary (see also [3]).

Corollary 3. *If* $A, B \in \mathcal{B}_+(H)$, $A \geq B > 0$ *such that* $\mathsf{Sp}(A) \subseteq [n, N]$ *and* $\mathsf{Sp}(B) \subseteq [m, M]$ *for some scalars* $0 < n < N$ *and* $0 < m < M$, *then*

$$\begin{aligned} K(n, N, p)A^p \geq B^p > 0 \qquad &\text{for all} \quad p > 1, \\ K(m, M, p)A^p \geq B^p > 0 \qquad &\text{for all} \quad p > 1, \\ K(n, N, p)B^p \geq A^p > 0 \qquad &\text{for all} \quad p < -1 \end{aligned}$$

and

$$K(m, M, p)B^p \geq A^p > 0 \qquad \text{for all} \quad p < -1,$$

where

$$K(m, M, p) = \frac{Mm^p - mM^p}{(1 - p)(M - m)}\left(\frac{1 - p}{p}\frac{M^p - m^p}{Mm^p - mM^p}\right)^p \qquad \text{for } p \in \mathbf{R}. \tag{4}$$

We cite the following known Jensen's inequality for an operator convex function [1, 2].

Lemma J *Let* $\Phi_j \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, $A_j \in \mathcal{B}_+(H)$ *with* $\mathsf{Sp}(A_j) \subseteq [m, M]$ *for some scalars* $0 < m < M$ *and* $\omega_j \in \mathbf{R}_+$ *such that* $\sum_{j=1}^k \omega_j = 1$ $(j = 1, \ldots, k)$. *If* f *is an operator convex function on* $[m, M]$, *then*

$$f(\sum_{j=1}^k \omega_j \ \Phi_j(A_j)) \leq \sum_{j=1}^k \omega_j \ \Phi_j(f(A_j))$$

holds.

In the next theorems we have reverse inequalities of Jensen's inequality for a convex function.

Theorem 4. *Let* $\Phi_j \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, $A_j \in \mathcal{B}_+(H)$ *with* $\mathsf{Sp}(A_j) \subseteq [m, M]$ *for some scalars* $0 < m < M$ *and* $\omega_j \in \mathbf{R}_+$ *such that* $\sum_{j=1}^k \omega_j = 1$ $(j = 1, \ldots, k)$. *If* $f, g \in \mathcal{C}([m, M])$ *and* f *is a convex function, then for a given* $\alpha \in \mathbf{R}$

$$\sum_{j=1}^k \omega_j \Phi_j(f(A_j)) \leq \alpha g(\sum_{j=1}^k \omega_j \Phi_j(A_j)) + \beta I$$

holds for

$$\beta = \max_{m \leq t \leq M} \left\{ f(m) + \frac{f(M) - f(m)}{M - m}(t - m) - \alpha g(t) \right\}.$$

Proof. Since f is a convex function, we have $f(t) \leq f(m) + \frac{f(M)-f(m)}{M-m}(t - m)$ for every $t \in [m, M]$. Hence it follows that $f(A_j) \leq f(m)I + \frac{f(M)-f(m)}{M-m}(A_j - mI)$ for each j. Since Φ_j is a normalized positive linear map, it follows that $\Phi_j(f(A_j)) \leq f(m)I + \frac{f(M)-f(m)}{M-m}(\Phi_j(A_j) - mI)$. Further, it multiplies with $\omega_j \in \mathbf{R}_+$, summing of all $j = 1, \ldots, k$, and using $\sum_{j=1}^k \omega_j = 1$ we have

$$\sum_{j=1}^k \omega_j \Phi_j \left(f(A_j)\right) \leq f(m)I + \frac{f(M) - f(m)}{M - m}(\sum_{j=1}^k \omega_j \Phi_j(A_j) - mI).$$

Besides, it follows from $mI \leq A_j \leq MI$ that $mI \leq \sum_{j=1}^k \omega_j \Phi_j(A_j) \leq MI$, i.e. $\mathsf{Sp}\left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right) \subseteq [m, M]$. Now we obtain

$$\begin{aligned} &\textstyle\sum_{j=1}^k \omega_j \Phi_j(f(A_j)) - \alpha g(\sum_{j=1}^k \omega_j \Phi_j(A_j)) \\ &\leq f(m)I + \tfrac{f(M)-f(m)}{M-m}(\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j) - mI) - \alpha g(\sum_{j=1}^k \omega_j \Phi_j(A_j)) \\ &\leq \max_{m \leq t \leq M} \left\{ f(m) + \frac{f(M) - f(m)}{M - m}(t - m) - \alpha g(t) \right\} I, \end{aligned}$$

which is the desired inequality. □

Remark 5. *Assume that conditions of Theorem 4 hold and let* g *be a differentiable function on* $[m, M]$. *Moreover, suppose that either of the following additional conditions holds:* (a) g *is strictly convex if* $\alpha > 0$, *or* (b) g *is strictly concave if* $\alpha < 0$.

Then β can be written explicitly [4, Remark 2.3]

$$\beta = f(m) + \mu(t_0 - m) - \alpha g(t_0),$$

where

$$t_0 = \begin{cases} \text{the unique solution of } g'(t) = \frac{\mu}{\alpha} & \text{if } \alpha g'(m) \le \mu \le \alpha g'(M) \\ M & \text{if } \mu > \alpha g'(M) \\ m & \text{if } \alpha g'(m) > \mu \end{cases}$$

and $\mu = \frac{f(M)-f(m)}{M-m}$.

Theorem 6. *Let $\Phi_j \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$ for some scalars $0 < m < M$ and $\omega_j \in \mathbf{R}_+$ such that $\sum_{j=1}^k \omega_j = 1$ $(j = 1, \ldots, k)$. Let $f, g \in \mathcal{C}([m, M])$ and f be a convex function. Suppose that either of the following conditions holds: (I) $g > 0$ on $[m, M]$ or (II) $g < 0$ on $[m, M]$. Then*

$$\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) \le \alpha_o\, g(\sum_{j=1}^{k} \omega_j \Phi_j(A_j))$$

holds for

$$\alpha_o = \max_{m \le t \le M} \left\{ (f(m) + \frac{f(M)-f(m)}{M-m}(t-m))/g(t) \right\} \qquad \text{in the case } (I)$$

or

$$\alpha_o = \min_{m \le t \le M} \left\{ (f(m) + \frac{f(M)-f(m)}{M-m}(t-m))/g(t) \right\} \qquad \text{in the case } (II).$$

Proof. If we choose α such that $\beta = 0$ in Theorem 4 then we have this theorem. □

Remark 7. *Assume that conditions of Theorem 6 hold and moreover suppose that the additional condition holds: (I_+) $f(m) > 0$, $f(M) > 0$ if $g > 0$ or (II_+) $f(m) < 0$, $f(M) < 0$ if $g < 0$, and additionally let either of the following conditions be valid: (III) g is a strictly concave differentiable function or (IV) g is a strictly convex twice differentiable function.*

In the case (I_+) α_o can be written explicitly $\alpha_o = (\mu_f t_o + \nu_f)/g(t_o)$, where

$$t_o = \begin{cases} M & \text{if } \frac{\mu_f}{\mu_g}\nu_g \ge \nu_f, \\ m & \text{if } \frac{\mu_f}{\mu_g}\nu_g < \nu_f, \end{cases} \qquad \text{in the case } (III),$$

or

$$t_o = \begin{cases} \text{the solution of } \; \mu_f g(t) = (\mu_f t + \nu_f)\, g'(t) & \text{if } f(m)\frac{g'(m)}{g(m)} < \mu_f < f(M)\frac{g'(M)}{g(M)}, \\ M & \text{if } \mu_f \ge f(M)\frac{g'(M)}{g(M)}, \\ m & \text{if } \mu_f \le f(m)\frac{g'(m)}{g(m)}, \end{cases}$$

in the case (IV), where we denote $\mu_f = \frac{f(M)-f(m)}{M-m}$, $\nu_f = \frac{Mf(m)-mf(M)}{M-m}$ and similarly ν_g.

In the case (II_+) we have α_o as above with the opposite conditions.

3 Results for means. In this section we discuss the usual operator order among weighted power means defined by (1).

For the sake of convenience, we denote intervals from (i) to (iv) as in Table 1 (see Figure 1).

(i)	$r \le s,\ s \notin \langle -1,1\rangle,\ r \notin \langle -1,1\rangle$ or $1/2 \le r \le 1 \le s$ or $r \le -1 \le s \le -1/2$,
(ii)	$s \ge 1,\ -1 < r < 1/2,\ r \ne 0$ or $r \le -1,\ -1/2 < s < 1,\ s \ne 0$,
(iii)	$-1 \le -s \le r \le s \le 1,\ r \ne 0$ or $-1 \le r \le s \le r/2 < 0$,
(iv)	$-1/2 \le r/2 < s < -r \le 1,\ s \ne 0$.

