(α, β)-FUZZY SUBALGEBRAS IN LATTICE IMPLICATION ALGEBRAS

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ABSTRACT. Using the notion of "belongingness ($\in$)" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, the concept of (α, β)-fuzzy subalgebra where α, β are any two of $\{\in, \mathrm{q}, \in\vee\mathrm{q}, \imath n\wedge\mathrm{q}\}$ with $\alpha \neq \in\wedge\mathrm{q}$ is introduced, and related properties are investigated.

1. Introduction

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [4], played a vital role to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, $(\in, \in\vee\mathrm{q})$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, the concept of (α, β)-fuzzy (quasi) subalgebra of a lattice implication algebra is introduced and related results are discussed in the present paper.

2. Preliminaries

A *lattice implication algebra* is defined to be a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\multimap$ " satisfying the following axioms:

(I1) $x \multimap (y \multimap z) = y \multimap (x \multimap z)$,
(I2) $x \multimap x = 1$,
(I3) $x \multimap y = y' \multimap x'$,
(I4) $x \multimap y = y \multimap x = 1 \Rightarrow x = y$,
(I5) $(x \multimap y) \multimap y = (y \multimap x) \multimap x$,
(L1) $(x \vee y) \multimap z = (x \multimap z) \wedge (y \multimap z)$,
(L2) $(x \wedge y) \multimap z = (x \multimap z) \vee (y \multimap z)$,

for all $x, y, z \in L$

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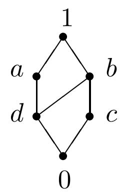
Example 2.1. [2] Let $L = \{0, a, b, c, d, 1\}$ be a set with Hasse diagram and Cayley tables as follows:



x	x'
0	1
a	c
b	d
c	a
d	b
1	0

$\multimap$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (x \multimap y) \multimap y, \ x \wedge y := ((x' \multimap y') \multimap y')',$$

for all $x, y \in L$. Then L is a lattice implication algebra.

We can define a partial ordering " $\leq$ " on a lattice implication algebra L by $x \leq y$ if and only if $x \multimap y = 1$.

In a lattice implication algebra L, the following hold (see [5]):

(z1) $0 \multimap x = 1$, $1 \multimap x = x$ and $x \multimap 1 = 1$.
(z2) $x \multimap y \leq (y \multimap z) \multimap (x \multimap z)$.
(z3) $x \leq y$ implies $y \multimap z \leq x \multimap z$ and $z \multimap x \leq z \multimap y$.
(z4) $x' = x \multimap 0$.
(z5) $x \vee y = (x \multimap y) \multimap y$.
(z6) $((y \multimap x) \multimap y')' = x \wedge y = ((x \multimap y) \multimap x')'$.
(z7) $x \leq (x \multimap y) \multimap y$.

A subset S of a lattice implication algebra L is called a *subalgebra* of L (see [6]) if it satisfies the following conditions

- $0 \in S$,
- $(\forall x, y \in S)\,(x \multimap y \in S)$.

A subset S of a lattice implication algebra L is called a *quasi subalgebra* of L (see [3]) if it is closed under the operation $\multimap$.

A fuzzy set A in a set L of the form

$$A(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set A in a set L, Pu and Liu [4] gave meaning to the symbol $x_t \alpha A$, where $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$.

To say that $x_t \in A$ (resp. $x_t \mathrm{q} A$) means that $A(x) \geq t$ (resp. $A(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set A.

To say that $x_t \in\vee\mathrm{q}\, A$ (resp. $x_t \in\wedge\mathrm{q}\, A$) means that $x_t \in A$ or $x_t \mathrm{q} A$ (resp. $x_t \in A$ and $x_t \mathrm{q} A$). For all $t_1, t_2 \in [0, 1]$, $\min\{t_1, t_2\}$ will be denoted by $M(t_1, t_2)$.

