EQUILIBRIUM IN TWO-PLAYER GAMES OF SHOWCASE SHOWDOWN

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ABSTRACT. There are games widely played in the routine world of gambles, roulette, quiz show and the sports excercises. The object of the game is to get the highest score among all of the players in the game, from one or two chances of sampling. The games of "Showcase Showdown" and "Risky Exchange" are investigated as continuous games on the unit square, and the optimal strategies for the two players and the winning probabilities they can obtain in the optimal play are derived.

1 The Game "Showcase Showdown". Consider the two players I and II (sometimes they are denoted by 1 and 2, respectively). Let $X_{ij}(i, j = 1, 2)$ be the random variable (r.v.) observed by player i at the j-th observation. We assume that X_{ij} s are *i.i.d.*, each with uniform distribution on $[0, 1]$. Player I (II) chooses his number $a(b)$, where $(a, b) \in [0, 1]^2$. Choices are made simultaneously and independently of the rival's choice. I (II) observes his first r.v. $X_{11}(X_{21})$, and

$$\text{I accepts (rejects) } X_{11} = x, \text{ if } x > (<) a;$$

$$\text{II accepts (rejects) } X_{21} = y, \text{ if } y > (<) b;$$

Define the *score* for player $i, i = 1, 2,$by

$$S_i(X_{i1}, X_{i2}) = \begin{cases} X_{i1} \\ (X_{i1} + X_{i2})I(X_{i1} + X_{i2} \le 1), \end{cases} \quad \text{if } X_{i1} \text{ is} \begin{cases} \text{accepted} \\ \text{rejected} \end{cases} \text{ by player } i \tag{1.1}$$

(*c.f.*, $I(e)$ is the indicator of the event e), which means that if X_{i1} is rejected, the second r.v. X_{i2} is resampled. A player with the higher score than his (or her) opponent is the *winner*. Player I (II) aims to choose his $a(b)$ which maximizes the probability of winning.

We call the above-mentioned game the "Showcase Showdown", which name comes from Ref.[1]. There are other games of the similar nature with the score functions different from (1.1). Consider the scores

$$S_i(X_{i1}, X_{i2}) = \begin{cases} X_{i1}, \\ \varphi(X_{i1}, X_{i2}) \end{cases}, \text{ if } X_{i1} \text{ is} \begin{cases} \text{accepted,} \\ \text{rejected} \end{cases} \text{ by player } i \tag{1.2}$$

where

$$\varphi(X_{i1}, X_{i2}) = X_{i2}, \tag{1.3}$$

$$\varphi(X_{i1}, X_{i2}) = \frac{1}{2}(X_{i1} + X_{i2}), \tag{1.4}$$

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and

$$\varphi(X_{i1}, X_{i2}) = X_{i2} I(X_{i1} \le X_{i2}). \tag{1.5}$$

We call the games with the scores (1.2)-(1.3), (1.2)-(1.4), and (1.2)-(1.5) "Keep-or-Exchange", "Competing Average" and "Risky Exchange", respectively.

The game "Keep-or-Exchange" is solved in Ref.[4;Theorem2]. The solution is given by the famous golden bisection number. The constant-sum game (*c.f.*, the sum is unity, since draw with positive probability doesn't occur) on the unit square $(a, b) \in [0, 1]^2$ has the unique saddle point (g, g) and the saddle value $\frac{1}{2}$, where $g \equiv \frac{1}{2}(\sqrt{5} - 1) \approx 0.61803$.

Here arises a natural question. Do there exist interesting threshould number like g, in the other game "Showcase Shoedown", "Competing Average" and "Risky Exchange"? The object of the present paper is to answer this question.

The answer is affirmatively given for the "Showcase Showdown" and the "Risky Exchange" by Theorem 1 in Section 2 and by Theorem 2 in Section 3, respectively. We find that the solution is given by the number $a^* \approx 0.54368$, which is a unique root in $[0, 1]$ of the cubic equation $1 = a + a^2 + a^3$. Since it is $1 = g + g^2$, in the "Keep-or-Exchange", similarity is striking. For the "Competing Average" the answer is not yet given.

Some remarks around this sort of problems are mentioned in Section 4.