Table 1: *Intervals from (i) to (iv)*

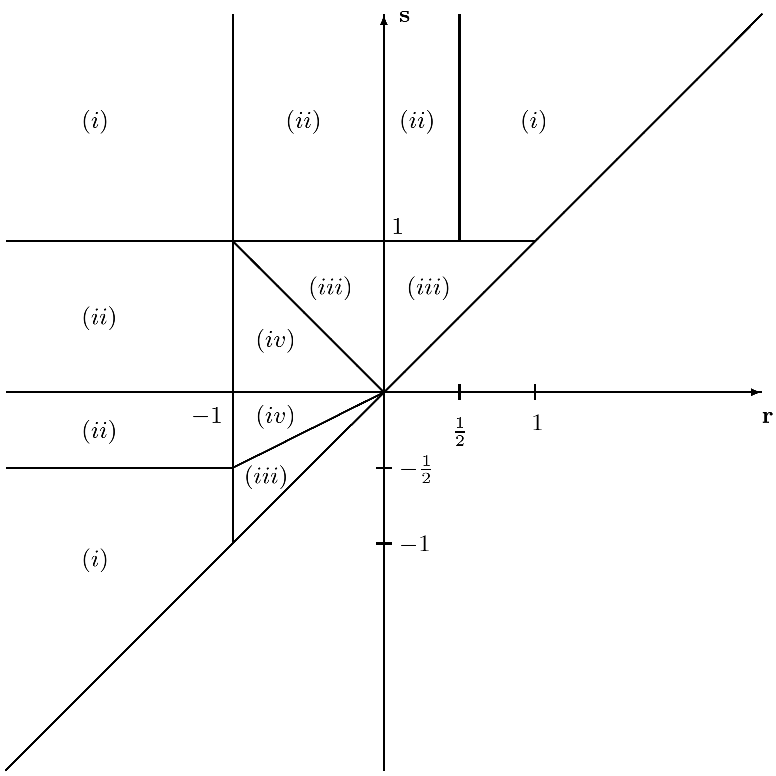

Figure 1

Our first result is given in the next theorem.

Theorem 8. *Let* $\Phi_j \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, $A_j \in \mathcal{B}_+(H)$ *with* $\mathsf{Sp}(A_j) \subseteq [m, M]$ *for some scalars* $0 < m < M$ *and* $\omega_j \in \mathbf{R}_+$ *such that* $\sum_{j=1}^k \omega_j = 1$ $(j = 1, \ldots, k)$.

(i) If $r \le s$, $s \notin \langle -1, 1\rangle$, $r \notin \langle -1, 1\rangle$ *or* $1/2 \le r \le 1 \le s$ *or* $r \le -1 \le s \le -1/2$ *then*

$$0 \le M_k^{[s]}(\mathbf{A}, \Phi, \omega) - M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le \Delta I.$$

(ii) If $s \ge 1$, $-1 < r < 1/2$, $r \ne 0$ *or* $r \le -1$, $-1/2 < s < 1$, $s \ne 0$ *then*

$$\Delta^* I \le M_k^{[s]}(\mathbf{A}, \Phi, \omega) - M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le \Delta I.$$

(iii) If $-1 \le -s \le r \le s \le 1$, $r \ne 0$ *or* $-1 \le r \le s \le r/2 < 0$ *then*

$$-C(m^r, M^r, \frac{1}{r})I \le M_k^{[s]}(\mathbf{A}, \Phi, \omega) - M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le \Delta I + C(m^r, M^r, \frac{1}{r})I.$$

(iv) If $-1/2 \le r/2 < s < -r \le 1$, $s \ne 0$ *then*

$$\Delta^* I - C(m^r, M^r, \frac{1}{r})I \le M_k^{[s]}(\mathbf{A}, \Phi, \omega) - M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le \Delta I + C(m^r, M^r, \frac{1}{r})I,$$

where

$$\Delta = \max_{\theta \in [0,1]} \left\{ [\theta M^s + (1-\theta) m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta) m^r]^{\frac{1}{r}} \right\},$$

$$\Delta^* = \min_{\theta \in [0,1] \cup [\frac{d}{M^r - m^r}, \frac{d}{M^r - m^r} + 1]} \left\{ [\theta M^s + (1-\theta) m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta) m^r - d]^{\frac{1}{r}} \right\},$$

$$d = \frac{M^s m^r - M^r m^s}{M^s - m^s} - \left(1 - \frac{r}{s}\right) \left(\frac{s}{r} \frac{M^r - m^r}{M^s - m^s} \right)^{\frac{r}{r-s}}$$

and $C(m, M, p)$ *is defined by* (3).

In order to prove Theorem 8, we need the following two lemmas.

Lemma 9. *Let* $\Phi_j \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, $A_j \in \mathcal{B}_+(H)$ *with* $\mathsf{Sp}(A_j) \subseteq [m, M]$ *for some scalars* $0 < m < M$ *and* $\omega_j \in \mathbf{R}_+$ *such that* $\sum_{j=1}^k \omega_j = 1$ $(j = 1, \ldots, k)$.

If $0 < p \le 1$ *then*

$$\mu_{t^p} \sum_{j=1}^k \omega_j \Phi_j(A_j) + \nu_{t^p} I \le \sum_{j=1}^k \omega_j \Phi_j(A_j^p) \le (\sum_{j=1}^k \omega_j \Phi_j(A_j))^p, \tag{5}$$

if $-1 \le p < 0$ *or* $1 \le p \le 2$ *then*

$$(\sum_{j=1}^k \omega_j \Phi_j(A_j))^p \le \sum_{j=1}^k \omega_j \Phi_j(A_j^p) \le \mu_{t^p} \sum_{j=1}^k \omega_j \Phi_j(A_j) + \nu_{t^p} I, \tag{6}$$

while if $p < -1$ *or* $p > 2$ *then*

(7)
$$\mu_{t^p} \sum_{j=1}^k \omega_j \Phi_j(A_j) + (1-p) \left(\frac{\mu_{t^p}}{p} \right)^{p/(p-1)} I \le \sum_{j=1}^k \omega_j \Phi_j(A_j^p) \le \mu_{t^p} \sum_{j=1}^k \omega_j \Phi_j(A_j) + \nu_{t^p} I,$$

where $\mu_{t^p} = \frac{M^p - m^p}{M - m}$ *and* $\nu_{t^p} = \frac{m^p M - M^p m}{M - m}$.

Proof. The right hand inequality of (5) and the left hand inequality of (6) follow from Lemma J, because $f(t)=t^p$ is an operator concave function if $0<p\le 1$ and an operator convex function if $-1\le p<0$ or $1\le p\le 2$. The left hand inequality of (5) and the right hand inequalities of (6) and (7) follow from Theorem 4 and Remark 5 if we put $f(t)=t^p$, $g(t)=t$ and $\alpha=\mu_{t^p}$.

Next we prove the left hand inequality of (7). Suppose that $p<-1$ or $p>2$. Since $f(t)=t^p$ is convex, then for each $s\in[m,M]$, $g_s(t)=f(s)+f'(s)(t-s)\le f(t)$ for all $t\in[m,M]$. Then it follows that $\sum_{j=1}^k \omega_j\Phi_j(A_j^p)-\mu_{t^p}\sum_{j=1}^k\omega_j\Phi_j(A_j)\ge\beta_2 I$, where $\beta_2=\max_{g_s\le f}\min_{m\le t\le M}\{g_s(t)-\mu_{t^p}t\}$. The linear function $g_s(t)-\mu_{t^p}t$ attained a minimum at m or M. We choose s which is the unique solution of $g_s(m)-\mu_{t^p}m=g_s(M)-\mu_{t^p}M$. We obtain $s=(\mu_{t^p}/p)^{1/(p-1)}$ and $\beta_2\ge g_s(m)-\mu_{t^p}m=(1-p)s^p=(1-p)\left(\mu_{t^p}/p\right)^{p/(p-1)}$. □

Lemma 10. *Let $\Phi_j\in P_N[\mathcal{B}(H),\mathcal{B}(K)]$, $A_j\in\mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j)\subseteq[m,M]$ for some scalars $0<m<M$ and $\omega_j\in\mathbf{R}_+$ such that $\sum_{j=1}^k\omega_j=1$ $(j=1,\dots,k)$.*

(a) If $1\le r\le s$ or $r\le -1\le s$ then

$$\left[\bar\mu\sum_{j=1}^k\omega_j\Phi_j(A_j^s)+\bar\nu I\right]^{1/r}\le\left(\sum_{j=1}^k\omega_j\Phi_j(A_j^r)\right)^{1/r}$$
$$\le\begin{cases}\left[\bar\mu\sum_{j=1}^k\omega_j\Phi_j(A_j^s)+\left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar\mu\right)^{\frac{r}{r-s}}I\right]^{1/r} & \text{if } -1/2<s<1,\ s\neq 0,\\ \left(\sum_{j=1}^k\omega_j\Phi_j(A_j^s)\right)^{1/s} & \text{otherwise.}\end{cases}$$

(b) If $r\le s\le -1$ or $r\le 1\le s$ then

$$\left[\frac{1}{\bar\mu}\sum_{j=1}^k\omega_j\Phi_j(A_j^r)-\frac{\bar\nu}{\bar\mu}I\right]^{1/s}\ge\left(\sum_{j=1}^k\omega_j\Phi_j(A_j^s)\right)^{1/s}$$
$$\ge\begin{cases}\left[\frac{1}{\bar\mu}\sum_{j=1}^k\omega_j\Phi_j(A_j^r)-\frac{1}{\bar\mu}\left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar\mu\right)^{\frac{r}{r-s}}I\right]^{1/s} & \text{if } -1<r<1/2,\ r\neq 0,\\ \left(\sum_{j=1}^k\omega_j\Phi_j(A_j^r)\right)^{1/r} & \text{otherwise.}\end{cases}$$

(c) If $-1\le -s\le r\le s\le 1$, $r\neq 0$ or $-1\le r\le s\le r/2<0$ then

$$\left[\bar\mu\sum_{j=1}^k\omega_j\Phi_j(A_j^s)+\bar\nu I\right]^{1/r}-C(m^r,M^r,\tfrac{1}{r})I\le\left(\sum_{j=1}^k\omega_j\Phi_j(A_j^r)\right)^{1/r}$$
$$\le\left(\sum_{j=1}^k\omega_j\Phi_j(A_j^s)\right)^{1/s}+C(m^r,M^r,\tfrac{1}{r})I.$$

(d) If $-1/2\le r/2<s<-r\le 1$, $s\neq 0$ then

$$\left[\bar\mu\sum_{j=1}^k\omega_j\Phi_j(A_j^s)+\bar\nu I\right]^{1/r}-C(m^r,M^r,\tfrac{1}{r})I\le\left(\sum_{j=1}^k\omega_j\Phi_j(A_j^r)\right)^{1/r}$$
$$\le\left[\bar\mu\sum_{j=1}^k\omega_j\Phi_j(A_j^s)+\left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar\mu\right)^{\frac{r}{r-s}}I\right]^{1/r}+C(m^r,M^r,\tfrac{1}{r})I,$$

where $\bar\mu=\frac{M^r-m^r}{M^s-m^s}$ and $\bar\nu=\frac{M^sm^r-M^rm^s}{M^s-m^s}$.