A fuzzy set A in a lattice implication algebra L is called a *fuzzy subalgebra* of L (see [7]) if it satisfies

(a1) $A(1) = A(0)$,
(a2) $(\forall x, y \in L)\,(A(x \multimap y) \geq M(A(x), A(y)))$.

If the condition (a1) does not hold, we say that A is a *fuzzy quasi subalgebra* of L.

Proposition 2.2. [7] *Let A be a fuzzy set in a lattice implication algebra L. Then A is a fuzzy subalgebra of L if and only if $A_t := \{x \in L \mid A(x) \geq t\}$ is a subalgebra of L for all $t \in (0, 1]$, for our convenience, the empty set $\emptyset$ is regarded as a subalgebra of L.*

3. (α, β)-FUZZY SUBALGEBRAS

In what follows let L denote a lattice implication algebra, and α and β will denote any one of $\in$, q, $\in\vee$q, or $\in\wedge$q unless otherwise specified. To say that $x_t\overline{\alpha}A$ means that $x_t\alpha A$ does not hold.

Proposition 3.1. *For any fuzzy set A in L, the condition* (a2) *is equivalent to the following condition*

(1) $$(\forall x, y \in L)\,(\forall t_1, t_2 \in (0,1])\,(x_{t_1}, y_{t_2} \in A \Rightarrow (x \multimap y)_{M(t_1,t_2)} \in A).$$

Proof. Assume that the condition (a2) is valid. Let $x, y \in L$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1}, y_{t_2} \in A$. Then $A(x) \geq t_1$ and $A(y) \geq t_2$, which imply from (a2) that

$$A(x \multimap y) \geq M(A(x), A(y)) \geq M(t_1, t_2).$$

Hence $(x \multimap y)_{M(t_1,t_2)} \in A$.

Conversely suppose that the condition (1) is valid. Note that $x_{A(x)} \in A$ and $y_{A(y)} \in A$ for all $x, y \in L$. Thus $(x \multimap y)_{M(A(x),A(y))} \in A$ by (1), and so $A(x \multimap y) \geq M(A(x), A(y))$. □

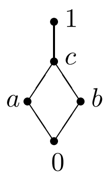
A fuzzy set A in L is said to be an (α, β)-*fuzzy subalgebra* of L, where $\alpha \neq \in\wedge$q, if it satisfies the following conditions:

(a1) $A(1) = A(0)$,
(a3) $(\forall x, y \in L)\,(\forall t_1, t_2 \in (0,1])\,(x_{t_1}\alpha A,\ y_{t_2}\alpha A \Rightarrow (x \multimap y)_{M(t_1,t_2)}\beta A.)$

If the condition (a1) does not hold, we say that A is an (α, β)-*fuzzy quasi subalgebra* of L.

Note that if A is a fuzzy set in L defined by $A(x) \leq 0.5$ for all $x \in L$, then the set $\{x_t \mid x_t \in\wedge\text{q}\, A\}$ is empty.

Example 3.2. Let $L = \{0, a, b, c, 1\}$ be a set with the following Hasse diagram as a partial ordering, and define a unary operation "$'$" and a binary operation "$\multimap$" as follows:



x	x'
0	1
a	b
b	a
c	c
1	0

$\multimap$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	c	1	1
b	a	c	1	1	1
c	c	c	c	1	1
1	0	a	b	c	1

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (x \multimap y) \multimap y \text{ and } x \wedge y := ((x' \multimap y') \multimap y')'$$

for all $x, y \in L$. Then L is a lattice implication algebra. Let A be a fuzzy set in L given by $A(1) = A(0) = 0.77$, $A(a) = 0.9$, $A(b) = 0.8$, and $A(c) = 0.7$. Then A is an $(\in, \in\vee$q$)$-fuzzy subalgebra of L.

For a fuzzy set A in L, we denote $L_0 := \{x \in L \mid A(x) > 0\}$.