2 Solution to the Game "Showcase Showdown".

Let the score be given by (1.1). Let $W_i, i = 0, 1, 2$ be the event that player i wins. Player zero means "nobody", that is, W_0 means the event that both players get zero scores. Denote by $M_i(a, b)$, the winning probability for player i, when I and II choose a and b, respectively. Evidently we have $\sum_{i=0,1,2} M_i(a, b) = 1, \ \forall (a, b) \in [0, 1]^2$,

$$\begin{aligned} M_0(a, b) &= P\{X_{11} < a, X_{21} < b, X_{11} + X_{12} > 1, X_{21} + X_{22} > 1\} \\ &= \frac{1}{2}a^2 \cdot \frac{1}{2}b^2 = \frac{1}{4}a^2 b^2 \end{aligned} \tag{2.1}$$

and, by symmetry,

$$M_1(a, a) = M_2(a, a) = \frac{1}{2}\left(1 - \frac{1}{4}a^4\right), \ \forall (a, b) \in [0, 1]^2. \tag{2.2}$$

Let p_{AA}, p_{AR}, *etc.*, denote the winning probability for I when the players' choice pair is A-A, A-R, *etc*,. Then clearly $M_1(a, b) = p_{AA} + p_{AR} + p_{RA} + p_{RR}$.

For the subsequent equations (2.3)∼(2.6), we denote $I(a < b), I(a = b)$ and $I(a > b)$ by ξ, η and ζ, respectively. Then we find that

$$\begin{aligned} p_{AA} &= P\{X_{11} > a, X_{21} > b, X_{11} > X_{21}\} \\ &= \xi \cdot \frac{1}{2}\bar{b}^2 + \eta \cdot \frac{1}{2}\bar{a}^2 + \zeta \cdot \frac{1}{2}(1 - a^2 - 2\bar{a}b), \end{aligned} \tag{2.3}$$

$$\begin{aligned} p_{AR} &= P[X_{11} > a, X_{21} < b, \{X_{21} + X_{22} > 1\} \cup \{X_{11} > X_{21} + X_{22}\}] \\ &= \frac{1}{2}\bar{a}b^2 + \iint_{s_1 > a, s_2 < b, s_1 > s_2} (s_1 - s_2) ds_1 ds_2 \\ &\qquad \text{(by denoting } s_1 = X_{11} \text{ and } s_2 = X_{21}\text{)} \\ &= \xi \cdot \left\{\frac{1}{2}(b - ab^2) + \frac{1}{6}(b^3 - a^3)\right\} + \eta \cdot \frac{1}{2}(a - a^3) + \zeta \cdot \frac{1}{2}(1 - a^2)b, \end{aligned} \tag{2.4}$$

since the double integral above is equal to

$$\xi\cdot\left\{\frac{1}{2}(b-b^2)+\frac{1}{6}(b^3-a^3)\right\}+\eta\cdot\frac{1}{2}(a-a^2)+\zeta\cdot\frac{1}{2}\left\{(1-a^2)b-(1-a)b^2\right\};$$

(2.5)
$$\begin{aligned} p_{RA} &= P\{X_{11}<a,\ \ b<X_{21}<X_{11}+X_{12}\le 1\} \\ &= \iint_{s_1<a,b<s_1+s_2<1}(s_1+s_2-b)ds_1ds_2 \\ &\qquad\qquad \text{(by denoting } s_1=X_{11} \text{ and } s_2=X_{12}) \\ &= \xi\cdot\frac{1}{2}a\bar{b}^2+\eta\cdot\frac{1}{2}a\bar{a}^2+\zeta\cdot\left[\frac{1}{2}\{(a^2-2a)b+a\}-\frac{1}{6}(a^3-b^3)\right]; \end{aligned}$$

and

(2.6)
$$\begin{aligned} p_{RR} &= P\{X_{11}<a,X_{11}+X_{12}\le 1\}\cdot P\{X_{21}<b,X_{21}+X_{22}>1\} \\ &\qquad +P\{X_{11}<a,X_{21}<b,1\ge X_{11}+X_{12}>X_{21}+X_{22}\} \\ &= \left(a-\frac{1}{2}a^2\right)\frac{1}{2}b^2+\iint_{1\ge s_1>s_2\ge 0}(s_1\wedge a)(s_2\wedge b)ds_1ds_2 \\ &= \frac{1}{4}(2a-a^2)b^2+I(a\le b)\left\{-\frac{1}{24}a^4+\frac{1}{6}ab(b^2-3b+3)\right\} \\ &\qquad +I(a>b)\left\{\frac{1}{24}b^4-\frac{1}{12}ab(2a^2-3ab-6\bar{b})\right\}. \end{aligned}$$

The double integral in the second expression in (2.6) is derived in this form since

$$\frac{d}{ds_1}P\{X_{11}+X_{12}\le s_1,X_{11}<a\}=\begin{cases}\dfrac{d}{ds_1}\dfrac{1}{2}s_1^2=s_1, & \text{if } s_1<a \\[2ex] \dfrac{d}{ds_1}\left\{\dfrac{1}{2}s_1^2-\dfrac{1}{2}(s_1-a)^2\right\}=a, & \text{if } s_1>a\end{cases}$$

and the computation is made in the two cases as shown in Figure 1.