Proof. Though the proof in the cases (a) and (b) is quite similar to [5, Corollary 5.8], we give a proof for the sake of completeness. Firstly we prove (a). Let $r\notin\langle -1,1\rangle$. We put $p=\frac{r}{s}$ in (5)-(7) and replace A_j by A_j^s $(j=1,\dots,k)$. Then

$$\left(\sum_{j=1}^k\omega_j\Phi_j(A_j^s)\right)^{r/s}\le\sum_{j=1}^k\omega_j\Phi_j(A_j^r)\le\bar\mu\sum_{j=1}^k\omega_j\Phi_j(A_j^s)+\bar\nu I$$

if $r \le -1$ and ($r \le s \le r/2$ or $-r \le s$),

$$\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\nu^*I \le \sum_{j=1}^{k}\omega_j\Phi_j(A_j^r) \le \bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\bar{\nu}I$$

if $r \le -1$, $r/2 < s < -r$, $s \ne 0$,

$$\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\bar{\nu}I \le \sum_{j=1}^{k}\omega_j\Phi_j(A_j^r) \le (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{r/s}$$

if $1 \le r \le s$, where $\nu^* = \left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar{\mu}\right)^{r/(r-s)}$. Using the fact that the function $f(t)=t^{\frac{1}{r}}$ is an operator increasing if $r \ge 1$ and an operator decreasing if $r \le -1$ (see Remark 2) we have

$$(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{1/s} \ge (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{1/r} \ge [\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\bar{\nu}I]^{1/r}$$

if $r \le -1$ and ($r \le s \le r/2$ or $-r \le s$),

$$[\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\nu^*I]^{1/r} \ge (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{1/r} \ge [\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\bar{\nu}I]^{1/r}$$

if $r \le -1$, $r/2 < s < -r$, $s \ne 0$,

$$[\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\bar{\nu}I]^{1/r} \le (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{1/r} \le (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{1/s}$$

if $1 \le r \le s$. If we put $s=-1$ or $r=-1$ in (2), then we have

$$(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{1/r} \le (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^{-1}))^{-1} \le (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{1/s}$$

for $r \le -1$ and $-1 \le s \le -1/2$. So, we obtain

$$[\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\bar{\nu}I]^{1/r} \le (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{1/r} \le (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{1/s}$$

if $r \le -1 \le s \le -1/2$ or ($r \le -1$, $1 \le s$) or $1 \le r \le s$,

$$[\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\bar{\nu}I]^{1/r} \le (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{1/r} \le [\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\nu^*I]^{1/r}$$

if $r \le -1$, $-1/2 < s < 1$, $s \ne 0$. Therefore we have (a).

Next we prove (b). Let $s \notin \langle -1,1\rangle$. We put $p=\frac{s}{r}$ in (5)-(7) and replace A_j by A_j^r $(j=1,\dots,k)$. Then

$$(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{s/r} \le \sum_{j=1}^{k}\omega_j\Phi_j(A_j^s) \le \tilde{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)+\tilde{\nu}I$$

if $s \geq 1$ and ($s/2 \leq r \leq s$ or $r \leq -s$),

$$\tilde{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)+\tilde{\nu}^*I \leq \sum_{j=1}^{k}\omega_j\Phi_j(A_j^s) \leq \tilde{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)+\tilde{\nu}I$$

if $s \geq 1$, $-s < r < s/2$, $r \neq 0$,

$$\tilde{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)+\tilde{\nu}I \leq \sum_{j=1}^{k}\omega_j\Phi_j(A_j^s) \leq (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{s/r}$$

if $r \leq s \leq -1$, where $\tilde{\mu} = \frac{M^s-m^s}{M^r-m^r} = \frac{1}{\bar{\mu}}$, $\quad \tilde{\nu} = \frac{M^r m^s - M^s m^r}{M^r-m^r} = -\frac{\bar{\nu}}{\bar{\mu}}$, $\quad \tilde{\nu}^* = \left(1-\frac{s}{r}\right)\left(\frac{r}{s}\tilde{\mu}\right)^{\frac{s}{s-r}} = -\frac{\nu^*}{\bar{\mu}}$. By raising above inequalities to the power $\frac{1}{s}$ we obtain

$$(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{1/r} \leq (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{1/s} \leq [\frac{1}{\bar{\mu}}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)-\frac{\bar{\nu}}{\bar{\mu}}I]^{1/s}$$

if $s \geq 1$ and ($s/2 \leq r \leq s$ or $r \leq -s$),

$$[\frac{1}{\bar{\mu}}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)-\frac{\nu^*}{\bar{\mu}}I]^{1/s} \leq (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{1/s} \leq [\frac{1}{\bar{\mu}}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)-\frac{\bar{\nu}}{\bar{\mu}}I]^{1/s}$$

if $s \geq 1$, $-s < r < s/2$, $r \neq 0$,

$$(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{1/r} \leq (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{1/s} \leq [\frac{1}{\bar{\mu}}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)-\frac{\bar{\nu}}{\bar{\mu}}I]^{1/s}$$

if $r < s \leq -1$. If we put $s = 1$ or $r = 1$ in (2), then we have

$$(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{1/r} \leq \sum_{j=1}^{k}\omega_j\Phi_j(A_j) \leq (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{1/s}$$

for $s \geq 1$ and $1/2 \leq r \leq 1$. So, we obtain

$$(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r))^{1/r} \leq (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{1/s} \leq [\frac{1}{\bar{\mu}}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)-\frac{\bar{\nu}}{\bar{\mu}}I]^{1/s}$$

if $1/2 \leq r \leq 1 \leq s$ or ($s \geq 1$, $r \leq -1$) or $r \leq s \leq -1$,

$$[\frac{1}{\bar{\mu}}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)-\frac{\nu^*}{\bar{\mu}}I]^{1/s} \leq (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{1/s} \leq [\frac{1}{\bar{\mu}}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)-\frac{\bar{\nu}}{\bar{\mu}}I]^{1/s}$$

if $s \geq 1$, $-1 < r < 1/2$, $r \neq 0$. Therefore we have (b).

Next we prove (c). If $0 < r \leq s \leq 1$ then $0 < \frac{r}{s} \leq 1$ and by (5) we have

$$\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\bar{\nu}I \leq \sum_{j=1}^{k}\omega_j\Phi_j(A_j^r) \leq (\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s))^{r/s}.$$

Using Corollary 1 for $p = \frac{1}{r}$, and because $m^r I \leq \bar{\mu}\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar{\nu} I \leq M^r I$ and $m^r I \leq \sum_{j=1}^k \omega_j \Phi_j(A_j^r) \leq M^r I$ we obtain

$$\begin{aligned}[\bar{\mu}\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar{\nu} I]^{1/r} - C(m^r, M^r, \frac{1}{r})I &\leq (\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^r))^{1/r} \\ &\leq (\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^s))^{1/s} + C(m^r, M^r, \frac{1}{r})I.\end{aligned}$$

If ($-1 \leq r < 0 < s \leq 1$ and $-1 \leq \frac{r}{s} < 0$) or ($-1 \leq r \leq s < 0$ and $1 \leq \frac{r}{s} \leq 2$) then by (6) we have

$$(\sum_{j=1}^k \omega_j \Phi_j(A_j^s))^{r/s} \leq \sum_{j=1}^k \omega_j \Phi_j(A_j^r) \leq \bar{\mu}\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar{\nu} I.$$

Using Corollary 1 for $p = \frac{1}{r} < -1$, and because $m^r I \leq \bar{\mu}\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar{\nu} I \leq M^r I$ and $m^r I \leq \sum_{j=1}^k \omega_j \Phi_j(A_j^r) \leq M^r I$ we obtain

$$\begin{aligned}(\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^s))^{1/s} + C(M^r, m^r, \frac{1}{r})I &\geq (\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^r))^{1/r} \\ &\geq [\bar{\mu}\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar{\nu} I]^{1/r} - C(M^r, m^r, \frac{1}{r})I.\end{aligned}$$

Since $C(M^r, m^r, \frac{1}{r}) = (\frac{1}{r} - 1)\left(r\mu_{t^{\frac{1}{r}}}\right)^{\frac{1}{r}/(\frac{1}{r}-1)} + \nu_{t^{\frac{1}{r}}} = (\frac{1}{r} - 1)\left(r\frac{m-M}{m^r - M^r}\right)^{\frac{1}{1-r}} + \frac{Mm^r - mM^r}{m^r - M^r} = (\frac{1}{r} - 1)\left(r\frac{M-m}{M^r - m^r}\right)^{\frac{1}{1-r}} + \frac{mM^r - Mm^r}{M^r - m^r} = C(m^r, M^r, \frac{1}{r})$, we have that

$$\begin{aligned}[\bar{\mu}\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar{\nu} I]^{1/r} - C(m^r, M^r, \frac{1}{r})I &\leq (\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^r))^{1/r} \\ &\leq (\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^s))^{1/s} + C(m^r, M^r, \frac{1}{r})I\end{aligned}$$

holds if $-1 \leq -s \leq r \leq s \leq 1$, $r \neq 0$ or $-1 \leq r \leq s \leq r/2 < 0$. Therefore we have (c).