Theorem 3.3. *If A is a nonzero $(\in, \in)$-fuzzy subalgebra of L, then the set L_0 is a subalgebra of L.*

Proof. Assume that $A(0) = 0$. Since A is nonzero, there exists $x \in L$ such that $A(x) = t > 0$. It follows that $x_t \in A$ so that $1_t = (x \multimap x)_{M(t,t)} \in A$ by (I2) and (a3). Hence $0 = A(0) = A(1) \geq t > 0$, which is a contradiction. Therefore $A(0) > 0$ and so $0 \in L_0$. Let $x, y \in L_0$. Then $A(x) > 0$ and $A(y) > 0$. Suppose that $A(x \multimap y) = 0$. Note that $x_{A(x)} \in A$ and $y_{A(y)} \in A$, but $(x \multimap y)_{M(A(x),A(y))}\overline{\in}A$ because $A(x \multimap y) = 0 < M(A(x), A(y))$. This is a contradiction, and thus $A(x \multimap y) > 0$, which shows that $x \multimap y \in L_0$. Consequently L_0 is a subalgebra of L. □

Theorem 3.4. *If A is a nonzero $(\in, \mathrm{q})$-fuzzy subalgebra of L, then the set L_0 is a subalgebra of L.*

Proof. Assume that $A(0) = 0$. Since A is nonzero, there exists $x \in L$ such that $A(x) = t > 0$. Then $x_t \in A$, and so

$$A(x \multimap x) + M(t, t) = A(1) + t = A(0) + t = t \leq 1.$$

This means that $(x \multimap x)_{M(t,t)} \overline{\mathrm{q}} A$, which is a contradiction. Hence $A(0) > 0$ and $0 \in L_0$. Let $x, y \in L_0$. Then $A(x) > 0$ and $A(y) > 0$. If $A(x \multimap y) = 0$, then

$$A(x \multimap y) + M(A(x), A(y)) = M(A(x), A(y)) \leq 1.$$

Hence $(x \multimap y)_{M(A(x),A(y))} \overline{\mathrm{q}} A$, which is a contradiction si nce $x_{A(x)} \in A$ and $y_{A(y)} \in A$. Thus $A(x \multimap y) > 0$, and so $x \multimap y \in L_0$. Therefore L_0 is a subalgebra of L. □

Theorem 3.5. *If A is a nonzero $(\mathrm{q}, \in)$-fuzzy subalgebra of L, then the set L_0 is a subalgebra of L.*

Proof. Assume that $A(0) = 0$. Since A is nonzero, there exists $x \in L$ such that $A(x) = t > 0$. Then $A(x) + 1 = t + 1 > 1$, and so $x_1 \, \mathrm{q} \, A$. But, since

$$A(x \multimap x) = A(1) = A(0) = 0 < 1 = M(1, 1),$$

we have $(x \multimap x)_{M(1,1)} \overline{\in} A$. This is impossible, and thus $A(0) > 0$, i.e., $0 \in L_0$. Let $x, y \in L_0$. Then $A(x) > 0$ and $A(y) > 0$. Thus $A(x) + 1 > 1$ and $A(y) + 1 > 1$, which imply that $x_1 \, \mathrm{q} \, A$ and $y_1 \, \mathrm{q} \, A$. If $A(x \multimap y) = 0$, then $A(x \multimap y) < 1 = M(1, 1)$. Therefore $(x \multimap y)_{M(1,1)} \overline{\in} A$, which is a contradiction. It follows that $A(x \multimap y) > 0$ so that $x \multimap y \in L_0$. This completes the proof. □

Theorem 3.6. *If A is a nonzero (q, q)-fuzzy subalgebra of L, then the set L_0 is a subalgebra of L.*