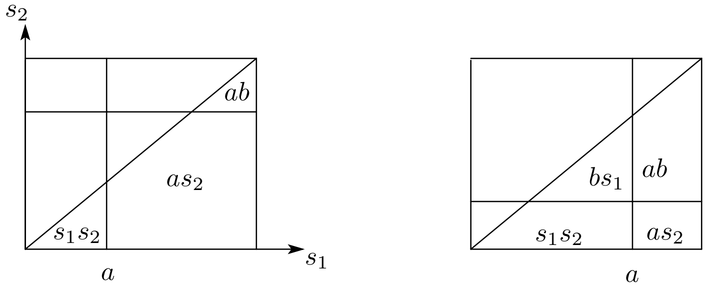

Figure 1a. Case $a<b$ Figure 1b. Case $a>b$

Therefore (2.6) becomes

(2.6′)
$$p_{RR}=\xi\cdot\left\{-\frac{1}{24}a^4-\frac{1}{4}a^2b^2+\frac{1}{2}ab+\frac{1}{6}ab^3\right\}$$

$$+\eta \cdot \left(\frac{1}{2}a^2 - \frac{1}{8}a^4\right) + \zeta \cdot \left(-\frac{1}{24}b^4 - \frac{1}{6}a^3b + \frac{1}{2}ab\right).$$

Now we first have to make sure that (2.2) is ture. By collecting the coefficients of η in (2.3) $\sim$ (2.6′), we have

$$\begin{aligned} M_1(a,a) &= [p_{AA} + p_{AR} + p_{RA} + p_{RR}]_{a=b} \\ &= \frac{1}{2}\bar{a}^2 + \frac{1}{2}(a - a^3) + \frac{1}{2}a\bar{a}^2 + \left(\frac{1}{2}a^2 - \frac{1}{8}a^4\right) \end{aligned}$$

which is easily shown to be equal to $\frac{1}{2}\left(1 - \frac{1}{4}a^4\right)$.

For each of ξ and ζ, collect the partial derivatives $\frac{\partial}{\partial a}$ of each of (2.3) $\sim$ (2.6′). Then we obtain

$$(2.7) \qquad \frac{\partial M_1(a,b)}{\partial a} = \begin{cases} \frac{1}{2}(1 - b - a^2 - ab^2) + \frac{1}{6}(b^3 - a^3), & \text{if } a < b \\ \frac{1}{2}(1 + b - a^2 - a^2b) - a, & \text{if } a > b \end{cases}$$

The expression in the r.h.s. is continuous on the diagonal $a = b$.

Theorem 1 *Solution to the game "Showcase Showdown". Let $a^* \approx 0.54368$ be a unique root of the cubic equation*

$$(2.8) \qquad 1 - a - a^2 - a^3 = 0.$$

Then the game has a unique equilibrium point (a^*, a^*) and the equilibrium payoffs are

$$(2.9) \qquad M_0(a^*, a^*) = \frac{1}{4}a^{*4} \approx 0.02184,$$

$$(2.10) \qquad M_1(a^*, a^*) = M_2(a^*, a^*) = \frac{1}{2}\left(1 - \frac{1}{4}a^{*4}\right) \approx 0.48908.$$

Proof. Consider Eq.(2.7) for $b = a^*$. Then we have, for $a < a^*$,

$$\begin{aligned} \frac{\partial M_1(a,a^*)}{\partial a} &= \frac{1}{2}(a^{*2} + a^{*3} - a^2 - aa^{*2}) + \frac{1}{6}(a^{*3} - a^3) \\ &> \frac{1}{2}(a^{*2} + a^{*3} - a^2 - aa^{*2}) > \frac{1}{2}a^{*2}(a^* - a); \end{aligned}$$

and, for $a > a^*$,

$$\begin{aligned} \frac{\partial M_1(a,a^*)}{\partial a} &= \frac{1}{2}(1 + a^* - a^2 - a^2a^*) - a = \frac{1}{2}\left\{(1 + a^*)(1 - a^2) - 2a\right\} \\ &< \frac{1}{2}(1 - a - a^2 - a^3) < \frac{1}{2}(1 - a^* - a^{*2} - a^{*3}) = 0. \end{aligned}$$

Hence

$$\frac{\partial M_1(a,a^*)}{\partial a} > (=,<)0, \quad \text{if } a < (=,>)a^*,$$

that is,

$$(2.11) \qquad \max_{a\in[0,1]} M_1(a,a^*) = M_1(a^*,a^*).$$

Next we want to show that $\max_{b\in[0,1]} M_2(a^*, b) = M_2(a^*, a^*)$.