Next we prove (d). If ($-1 \leq r < 0 < s \leq 1$ and $\frac{r}{s} < -1$) or ($-1 \leq r < s < 0$ and $\frac{r}{s} > 2$) then by (7) we obtain

$$\bar{\mu}\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \nu^* I \leq \sum_{j=1}^k \omega_j \Phi_j(A_j^r) \leq \bar{\mu}\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar{\nu} I.$$

Using Corollary 1 for $p = \frac{1}{r}$ we have that

$$\begin{aligned}[\bar{\mu}\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar{\nu} I]^{1/r} - C(m^r, M^r, \frac{1}{r})I &\leq (\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^r))^{1/r} \\ &\leq [\bar{\mu}\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \nu^* I]^{1/r} + C(m^r, M^r, \frac{1}{r})I\end{aligned}$$

holds if $-1/2 \leq r/2 < s < -r \leq 1$, $s \neq 0$. Therefore we have (d). □

Proof of Theorem 8. Though the proof in the cases (i) and (ii) is quite similar to [5, Theorem 5.9], we give a proof for the sake of completeness. It follows from Lemma 10 (a) that

$$\begin{aligned}M_k^{[s]}(\mathbf{A}, \Phi, \omega) - \left[\bar{\mu}\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \nu^* I\right]^{1/r} &\leq M_k^{[s]}(\mathbf{A}, \Phi, \omega) - M_k^{[r]}(\mathbf{A}, \Phi, \omega) \\ &\leq M_k^{[s]}(\mathbf{A}, \Phi, \omega) - \left[\bar{\mu}\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar{\nu} I\right]^{1/r}\end{aligned} \tag{8}$$

holds if $-1/2 < s < 1$, $s \neq 0$, $r \leq -1$ and

$$\begin{aligned} 0 = M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_s^{[s]}(\mathbf{A},\Phi,\omega) &\leq M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) \\ &\leq M_k^{[s]}(\mathbf{A},\Phi,\omega) - \left[\bar\mu \sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar\nu I\right]^{1/r} \end{aligned} \tag{9}$$

holds if $1 \leq r \leq s$ or $(r \leq -1,\ s \geq 1)$ or $r \leq -1 \leq s \leq -1/2$.

It follows from Lemma 10 (b) that

$$\begin{aligned} [\tfrac{1}{\bar\mu} \sum_{j=1}^k \omega_j \Phi_j(A_j^r) - \tfrac{\nu^*}{\bar\mu} I]^{1/s} - M_k^{[r]}(\mathbf{A},\Phi,\omega) &\leq M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) \\ &\leq [\tfrac{1}{\bar\mu} \sum_{j=1}^k \omega_j \Phi_j(A_j^r) - \tfrac{\bar\nu}{\bar\mu} I]^{1/s} - M_k^{[r]}(\mathbf{A},\Phi,\omega) \end{aligned} \tag{10}$$

holds if $-1 < r < 1/2$, $r \neq 0$, $s \geq 1$ and

$$\begin{aligned} 0 = M_k^{[r]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) &\leq M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) \\ &\leq [\tfrac{1}{\bar\mu} \sum_{j=1}^k \omega_j \Phi_j(A_j^r) - \tfrac{\bar\nu}{\bar\mu} I]^{1/s} - M_k^{[r]}(\mathbf{A},\Phi,\omega) \end{aligned} \tag{11}$$

holds if $r \leq s \leq -1$ or $(r \leq -1,\ s \geq 1)$ or $1/2 \leq r \leq 1 \leq s$.

It follows from the right hand inequalities of (8) and (9) that

$$\begin{aligned} M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) &\leq M_k^{[s]}(\mathbf{A},\Phi,\omega) - \left[\bar\mu \sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar\nu I\right]^{1/r} \\ &\leq \max_{t \in \bar T} \left\{ t^{1/s} - [\bar\mu\, t + \bar\nu]^{1/r} \right\} I \end{aligned}$$

holds, where $\bar T$ denotes the close interval joining m^s to M^s. We set $t = \theta M^s + (1-\theta) m^s$ for some $\theta \in [0,1]$. Then we have $\bar\mu \cdot t + \bar\nu = \theta M^r + (1-\theta) m^r$ and hence $\max_{t\in\bar T}\{t^{1/s} - [\bar\mu\, t + \bar\nu]^{1/r}\} = \Delta$. Therefore, we obtain $M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) \leq \Delta I$ if $1 \leq r \leq s$ or $r \leq -1 \leq s$.

It follows from the right hand inequalities of (10) and (11) that

$$\begin{aligned} M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) &\leq [\tfrac{1}{\bar\mu} \sum_{j=1}^k \omega_j \Phi_j(A_j^r) - \tfrac{\bar\nu}{\bar\mu} I]^{1/s} - M_k^{[r]}(\mathbf{A},\Phi,\omega) \\ &\leq \max_{t\in\bar T_1} \left\{ \left[\tfrac{1}{\bar\mu}\, t - \tfrac{\bar\nu}{\bar\mu}\right]^{1/s} - t^{1/r} \right\} I \end{aligned}$$

holds, where $\bar T_1$ denotes the close interval joining m^r to M^r. We set $t = \theta M^r + (1-\theta) m^r$ for some $\theta \in [0,1]$. Then we have $\frac{1}{\bar\mu} \cdot t - \frac{\bar\nu}{\bar\mu} = \theta M^s + (1-\theta) m^s$ and hence $\max_{t\in\bar T_1}\{\left[\frac{1}{\bar\mu}\, t - \frac{\bar\nu}{\bar\mu}\right]^{1/s} - t^{1/r}\} = \Delta$. Therefore, we obtain $M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) \leq \Delta I$ if $r \leq s \leq -1$ and $r \leq 1 \leq s$.

Then we have the right hand inequalities of (i) and (ii) in this theorem.

From the left hand inequalities of (9) and (11) we have the left hand inequality of (i).

From the left hand inequality of (8) we obtain that

$$\begin{aligned} M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) &\geq M_k^{[s]}(\mathbf{A},\Phi,\omega) - \left[\bar\mu \sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \nu^* I\right]^{1/r} \\ &\geq \min_{t\in\bar T} \left\{ t^{1/s} - [\bar\mu\, t + \nu^*]^{1/r} \right\} I = \min_{t\in\bar T} \left\{ t^{1/s} - [\bar\mu\, t + \bar\nu - d]^{1/r} \right\} I \\ &= \min_{\theta\in[0,1]} \left\{ [\theta M^s + (1-\theta) m^s]^{1/s} - [\theta M^r + (1-\theta) m^r - d]^{1/r} \right\} I \geq \Delta^* I \end{aligned} \tag{12}$$

holds if $-1/2 < s < 1$, $s \neq 0$, $r \leq -1$.

From the left hand inequality of (10) we obtain that

$$
\begin{aligned}
M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) &\geq \left[\frac{1}{\bar\mu}\sum_{j=1}^k \omega_j \Phi_j(A_j^r) - \frac{\nu^*}{\bar\mu} I\right]^{1/s} - M_k^{[r]}(\mathbf{A},\Phi,\omega) \\
&\geq \min_{t\in \bar T_1} \left\{ \left[\frac{1}{\bar\mu} t - \frac{\nu^*}{\bar\mu}\right]^{1/s} - t^{1/r} \right\} I \\
&= \min_{\theta\in[0,1]} \left\{ [\theta M^s + (1-\theta) m^s + \tfrac{d}{\bar\mu}]^{1/s} - [\theta M^r + (1-\theta) m^r]^{1/r} \right\} I \\
&= \min_{\theta\in[\frac{d}{M^r-m^r},\frac{d}{M^r-m^r}+1]} \left\{ [\theta M^s + (1-\theta) m^s]^{1/s} - [\theta M^r + (1-\theta) m^r - d]^{1/r} \right\} I \\
&\geq \Delta^* I \qquad (13)
\end{aligned}
$$

holds if $-1 < r < 1/2$, $r \neq 0$, $s \geq 1$.

From inequalities (12) and (13) we have the left hand inequality of (ii) in this theorem. Therefore we have (i) and (ii) in this theorem.