Proof. Assume that $A(0) = 0$. Since A is nonzero, there exists $x \in L$ such that $A(x) = t > 0$. Then $A(x) + 1 = t + 1 > 1$, and so $x_1 \, \mathrm{q} \, A$. Since

$$A(x \multimap x) + M(1, 1) = A(1) + 1 = A(0) + 1 = 1,$$

we get $(x \multimap x)_{M(1,1)} \, \overline{\mathrm{q}} \, A$. This is a contradiction. Hence $A(0) > 0$, and thus $0 \in L_0$. Now let $x, y \in L_0$. Then $A(x) > 0$ and $A(y) > 0$. Thus $A(x) + 1 > 1$ and $A(y) + 1 > 1$, and therefore $x_1 \, \mathrm{q} \, A$ and $y_1 \, \mathrm{q} \, A$. If $A(x \multimap y) = 0$, then $A(x \multimap y) + M(1, 1) = 0 + 1 = 1$, and so $(x \multimap y)_{M(1,1)} \, \overline{\mathrm{q}} \, A$. This is impossible, and hence $A(x \multimap y) > 0$, i.e., $x \multimap y \in L_0$. This completes the proof. □

Corollary 3.7. *If A is one of the following*

(i) *a nonzero $(\in, \in \wedge \, \mathrm{q})$-fuzzy subalgebra of L,*
(ii) *a nonzero $(\in, \in \vee \, \mathrm{q})$-fuzzy subalgebra of L,*
(iii) *a nonzero $(\in \vee \, \mathrm{q}, \mathrm{q})$-fuzzy subalgebra of L,*
(iv) *a nonzero $(\in \vee \, \mathrm{q}, \in)$-fuzzy subalgebra of L,*
(v) *a nonzero $(\in \vee \, \mathrm{q}, \in \wedge \, \mathrm{q})$- fuzzy subalgebra of L,*
(vi) *a nonzero $(\mathrm{q}, \in \wedge \, \mathrm{q})$-fuzzy subalgebra of L,*
(vii) *a nonzero $(\mathrm{q}, \in \vee \, \mathrm{q})$-fuzzy) subalgeb ra of L,*

then the set $L_0 := \{x \in L \mid A(x) > 0\}$ is a subalgebra of L.

Proof. The proof is similar to the proof of Theorems 3.3, 3.4, 3.5, and/or 3.6. □

Theorem 3.8. *If A is a nonzero (q, q)-fuzzy subalgebra of L, then it is constant on L_0.*

Proof. Assume that A is not constant on L_0. Then there exists $y \in L_0$ such that $t_y = A(y) \neq A(0) = t_0$. Then either $t_y > t_0$ or $t_y < t_0$. Suppose $t_y < t_0$ and choose $t_1, t_2 \in (0, 1]$ such that $1 - t_0 < t_1 < 1 - t_y < t_2$. Then $A(1) + t_1 = A(0) + t_1 = t_0 + t_1 > 1$ and $A(y) + t_2 = t_y + t_2 > 1$, and so $1_{t_1} \mathrm{q} A$ and $y_{t_2} \mathrm{q} A$. Since

$$A(1 \multimap y) + M(t_1, t_2) = A(y) + t_1 = t_y + t_1 < 1,$$

we have $(1 \multimap y)_{M(t_1,t_2)} \overline{\mathrm{q}} A$, which is a contradiction. Next assume that $t_y > t_0$. Then $A(y) + (1 - t_0) = t_y + 1 - t_0 > 1$ and so $y_{1-t_0} \mathrm{q} A$. Since

$$A(y \multimap y) + (1 - t_0) = A(1) + 1 - t_0 = A(0) + 1 - t_0 = t_0 + 1 - t_0 = 1,$$

we get $(y \multimap y)_{M(1-t_0,1-t_0)} \overline{\mathrm{q}} A$. This is impossible. Therefore A is constant on L_0. □

Theorem 3.9. *Let H be a subalgebra of L and let A be a fuzzy set in L such that $A(x) = 0$ for all $x \in L \setminus H$. If any one of the following holds:*

(i) *A is nonzero constant on H,*
(ii) *$A(0) = A(1)$ and $A(x) \geq 0.5$ for all $x \in H$,*

then A is a $(\mathrm{q}, \in\vee\mathrm{q})$-fuzzy subalgebra of L.