For each of ξ and ζ, collect the partial derivatives $\frac{\partial}{\partial b}$ of each of (2.3) $\sim$ (2.6′), we find that

$$\frac{\partial M_1(a,b)}{\partial b} = \begin{cases} \frac{1}{2}(ab^2 - a^2b + b^2 + 2b - a - 1), & \text{if } a < b \\ \frac{1}{2}(b^2 + a - 1) - \frac{1}{6}(a^3 - b^3), & \text{if } a > b \end{cases} \tag{2.12}$$

where the r.h.s. is continuous on the diagonal $a = b$.

Therefore it follows, from (2.11) and (2.12), that

$$\begin{aligned} \frac{\partial M_2(a,b)}{\partial b} &= \frac{\partial}{\partial b}\{1 - M_0(a,b) - M_1(a,b)\} = -\frac{1}{2}a^2b - \frac{\partial}{\partial b}M_1(a,b) \\ &= \begin{cases} \frac{1}{2}(-a^2b - b^2 - 2b + a + 1), & \text{if } a < b \\ -\frac{1}{2}a^2b + \frac{1}{6}(a^3 - b^3) + \frac{1}{2}(1 - b^2 - a), & \text{if } a > b \end{cases} \end{aligned} \tag{2.13}$$

Consider Eq.(2.13) for $a = a^*$. Then, for $b < a^*$,

$$\begin{aligned} \frac{\partial M_2(a^*,b)}{\partial b} &= -\frac{1}{2}a^{*2}b + \frac{1}{6}(a^{*3} - b^3) + \frac{1}{2}(1 - b^2 - a^*) \\ &> \frac{1}{2}(1 - a^* - b^2 - a^{*2}b) > 0 \end{aligned}$$

since the quadratic equation of b, here, is decreasing with values $\frac{1}{2}(1 - a^*)$ at $b = 0$,and 0 at $b = a^*$.

Also, for $b > a^*$,

$$\frac{\partial M_2(a^*,b)}{\partial b} = \frac{1}{2}(-a^{*2}b - b^2 - 2b + a^* + 1) < 0,$$

since the above quadratic function is decreasing with values 0 at $b = a^*$ and $-1 + \frac{1}{2}a^*\bar{a}^*$ at $b = 1$. Hence

$$\frac{\partial M_2(a^*,b)}{\partial b} > (=,<), \quad \text{if } b < (=,>)a^*,$$

that is,

$$\max_{b\in[0,1]} M_2(a^*,b) = M_2(a^*,a^*). \tag{2.14}$$

The theorem is thus proven by (2.11) and (2.14). □

3 Solution to the game "Risky Exchange". Let $W_i, M_i(a,b), i = 0,1,2$, and ξ,η,ζ have the same meanings as in Section 2. Evidently, we have $\sum_{i=0,1,2} M_i(a,b) = 1$, and, from (1.2)-(1.5),

$$M_0(a,b) = P\{X_{12} < X_{11} < a, X_{22} < X_{21} < b\} = \frac{1}{4}a^2b^2, \tag{3.1}$$

$$M_1(a,a) = M_2(a,a) = \frac{1}{2}\left(1 - \frac{1}{4}a^4\right), \quad \forall a \in [0,1]. \tag{3.2}$$

Surprizingly we find that all four probabilities p_{AA}, *etc.*, remain unchanged as in the game Showcase Showdown in Section 2. Actually we have

$$p_{AA} = P\{X_{11} > a, X_{21} > b, X_{11} > X_{21}\} = \text{same as in } (2.3), \tag{3.3}$$

$$
\begin{aligned}
(3.4)\quad p_{AR} &= P[X_{11} > a, \{X_{22} < X_{21} < b\} \cup \{X_{21} < b, X_{11} > X_{22} > X_{21}\}] \\
&= P\{X_{11} > a, X_{22} < X_{21} < b\} + P\{X_{11} > a, X_{21} < b, X_{11} > X_{22} > X_{21}\} \\
&= \frac{1}{2}\bar{a}b^2 + \iint_{s_1 > a, s_2 < b, s_1 > s_2} (s_1 - s_2) ds_1 ds_2 = \text{same as in (2.4)},
\end{aligned}
$$