By Lemma 10 (c) we obtain

$$
\begin{aligned}
-C(m^r, M^r, \tfrac{1}{r}) I &\leq M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) \\
&\leq M_k^{[s]}(\mathbf{A},\Phi,\omega) - \left[\bar\mu \sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar\nu I\right]^{1/r} + C(m^r, M^r, \tfrac{1}{r}) I \\
&\leq \max_{t\in\bar T} \left\{ t^{1/s} - [\bar\mu\, t + \bar\nu]^{1/r} \right\} I + C(m^r, M^r, \tfrac{1}{r}) I = \Delta I + C(m^r, M^r, \tfrac{1}{r}) I
\end{aligned}
$$

if $-1 \leq -s \leq r \leq s \leq 1$, $r \neq 0$ or $-1 \leq r \leq s \leq r/2 < 0$. Then we have (iii) in this theorem.

By Lemma 10 (d) we obtain

$$
\begin{aligned}
M_k^{[s]}(\mathbf{A},\Phi,\omega) - \left[\bar\mu \sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \nu^* I\right]^{1/r} - C(m^r, M^r, \tfrac{1}{r}) I \\
\leq M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) \\
\leq M_k^{[s]}(\mathbf{A},\Phi,\omega) - \left[\bar\mu \sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar\nu I\right]^{1/r} + C(m^r, M^r, \tfrac{1}{r}) I
\end{aligned}
$$

if $-1/2 \leq r/2 < s < -r \leq 1$, $s \neq 0$. Then

$$
\begin{aligned}
\Delta^* I - C(m^r, M^r, \tfrac{1}{r}) I &\leq \min_{t\in\bar T} \left\{ t^{1/s} - [\bar\mu\, t + \bar\nu - d]^{1/r} \right\} I - C(m^r, M^r, \tfrac{1}{r}) I \\
&\leq M_k^{[s]}(\mathbf{A},\Phi,\omega) - M_k^{[r]}(\mathbf{A},\Phi,\omega) \\
&\leq \max_{t\in\bar T} \left\{ t^{1/s} - [\bar\mu\, t + \bar\nu]^{1/r} \right\} I + C(m^r, M^r, \tfrac{1}{r}) I = \Delta I + C(m^r, M^r, \tfrac{1}{r}) I
\end{aligned}
$$

and we have (iv) in this theorem. □

Our second result is given in the next theorem.

Theorem 11. *Let $\Phi_j \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$ for some scalars $0 < m < M$ and $\omega_j \in \mathbf{R}_+$ such that $\sum_{j=1}^k \omega_j = 1$ $(j = 1, \ldots, k)$.*

(i) If $r \leq s$, $s \notin \langle -1, 1\rangle$, $r \notin \langle -1, 1\rangle$ or $1/2 \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -1/2$ then

$$\tilde\Delta(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A},\Phi,\omega) \leq M_k^{[r]}(\mathbf{A},\Phi,\omega) \leq M_k^{[s]}(\mathbf{A},\Phi,\omega).$$

(ii) If $s \geq 1$, $-1 < r < 1/2$, $r \neq 0$ or $r \leq -1$, $-1/2 < s < 1$, $s \neq 0$ then

$$\tilde\Delta(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A},\Phi,\omega) \leq M_k^{[r]}(\mathbf{A},\Phi,\omega) \leq \tilde\Delta(\kappa, r, s) M_k^{[s]}(\mathbf{A},\Phi,\omega).$$

(iii) If $-1 \le -s \le r \le s \le 1,\ r \ne 0$ *or* $-1 \le r \le s \le r/2 < 0$ *then*

$$\tilde{\Delta}(\kappa, r, 1)^{-1}\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \le M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le \tilde{\Delta}(\kappa, r, 1) M_k^{[s]}(\mathbf{A}, \Phi, \omega).$$

(iv) If $-1/2 \le r/2 < s < -r \le 1,\ s \ne 0$ *then*

$$\tilde{\Delta}(\kappa, s, 1)^{-1}\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \le M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le \tilde{\Delta}(\kappa, s, 1) M_k^{[s]}(\mathbf{A}, \Phi, \omega),$$

where

$$\tilde{\Delta}(\kappa, r, s) = \left\{ \frac{r(\kappa^s - \kappa^r)}{(s-r)(\kappa^r - 1)} \right\}^{\frac{1}{s}} \left\{ \frac{s(\kappa^r - \kappa^s)}{(r-s)(\kappa^s - 1)} \right\}^{-\frac{1}{r}}, \qquad \kappa = \frac{M}{m}.$$

In order to prove Theorem 11, we need the following lemma.

Lemma 12. *Let* $\Phi_j \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, $A_j \in \mathcal{B}_+(H)$ *with* $\mathsf{Sp}(A_j) \subseteq [m, M]$ *for some scalars* $0 < m < M$ *and* $\omega_j \in \mathbf{R}_+$ *such that* $\sum_{j=1}^k \omega_j = 1$ $(j = 1, \dots, k)$.

(a) If $0 < p \le 1$ *then*

$$K(m, M, p)\Big(\sum_{j=1}^k \omega_j \Phi_j(A_j)\Big)^p \le \sum_{j=1}^k \omega_j \Phi_j(A_j^p) \le \Big(\sum_{j=1}^k \omega_j \Phi_j(A_j)\Big)^p.$$

(b) If $-1 \le p < 0$ *or* $1 \le p \le 2$ *then*

$$\Big(\sum_{j=1}^k \omega_j \Phi_j(A_j)\Big)^p \le \sum_{j=1}^k \omega_j \Phi_j(A_j^p) \le K(m, M, p)\Big(\sum_{j=1}^k \omega_j \Phi_j(A_j)\Big)^p.$$

(c) If $p < -1$ *or* $p > 2$ *then*

$$K(m, M, p)^{-1}\Big(\sum_{j=1}^k \omega_j \Phi_j(A_j)\Big)^p \le \sum_{j=1}^k \omega_j \Phi_j(A_j^p) \le K(m, M, p)\Big(\sum_{j=1}^k \omega_j \Phi_j(A_j)\Big)^p,$$

where $K(m, M, p)$ *is defined by* (4).

Proof. Since $f(t) = t^p$ is an operator concave function for $0 < p \le 1$, then by Lemma J we have the right hand inequality of (a). By converse of Theorem 6 for $f(t) \equiv g(t) = t^p$, $0 < p \le 1$ and Remark 7 we have $\sum_{j=1}^k \omega_j \Phi_j(A_j^p) \ge \alpha_o \left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right)^p$, when $\alpha_o = K(m, M, p)$ and the left hand inequality of (a) follows. Because $f(t) = t^p$ is an operator convex function if $-1 \le p < 0$ or $1 \le p \le 2$ then by Lemma J we have the left hand inequality of (b) and by Theorem 6 for $f(t) \equiv g(t) = t^p$ and Remark 7 we have the right hand inequality of (b). Because $f(t) = t^p$ is a convex function if $p < -1$ or $p > 2$ then by Theorem 6 for $f(t) \equiv g(t) = t^p$ and Remark 7 we have the right hand inequality of (c). Next we prove the left hand inequality of (c). Suppose that $p < -1$ or $p > 2$. Since $f(t) = t^p$ is convex, then for each $s \in [m, M]$, $g_s(t) \equiv f(s) + f'(s)(t - s) \le f(t)$ for all $t \in [m, M]$. Then the following inequality holds (see [5, Remark 4.13]):

$$\sum_{j=1}^k \omega_j \Phi_j(f(A_j)) \ge \alpha_2\, f\left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right) \qquad \text{with} \qquad \alpha_2 = \max_{0 \le g_s \le f} \ \min_{m \le t \le M} \ \frac{g_s(t)}{f(t)}.$$

The function $\frac{g_s(t)}{f(t)}$ is concave and its minimum is attained at m or M. We choose s which is the unique solution of $\frac{g_s(m)}{f(m)} = \frac{g_s(M)}{f(M)}$. Then we obtain $s = \frac{\nu p}{\mu(1-p)}$, where

$\mu = (M^p - m^p)/(M-m)$, $\nu = (Mm^p - mM^p)(M-m)$ and it follows $\alpha_2 \ge s^{p-1}\,\frac{s(1-p)+pm}{m^p} = (\frac{\nu p}{\mu(1-p)})^{p-1}\,\frac{p\nu/\mu+pm}{m^p} = (\frac{\nu p}{\mu(1-p)})^p\,\frac{1-p}{\nu} = K(m,M,p)^{-1}$. □

Proof of Theorem 11. Though the proof in the cases (i) and (ii) is quite similar to [5, Theorem 5.7], we give a proof for the sake of completeness. Suppose that $s \ge 1$ and $r < 1$. We put $p = \frac{s}{r}$. If $r > 0$ then Lemma 12 (b) and (c) gives

$$(\sum_{j=1}^{k} \omega_j \Phi_j(A_j))^{s/r} \le \sum_{j=1}^{k} \omega_j \Phi_j(A_j^{s/r}) \le K(m,M,\frac{s}{r})(\sum_{j=1}^{k} \omega_j \Phi_j(A_j))^{s/r}$$

if $s/2 \le r \le s$ and

$$K(m,M,\frac{s}{r})^{-1}(\sum_{j=1}^{k} \omega_j \Phi_j(A_j))^{s/r} \le \sum_{j=1}^{k} \omega_j \Phi_j(A_j^{s/r}) \le K(m,M,\frac{s}{r})(\sum_{j=1}^{k} \omega_j \Phi_j(A_j))^{s/r}$$

if $0 < r < s/2$. Replacing A_j by A_j^r $(j = 1, \dots, k)$ we have

$$(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r))^{s/r} \le \sum_{j=1}^{k} \omega_j \Phi_j(A_j^{s}) \le K(m^r,M^r,\frac{s}{r})(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r))^{s/r}$$

if $s/2 \le r \le s$ and

$$K(m^r,M^r,\frac{s}{r})^{-1}(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r))^{s/r} \le \sum_{j=1}^{k} \omega_j \Phi_j(A_j^{s}) \le K(m^r,M^r,\frac{s}{r})(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r))^{s/r}$$

if $0 < r < s/2$, where

$$\begin{aligned} K(m^r,M^r,\frac{s}{r}) &= \frac{m^r(M^r)^{\frac{s}{r}} - M^r(m^r)^{\frac{s}{r}}}{(\frac{s}{r}-1)(M^r-m^r)} \left(\frac{(\frac{s}{r}-1)((M^r)^{\frac{s}{r}} - (m^r)^{\frac{s}{r}})}{\frac{s}{r}(m^r(M^r)^{\frac{s}{r}} - M^r(m^r)^{\frac{s}{r}})}\right)^{\frac{s}{r}} \\ &= \frac{r(\kappa^s - \kappa^r)}{(s-r)(\kappa^r-1)} \left(\frac{s(\kappa^r-\kappa^s)}{(r-s)(\kappa^s-1)}\right)^{-\frac{s}{r}}. \end{aligned}$$