Proof. Assume that A is nonzero constant on H. Obviously $A(0) = A(1)$ because $0, 1 \in H$. Let $x, y \in L$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \mathrm{q} A$ and $y_{t_2} \mathrm{q} A$. Then either $x \multimap y \in L \setminus H$ or $x \multimap y \in H$. In the first case we have $x \in L \setminus H$ or $y \in L \setminus H$. Thus $A(x) = 0$ or $A(y) = 0$, and thus $t_1 > 1$ or $t_2 > 1$. This is a contradiction. Hence we know that $x \multimap y \in H$, and $A(x \multimap y) = t > 0$. If $A(x \multimap y) = t < M(t_1, t_2)$, then

$$A(x \multimap y) + M(t_1, t_2) = t + M(t_1, t_2) > 1.$$

Hence $(x \multimap y)_{M(t_1,t_2)} \in\vee\mathrm{q}\, A$, and consequently A is a $(\mathrm{q}, \in\vee\mathrm{q})$-fuzzy subalgebra of L. Now suppose that $A(0) = A(1)$ and $A(x) \geq 0.5$ for all $x \in H$. Let $x, y \in L$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \mathrm{q} A$ and $y_{t_2} \mathrm{q} A$. Then $x, y \in H$, and so $x \multimap y \in H$. If $M(t_1, t_2) > 0.5$, then $A(x \multimap y) + M(t_1, t_2) > 1$, i.e., $(x \multimap y)_{M(t_1,t_2)} \mathrm{q} A$. If $M(t_1, t_2) \leq 0.5$, then $A(x \multimap y) \geq M(t_1, t_2)$, i.e., $(x \multimap y)_{M(t_1,\, t_2)} \in A$. Thus in any case we have $(x \multimap y)_{M(t_1,\, t_2)} \in\vee\mathrm{q} A$. Therefore A is a $(\mathrm{q}, \in\vee\mathrm{q})$-fuzzy subalgebra of L. □

Theorem 3.10. *Let A be a $(\mathrm{q}, \in\vee\mathrm{q})$-fuzzy subalgebra of L such that A is not constant on L_0. Then*

(i) *there exists $x \in L$ such that $A(x) \geq 0.5$.*
(ii) *$A(x) \geq 0.5$ for all $x \in L_0$.*

Proof. (i) Assume that $A(x) < 0.5$ for all $x \in L$. Since A is not constant on L_0, there exists $x \in L_0$ such that $t_x = A(x) \neq A(0) = t_0$. Then either $t_0 < t_x$ or $t_0 > t_x$. If $t_0 < t_x$, choose $\delta > 0.5$ such that $t_0 + \delta < 1 < t_x + \delta$. Then $x_\delta \mathrm{q} A$. Now

$$A(x \multimap x) = A(1) = A(0) = t_0 < \delta = M(\delta, \delta)$$

and

$$A(x \multimap x) + M(\delta, \delta) = A(1) + \delta = A(0) + \delta = t_0 + \delta < 1.$$

Thus $(x \multimap x)_{M(\delta,\, \delta)} \overline{\in\vee\mathrm{q}} A$, which is impossible. Next if $t_0 > t_x$, we can choose $\delta > 0$ such that $t_x + \delta < 1 < t_0 + \delta$. Then $1_\delta \mathrm{q} A$ and $x_1\, \mathrm{q}\, A$, but $(1 \multimap x)_{M(\delta,1)} = x_\delta \overline{\in\vee\mathrm{q}} A$ because $A(x) < 0.5 < \delta$ and $A(x) + \delta = t_x + \delta < 1$. This leads a contradiction. Hence (i) is valid.