$$
\begin{aligned}
(3.5)\quad p_{RA} &= P\{X_{11} < a \wedge X_{12},\ b < X_{21} < X_{12}\} \\
&= \int_b^1 (a \wedge t)(t - b) dt \qquad \text{(by denoting } X_{12} = t) \\
&= \xi \cdot \frac{1}{2}a\bar{b}^2 + \eta \cdot \frac{1}{2}a\bar{a}^2 + \zeta \cdot \left\{\frac{1}{2}(a^2 b - 2ab + a) - \frac{1}{6}(a^3 - b^3)\right\} \\
&= \text{same as in (2.5)},
\end{aligned}
$$

and

$$
\begin{aligned}
(3.6)\quad p_{RR} &= P[X_{11} < a \wedge X_{12}, \\
&\qquad \{X_{22} < X_{21} < b\} \cup \{X_{21} < b \wedge X_{22}, X_{12} > X_{22}\}] \\
&= \frac{1}{4}(2a - a^2)b^2 + \iint_{1 \ge s_1 > s_2 \ge 0} (s_1 \wedge a)(s_2 \wedge b) ds_1 ds_2 \\
&\qquad \text{(by denoting } X_{12} = s_1 \text{ and } X_{22} = s_2) \\
&= \text{same as in (2.6)}.
\end{aligned}
$$

Therefore we obtain

Theorem 2 *Solution to the game "Risky Exchange" is the same as that of the game "Showcase Shoedown".*

The two score functions (1.1) and (1.5) are seemingly different. However, probabilities (2.3) $\sim$ (2.6) remain unchanged from probabilities (3.3) $\sim$ (3.6). This result may possibly be "not surprizing", since it is due to the two facts : (a) If $X \sim \mathbf{U}_{[0,1]}$ (*i.e.*, X is uniformly distributed on $[0, 1]$), then so is $1 - X$, and (b) If X_1 and X_2 are *iid* with $\mathbf{U}_{[0,1]}$, then $X_1 + X_2(\text{mod}.1) \sim \mathbf{U}_{[0,1]}$.

4 Remarks.

Remark 1. Consider the one-player version of Showcase Showdown, where player aims to maximize his expected score. Let a and $M(a)$ be the player's threshold number and the expected score obtained by employing it, respectively. Then we have

$$
\begin{aligned}
M(a) &= \int_a^1 x dx + \int_0^a E\{(x + X_{12}) I(x + X_{12} \le 1)\} dx \\
&= \frac{1}{2}(1 - a^2) + \int_0^a dx \int_0^{1-x} (x + t) dt = \frac{1}{2}\left(1 + a - a^2 - \frac{1}{3}a^3\right)
\end{aligned}
$$

which is maximized at $a = a^* = \sqrt{2} - 1 \approx 0.41421$, and $M(a^*) = \frac{1}{3}(2\sqrt{2} - 1) \approx 0.60948$.

We can easily find that one-player version of Risky Exchange also has the same solution as above.

Remark2. It is interesting to consider the sequential-move version of the game.
There appears the unfair acquisition of information by players. The game is played in two stages :

In the first stage, I observes that $X_{11} = x$ and choose one of either A_1 (*i.e.*, I accepts x) or R_1 (*i.e.*, I reject x, and resamples a new r.v. X_{12}). The observed value x and I's choice of eigher A_1 or R_1 are *informed to* II. But the observed value of X_{12} is *not informed to* II.

In the second stage, II observes that $X_{21} = y$ and chooses either one of A_2 (*i.e.*, II accepts y) or R_2 (*i.e.*, II rejects y and resamples a new r.v. X_{22}).

After the second stage is over, showdown is made, the scores are compared and the player with the higher score than opponent's becomes the winner. Each player aims to maximize the probability of his (or her) winning.

The sequential-move version of the games Showcase Showdown, Keep-or-Exchange and Competing Average are solved in Ref.[4]. As would be expected, the second-mover has an advantage over the first-mover, and this intuition is proven to be true. In Showcase Showdown, for example, player' winning probabilities and probability of draw are

$$P(W_1) \approx 0.4768, \quad P(W_2) \approx 0.5124, \quad P(D) \approx 0.0108$$

in the sequential-move version, and

$$P(W_1) = P(W_2) \approx 0.48908, \quad P(D) \approx 0.02184$$

in the simultaneous-move version. See Ref.[4 : Theorem 4] and Theorem 1 of the present article.

Remark3. Extention to the three-player games should be investigated. As the first result obtained in this area, the solution to the three-player Keep-or Exchange is derived in Ref.[5]. Closely related works to the present paper are found in Ref.[2, 3], where the last mover doesn't stand most advantageous in some three-player sequential-move games.

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