By raising above inequalities to the power $1/s$ it follows from the Löwner-Heinz theorem that

$$M_k^{[r]}(\mathbf{A},\Phi,\omega) \le M_k^{[s]}(\mathbf{A},\Phi,\omega) \le K(m^r,M^r,\frac{s}{r})^{1/s} M_k^{[r]}(\mathbf{A},\Phi,\omega)$$

holds if $s/2 \le r \le s$ and

$$K(m^r,M^r,\frac{s}{r})^{-1/s} M_k^{[r]}(\mathbf{A},\Phi,\omega) \le M_k^{[s]}(\mathbf{A},\Phi,\omega) \le K(m^r,M^r,\frac{s}{r})^{1/s} M_k^{[r]}(\mathbf{A},\Phi,\omega)$$

holds if $0 < r < s/2$, where

$$K(m^r,M^r,\frac{s}{r})^{1/s} = \left\{\frac{r(\kappa^s-\kappa^r)}{(s-r)(\kappa^r-1)}\right\}^{1/s} \left\{\frac{s(\kappa^r-\kappa^s)}{(r-s)(\kappa^s-1)}\right\}^{-1/r} = \tilde{\Delta}(\kappa,r,s).$$

If we put $s = 1$ or $r = 1$ in (2), then we have

$$(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r))^{1/r} \le \sum_{j=1}^{k} \omega_j \Phi_j(A_j) \le (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^s))^{1/s}$$

for $1/2 \leq r \leq 1$ and $s \geq 1$. Therefore for $s > 1$ we have

$$M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}(\kappa, r, s) M_k^{[r]}(\mathbf{A}, \Phi, \omega)$$

if $1/2 \leq r \leq 1$ and

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}(\kappa, r, s) M_k^{[r]}(\mathbf{A}, \Phi, \omega)$$

if $0 < r < 1/2$. So, we obtain

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $1/2 \leq r \leq 1$, $s \geq 1$ and

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}(\kappa, r, s) M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $0 < r < 1/2$, $s \geq 1$.

If $r < 0$ then Lemma 12 (b) and (c) with the Löwner-Heinz theorem gives

$$M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq K(M^r, m^r, \frac{s}{r})^{1/s} M_k^{[r]}(\mathbf{A}, \Phi, \omega)$$

if $r \leq -s$ and

$$K(M^r, m^r, \frac{s}{r})^{-1/s} M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq K(M^r, m^r, \frac{s}{r})^{1/s} M_k^{[r]}(\mathbf{A}, \Phi, \omega)$$

if $-s < r < 0$, where

$$\begin{aligned} K(M^r, m^r, \frac{s}{r})^{1/s} = &= \left\{ \frac{r(\kappa^{-s} - \kappa^{-r})}{(s-r)(\kappa^{-r} - 1)} \right\}^{1/s} \left\{ \frac{s(\kappa^{-r} - \kappa^{-s})}{(r-s)(\kappa^{-s} - 1)} \right\}^{-1/r} \\ &= \left\{ \frac{r(\kappa^{s} - \kappa^{r})}{(s-r)(\kappa^{r} - 1)} \right\}^{1/s} \frac{1}{\kappa} \left\{ \frac{s(\kappa^{r} - \kappa^{s})}{(r-s)(\kappa^{s} - 1)} \right\}^{-1/r} \kappa \\ &= \tilde{\Delta}(\kappa, r, s). \end{aligned}$$

Therefore, similarly to above we have

$$M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}(\kappa, r, s) M_k^{[r]}(\mathbf{A}, \Phi, \omega)$$

if $r \leq -1$ and

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}(\kappa, r, s) M_k^{[r]}(\mathbf{A}, \Phi, \omega)$$

if $-1 < r < 0$. So, we obtain

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $r \leq -1$, $s \geq 1$ and

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}(\kappa, r, s) M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $-1 < r < 0$, $s \geq 1$.

Now, suppose that $1 \le r \le s$. We put $p = \frac{r}{s}$. Then Lemma 12 (a) with the Löwner-Heinz theorem gives

$$K(m^s, M^s, \frac{r}{s})^{1/r} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \le M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le M_k^{[s]}(\mathbf{A}, \Phi, \omega).$$

Since $K(m^s, M^s, \frac{r}{s})^{1/r} = \tilde{\Delta}(\kappa, r, s)^{-1}$, we obtain

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \le M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $1 \le r \le s$.

Therefore, we have desired results in the cases (i) and (ii) for $s \ge 1$.

Next we prove desired results in the cases (i) and (ii) for $r \le -1$.

If $-1 < s < 1$ we put $p = \frac{r}{s}$. If $0 < s < 1$ then Lemma 12 (c) gives

$$K(m^s, M^s, \frac{r}{s})^{-1} (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^s))^{r/s} \le (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r)) \le K(m^s, M^s, \frac{r}{s}) (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^s))^{r/s}.$$

Since the function $f(t) = t^{1/r}$ is an operator decreasing for $r \le -1$, then we obtain

$$K(m^s, M^s, \frac{r}{s})^{-1/r} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \ge M_k^{[r]}(\mathbf{A}, \Phi, \omega) \ge K(m^s, M^s, \frac{r}{s})^{1/r} M_k^{[s]}(\mathbf{A}, \Phi, \omega),$$

so we have

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \le M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le \tilde{\Delta}(\kappa, r, s) M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $0 < s < 1$, $r \le -1$.

If $-1 < s < 0$ then Lemma 12 (b) and (c), with the fact that the function $f(t) = t^{1/r}$ is an operator decreasing for $r \le -1$, gives

$$M_k^{[s]}(\mathbf{A}, \Phi, \omega) \ge M_k^{[r]}(\mathbf{A}, \Phi, \omega) \ge K(M^s, m^s, \frac{r}{s})^{1/r} M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $-1 < s \le r/2$ and

$$K(M^s, m^s, \frac{r}{s})^{-1/r} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \ge M_k^{[r]}(\mathbf{A}, \Phi, \omega) \ge K(M^s, m^s, \frac{r}{s})^{1/r} M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $r/2 < s < 0$, where $K(M^s, m^s, \frac{r}{s})^{1/r} = \tilde{\Delta}(\kappa, r, s)^{-1}$. If we put $s = -1$ or $r = -1$ in (2), then we have

$$(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r))^{1/r} \le (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^{-1}))^{-1} \le (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^s))^{1/s}$$

for $r \le -1$ and $-1 < s \le -1/2$, so we have

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \le M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $-1 < s \le -1/2$, $r \le -1$ and

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \le M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le \tilde{\Delta}(\kappa, r, s) M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $-1/2 < s < 0$, $r \le -1$.

If $r \leq s \leq -1$ then we put $p = \frac{s}{r}$. Lemma 12 (a), with the fact that the function $f(t) = t^{1/s}$ is an operator decreasing for $s \leq -1$, gives

$$K(M^r, m^r, \frac{s}{r})^{1/s} M_k^{[r]}(\mathbf{A}, \Phi, \omega) \geq M_k^{[s]}(\mathbf{A}, \Phi, \omega) \geq M_k^{[r]}(\mathbf{A}, \Phi, \omega),$$

so we have

$$\tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $r \leq s \leq -1$.

We have the desired results in the cases (i) and (ii) for $r \leq -1$.

(iii) If $0 < r \leq s \leq 1$ then $0 < \frac{r}{s} \leq 1$. If we put $p = \frac{r}{s}$ in Lemma 12 (a) and replace A_j by A_j^s $(j = 1, \ldots, k)$, then we obtain

$$K(m^s, M^s, \frac{r}{s}) (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^s))^{r/s} \leq \sum_{j=1}^{k} \omega_j \Phi_j(A_j^r) \leq (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^s))^{r/s}.$$

By raising above inequality to the power $1/r (\geq 1)$ it follows from Corollary 3 that

$$\begin{aligned} K(m^r, M^r, \tfrac{1}{r})^{-1} K(m^s, M^s, \tfrac{r}{s})^{1/r} M_k^{[s]}(\mathbf{A}, \Phi, \omega) &\leq M_k^{[r]}(\mathbf{A}, \Phi, \omega) \\ &\leq K(m^r, M^r, \tfrac{1}{r}) M_k^{[s]}(\mathbf{A}, \Phi, \omega). \end{aligned}$$

So, we have

$$\tilde{\Delta}(\kappa, r, 1)^{-1} \tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}(\kappa, r, 1) M_k^{[s]}(\mathbf{A}, \Phi, \omega).$$

If $-1 \leq -s \leq r < 0$ then $-1 \leq \frac{r}{s} < 0$, but if $-1 \leq r \leq s \leq r/2 < 0$ then $1 \leq \frac{r}{s} \leq 2$. If we put $p = \frac{r}{s}$ in Lemma 12 (b) and replace A_j by A_j^s $(j = 1, \ldots, k)$, then we obtain

$$(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^s))^{r/s} \leq \sum_{j=1}^{k} \omega_j \Phi_j(A_j^r) \leq K(m^s, M^s, \frac{r}{s}) (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^s))^{r/s}$$

if $-1 \leq -s \leq r < 0$ and

$$(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^s))^{r/s} \leq \sum_{j=1}^{k} \omega_j \Phi_j(A_j^r) \leq K(M^s, m^s, \frac{r}{s}) (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^s))^{r/s}$$

if $-1 \leq r \leq s \leq r/2 < 0$. Using that $K(M^s, m^s, \frac{r}{s}) = K(m^s, M^s, \frac{r}{s})$ and by raising above inequalities to the power $1/r$ then it follows from Corollary 3 that we obtain

$$\begin{aligned} K(m^r, M^r, \tfrac{1}{r}) M_k^{[s]}(\mathbf{A}, \Phi, \omega) &\geq M_k^{[r]}(\mathbf{A}, \Phi, \omega) \\ &\geq K(m^r, M^r, \tfrac{1}{r})^{-1} K(m^s, M^s, \tfrac{r}{s})^{1/r} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \end{aligned}$$

if $-1 \leq -s \leq r < 0$ or $-1 \leq r \leq s \leq r/2 < 0$. So, we have

$$\tilde{\Delta}(\kappa, r, 1)^{-1} \tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \leq M_k^{[r]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}(\kappa, r, 1) M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $-1 \leq -s \leq r \leq s \leq 1$, $r \neq 0$ or $-1 \leq r < s \leq r/2 < 0$.

(iv) Next, let $-1 \le r < -s < 0$ or $-1/2 \le r/2 < s < 0$. Then $-1 < \frac{s}{r} < 0$ or $0 < \frac{s}{r} < \frac{1}{2}$. If we put $p = \frac{s}{r}$ in Lemma 12 (b) and (a) and replace A_j by A_j^r $(j = 1, \dots, k)$, then we obtain

$$(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r))^{s/r} \le \sum_{j=1}^{k} \omega_j \Phi_j(A_j^s) \le K(M^r, m^r, \frac{s}{r})(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r))^{s/r}$$

if $-1 \le r < -s < 0$ and

$$K(M^r, m^r, \frac{s}{r})(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r))^{s/r} \le \sum_{j=1}^{k} \omega_j \Phi_j(A_j^s) \le (\sum_{j=1}^{k} \omega_j \Phi_j(A_j^r))^{s/r}$$

if $-1/2 \le r/2 < s < 0$. By raising above inequalities to the power $1/s$ it follows from Corollary 3 that

$$\begin{aligned} &K(m^s, M^s, \tfrac{1}{s})^{-1} M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le M_k^{[s]}(\mathbf{A}, \Phi, \omega) \\ &\le K(m^s, M^s, \tfrac{1}{s}) K(M^r, m^r, \tfrac{s}{r})^{1/s} M_k^{[r]}(\mathbf{A}, \Phi, \omega) \end{aligned}$$

if $-1 \le r < -s < 0$ and

$$\begin{aligned} K(M^s, m^s, \tfrac{1}{s}) K(M^r, m^r, \tfrac{s}{r})^{1/s} M_k^{[r]}(\mathbf{A}, \Phi, \omega) &\ge M_k^{[s]}(\mathbf{A}, \Phi, \omega) \\ &\ge K(M^s, m^s, \tfrac{1}{s})^{-1} M_k^{[r]}(\mathbf{A}, \Phi, \omega) \end{aligned}$$

if $-1/2 \le r/2 < s < 0$. Since $K(M^s, m^s, \frac{1}{s}) = K(m^s, M^s, \frac{1}{s}) = \tilde{\Delta}(\kappa, 1, s)^{-1} = \tilde{\Delta}(\kappa, s, 1)$ we have

$$\tilde{\Delta}(\kappa, s, 1)^{-1} M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le M_k^{[s]}(\mathbf{A}, \Phi, \omega) \le \tilde{\Delta}(\kappa, s, 1) \tilde{\Delta}(\kappa, r, s) M_k^{[r]}(\mathbf{A}, \Phi, \omega)$$

if $-1/2 \le r/2 < s < -r \le 1$, $s \ne 0$. So, we have

$$\tilde{\Delta}(\kappa, s, 1)^{-1} \tilde{\Delta}(\kappa, r, s)^{-1} M_k^{[s]}(\mathbf{A}, \Phi, \omega) \le M_k^{[r]}(\mathbf{A}, \Phi, \omega) \le \tilde{\Delta}(\kappa, s, 1) M_k^{[s]}(\mathbf{A}, \Phi, \omega)$$

if $-1/2 \le r/2 < s < -r \le 1$, $s \ne 0$. □

4 Applications.

In this section, we show applications of Theorems 8 and 11 for some special maps. Firstly, we state the following two corollaries obtained by applying $\Phi_j = I$ $(j = 1, \dots, k)$ to this theorem. This is an extension of our results given in [7, Theorem 1].

Corollary 13. *Let $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$ $(j = 1, \dots, k)$ and $\omega_j \in \mathbf{R}_+$ such that $\sum_{j=1}^{k} \omega_j = 1$.*

(i) If $r \le s$, $s \notin \langle -1, 1\rangle$, $r \notin \langle -1, 1\rangle$ or $1/2 \le r \le 1 \le s$ or $r \le -1 \le s \le -1/2$ then

$$0 \le (\sum_{j=1}^{k} \omega_j A_j^s)^{1/s} - (\sum_{j=1}^{k} \omega_j A_j^r)^{1/r} \le \Delta I.$$

(ii) If $s \ge 1$, $-1 < r < 1/2$, $r \ne 0$ or $r \le -1$, $-1/2 < s < 1$, $s \ne 0$ then

$$\Delta^* I \le (\sum_{j=1}^{k} \omega_j A_j^s)^{1/s} - (\sum_{j=1}^{k} \omega_j A_j^r)^{1/r} \le \Delta I.$$

(iii) If $-1 \le -s \le r \le s \le 1,\ r \ne 0$ *or* $-1 \le r \le s \le r/2 < 0$ *then*

$$-CI \le (\sum_{j=1}^{k} \omega_j A_j^s)^{1/s} - (\sum_{j=1}^{k} \omega_j A_j^r)^{1/r} \le (\Delta + C)I.$$

(iv) If $-1/2 \le r/2 < s < -r \le 1,\ s \ne 0$ *then*

$$(\Delta^* - C)I \le (\sum_{j=1}^{k} \omega_j A_j^s)^{1/s} - (\sum_{j=1}^{k} \omega_j A_j^r)^{1/r} \le (\Delta + C)I,$$

where

$$\Delta = \max_{\theta\in[0,1]} \left\{ [\theta M^s + (1-\theta)m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta)m^r]^{\frac{1}{r}} \right\},$$

$$\Delta^* = \min_{\theta\in[0,1]\cup[\frac{d}{M^r-m^r},\frac{d}{M^r-m^r}+1]} \left\{ [\theta M^s + (1-\theta)m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta)m^r - d]^{\frac{1}{r}} \right\},$$

$$d = \frac{M^s m^r - M^r m^s}{M^s - m^s} - \left(1 - \frac{r}{s}\right)\left(\frac{s}{r}\frac{M^r - m^r}{M^s - m^s}\right)^{\frac{r}{r-s}},$$

$$C = \left(\frac{1}{r} - 1\right)\left(r\frac{M - m}{M^r - m^r}\right)^{\frac{1}{1-r}} + \frac{M^r m - m^r M}{M^r - m^r}.$$

Corollary 14. *Let* $A_j \in \mathcal{B}_+(H)$ *with* $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$ *(*$j = 1, \ldots, k$*) and* $\omega_j \in \mathbf{R}_+$ *such that* $\sum_{j=1}^{k} \omega_j = 1$.

(i) If $r \le s,\ s \notin \langle -1, 1\rangle,\ r \notin \langle -1, 1\rangle$ *or* $1/2 \le r \le 1 \le s$ *or* $r \le -1 \le s \le -1/2$ *then*

$$\tilde{\Delta}(\kappa, r, s)^{-1}(\sum_{j=1}^{k} \omega_j A_j^s)^{1/s} \le (\sum_{j=1}^{k} \omega_j A_j^r)^{1/r} \le (\sum_{j=1}^{k} \omega_j A_j^s)^{1/s}.$$

(ii) If $s \ge 1,\ -1 < r < 1/2,\ r \ne 0$ *or* $r \le -1,\ -1/2 < s < 1,\ s \ne 0$ *then*

$$\tilde{\Delta}(\kappa, r, s)^{-1}(\sum_{j=1}^{k} \omega_j A_j^s)^{1/s} \le (\sum_{j=1}^{k} \omega_j A_j^r)^{1/r} \le \tilde{\Delta}(\kappa, r, s)(\sum_{j=1}^{k} \omega_j A_j^s)^{1/s}.$$

(iii) If $-1 \le -s \le r \le s \le 1,\ r \ne 0$ *or* $-1 \le r \le s \le r/2 < 0$ *then*

$$\tilde{\Delta}(\kappa, r, 1)^{-1}\tilde{\Delta}(\kappa, r, s)^{-1}(\sum_{j=1}^{k} \omega_j A_j^s)^{1/s} \le (\sum_{j=1}^{k} \omega_j A_j^r)^{1/r} \le \tilde{\Delta}(\kappa, r, 1)(\sum_{j=1}^{k} \omega_j A_j^s)^{1/s}.$$

(iv) If $-1/2 \le r/2 < s < -r \le 1,\ s \ne 0$ *then*

$$\tilde{\Delta}(\kappa, s, 1)^{-1}\tilde{\Delta}(\kappa, r, s)^{-1}(\sum_{j=1}^{k} \omega_j A_j^s)^{1/s} \le (\sum_{j=1}^{k} \omega_j A_j^r)^{1/r} \le \tilde{\Delta}(\kappa, s, 1)(\sum_{j=1}^{k} \omega_j A_j^s)^{1/s},$$

where

$$\tilde{\Delta}(\kappa, r, s)\left\{\frac{r(\kappa^s - \kappa^r)}{(s-r)(\kappa^r - 1)}\right\}^{\frac{1}{s}} \left\{\frac{s(\kappa^r - \kappa^s)}{(r-s)(\kappa^s - 1)}\right\}^{-\frac{1}{r}}, \qquad \kappa = \frac{M}{m}.$$

Next we can state two results obtained by applying $\Phi_j(A_j) = (A_j x_j, x_j)/(x_j, x_j)$ $(j = 1, \dots, k)$ to Theorems 8 and 11. Because in this case $\Phi_j(A_j) \in \mathbf{R}$, then in statements where we demanded functions be operator convex or monotone, now we request some weaker conditions, i.e. we demand (real) convex or monotone functions. Hence we obtain better constants as follows.

Corollary 15. *Let $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$ $(j = 1, \dots, k)$ and $x_j \in H$ such that $\sum_{j=1}^k \|x_j\|^2 = 1$. Then for $r \le s$, $r, s \neq 0$*

$$0 \le (\sum_{j=1}^k (A_j^s x_j, x_j))^{1/s} - (\sum_{j=1}^k (A_j^r x_j, x_j))^{1/r} \le \Delta I,$$

where $\Delta = \max_{\theta \in [0,1]} \left\{ [\theta M^s + (1-\theta) m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta) m^r]^{\frac{1}{r}} \right\}$.

Corollary 16. *Let $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$ $(j = 1, \dots, k)$ and $x_j \in H$ such that $\sum_{j=1}^k \|x_j\|^2 = 1$. . Then for $r \le s$, $r, s \neq 0$*

$$(\sum_{j=1}^k (A_j^s x_j, x_j))^{1/s} \le (\sum_{j=1}^k (A_j^r x_j, x_j))^{1/r} \le \tilde{\Delta}(\kappa, r, s)(\sum_{j=1}^k (A_j^s x_j, x_j))^{1/s},$$

where $\tilde{\Delta}(\kappa, r, s) \left\{ \frac{r(\kappa^s - \kappa^r)}{(s-r)(\kappa^r - 1)} \right\}^{\frac{1}{s}} \left\{ \frac{s(\kappa^r - \kappa^s)}{(r-s)(\kappa^s - 1)} \right\}^{-\frac{1}{r}}$, $\kappa = \frac{M}{m}$.

Finally we state the following two corollaries obtained by putting $k = 1$ in Theorems 8 and 11. Then we apply these theorems to the following maps: $\Phi \in \mathbf{P}_N[\mathcal{B}(H), \mathcal{B}(H)]$, $\Phi(A) = X^* A X$, where $A \in \mathcal{B}_h(H)$ and $X \in \mathcal{B}(H)$ is unitary operator. Next we need to generalize all inequalities by sums.

Corollary 17. *Let $A_j \in \mathcal{B}_+(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$ $(j = 1, \dots, k)$ and $X_j \in \mathcal{B}(H)$ be contractions such that $\sum_{j=1}^k X_j^* X_j = I$.*

(i) If $r \le s$, $s \notin \langle -1, 1 \rangle$, $r \notin \langle -1, 1 \rangle$ or $1/2 \le r \le 1 \le s$ or $r \le -1 \le s \le -1/2$ then

$$0 \le (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s} - (\sum_{j=1}^k X_j^* A_j^r X_j)^{1/r} \le \Delta I.$$

(ii) If $s \ge 1$, $-1 < r < 1/2$, $r \neq 0$ or $r \le -1$, $-1/2 < s < 1$, $s \neq 0$ then

$$\Delta^* I \le (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s} - (\sum_{j=1}^k X_j^* A_j^r X_j)^{1/r} \le \Delta I.$$

(iii) If $-1 \le -s \le r \le s \le 1$, $r \neq 0$ or $-1 \le r \le s \le r/2 < 0$ then

$$-CI \le (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s} - (\sum_{j=1}^k X_j^* A_j^r X_j)^{1/r} \le (\Delta I + C)I.$$

(iv) If $-1/2 \le r/2 < s < -r \le 1$, $s \neq 0$ then

$$(\Delta^* - C)I \le (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s} - (\sum_{j=1}^k X_j^* A_j^r X_j)^{1/r} \le (\Delta + C)I,$$

where Δ, Δ^*, d *and* C *are defined as in Corollary 13.*

Corollary 18. *Let* $A_j \in \mathcal{B}_+(H)$ *with* $\mathsf{Sp}(A_j) \subseteq [m, M]$, $0 < m < M$ $(j = 1, \ldots, k)$ *and* $X_j \in \mathcal{B}(H)$ *be contractions such that* $\sum_{j=1}^k X_j^* X_j = I$.

(*i*) *If* $r \le s$, $s \notin \langle -1, 1\rangle$, $r \notin \langle -1, 1\rangle$ *or* $1/2 \le r \le 1 \le s$ *or* $r \le -1 \le s \le -1/2$ *then*

$$\tilde{\Delta}(\kappa, r, s)^{-1} (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s} \le (\sum_{j=1}^k X_j^* A_j^r X_j)^{1/r} \le (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s}.$$

(*ii*) *If* $s \ge 1$, $-1 < r < 1/2$, $r \ne 0$ *or* $r \le -1$, $-1/2 < s < 1$, $s \ne 0$ *then*

$$\tilde{\Delta}(\kappa, r, s)^{-1} (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s} \le (\sum_{j=1}^k X_j^* A_j^r X_j)^{1/r} \le \tilde{\Delta}(\kappa, r, s) (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s}.$$

(*iii*) *If* $-1 \le -s \le r \le s \le 1$, $r \ne 0$ *or* $-1 \le r \le s \le r/2 < 0$ *then*

$$\tilde{\Delta}(\kappa, r, 1)^{-1} \tilde{\Delta}(\kappa, r, s)^{-1} (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s} \le (\sum_{j=1}^k X_j^* A_j^r X_j)^{1/r} \le \tilde{\Delta}(\kappa, r, 1) (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s}.$$

(*iv*) *If* $-1/2 \le r/2 < s < -r \le 1$, $s \ne 0$ *then*

$$\tilde{\Delta}(\kappa, s, 1)^{-1} \tilde{\Delta}(\kappa, r, s)^{-1} (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s} \le (\sum_{j=1}^k X_j^* A_j^r X_j)^{1/r} \le \tilde{\Delta}(\kappa, s, 1) (\sum_{j=1}^k X_j^* A_j^s X_j)^{1/s},$$

where $\tilde{\Delta}(\kappa, r, s)$ *is defined as in Corollary 14.*

References

[1] M.D.Choi, *A Schwarz inequality for positive linear maps on C*-algebras*, Illinois J. Math., **18** (1974), 565–574.

[2] C.Davis, *A Schwarz inequality for convex operator functions*, Proc. Amer. Math. Soc., **8** (1957), 42–44.

[3] T.Furuta, *Operator inequalities assocaited with Holder-McCarthy and Kantorovich inequalities*, J. Inequal. Appl., **2** (1998), 137–148.

[4] J.Mićić, J.E.Pečarić and Y.Seo, *Function order of positive operators based on the Mond-Pečarić method*, Linear Alg. Appl., **360** (2003), 15–34.

[5] J. Mićić, J. Pečarić, Y. Seo, M. Tominaga, *Inequalities for positive linear maps on Hermitian matrices*, Math. Inequal. Appl. **3** (2000), 559–591.

[6] B. Mond, J. E. Pečarić, *Converses of Jensen's inequality for several operators*, Revue d'analyse numer. et de théorie de l'approxim. **23** (1994) 2, 179–183.

[7] J.Pečarić and J.Mićić, *Chaotic order among means of positive operators*, Sci. Math. Japon., **57** (2003), 139–148, :e7, 97–106.

[8] J.Pečarić and J.Mićić, *Some function reversing the order of positive operators*, manuscript

[9] T.Yamazaki, *An extension of Specht's theorem via Kantorovich inequality and related results*, Math. Inequal. Appl., **3** (2000), 89–96.

Jadranka Mićić
Electrical Engineering Department
Polytechnic of Zagreb
Konavoska 2, 10000 Zagreb, Croatia
e-mail: jmicic@public.srce.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb, Croatia
e-mail: pecaric@hazu.hr