(ii) We first show that $A(0) = A(1) \geq 0.5$. If possible, let $t_0 = A(0) < 0.5$. By (i), there exists $x \in L$ such that $t_x = A(x) \geq 0.5$. It follows that $t_0 < t_x$. Choose $t_1 > t_0$ such that $t_0 + t_1 < 1 < t_x + t_1$. Then $x_{t_1} \mathrm{q} A$ since $A(x) + t_1 = t_x + t_1 > 1$. Now we have

$$A(x \multimap x) + M(t_1, t_1) = A(1) + t_1 = A(0) + t_1 = t_0 + t_1 < 1$$

and

$$A(x \multimap x) = A(1) = A(0) = t_0 < t_1 = M(t_1, t_1).$$

Hence $(x \multimap x)_{M(t_1,t_1)} \overline{\in\vee\text{q}} A$, which leads a contradiction. Therefore $A(0) = A(1) \geq 0.5$. Assume that $t_x = A(x) < 0.5$ for some $x \in L_0$. Take $t > 0$ such that $t_x + t < 0.5$. Then $A(x) + 1 = t_x + 1 > 1$ and $A(1) + (0.5 + t) > 1$, and thus $x_1\,\text{q}\,A$ and $1_{0.5+t}\text{q}A$. But $(1 \multimap x)_{M(1,0.5+t)} = x_{0.5+t} \overline{\in\vee\text{q}}\, A$ since $A(1 \multimap x) = A(x) < 0.5 + t \leq M(1, 0.5 + t)$ and

$$A(1 \multimap x) + M(1, 0.5 + t) = A(x) + 0.5 + t = t_x + 0.5 + t < 0.5 + 0.5 = 1.$$

This is a contradiction. Hence $A(x) \geq 0.5$ for all $x \in L_0$. This completes the proof. □

Lemma 3.11. *Let A be a fuzzy set in L. Then the following are equivalent:*

(i) $(\forall x, y \in L)\,(\forall t_1, t_2 \in (0, 1])\,(x_{t_1},\, y_{t_2} \in A \Rightarrow (x \multimap y)_{M(t_1,t_2)} \in\vee\text{q}\, A)$.
(ii) $(\forall x, y \in L)\,(A(x \multimap y) \geq M(A(x), A(y), 0.5))$.

Proof. Suppose that (i) is valid. Let $x, y \in L$. If $M(A(x), A(y)) < 0.5$, then $A(x \multimap y) \geq M(A(x), A(y))$. For, assume that $A(x \multimap y) < M(A(x), A(y))$ and choose t such that $A(x \multimap y) < t < M(A(x), A(y))$. Then $x_t \in A$ and $y_t \in A$ but $(x \multimap y)_{M(t,t)} = (x \multimap y)_t \overline{\in\vee\text{q}}\, A$ which contradicts (i). Hence

$$A(x \multimap y) \geq M(A(x), A(y)).$$

Now if $M(A(x), A(y)) \geq 0.5$, then $A(x \multimap y) \geq 0.5$ because if not then $x_{0.5} \in A$ and $y_{0.5} \in A$ but $(x \multimap y)_{M(0.5,0.5)} = (x \multimap y)_{0.5} \overline{\in\vee\text{q}}\, A$, a contradiction. Consequently, (ii) is valid.

Conversely, assume that (ii) holds. Let $x, y \in L$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in A$ and $y_{t_2} \in A$. Then $A(x) \geq t_1$ and $A(y) \geq t_2$. Suppose that $A(x \multimap y) < M(t_1, t_2)$. Then $M(A(x), A(y)) \geq 0.5$ because if not, then

$$A(x \multimap y) \geq M(A(x), A(y), 0.5) \geq M(A(x), A(y)) \geq M(t_1, t_2)$$

which is a contradiction. It follows that

$$A(x \multimap y) + M(t_1, t_2) > 2A(x \multimap y) \geq 2M(A(x), A(y), 0.5) = 1.$$

This shows that $(x \multimap y)_{M(t_1,t_2)} \in\vee\,\text{q}\, A$. □

Theorem 3.12. *Let A be an $(\in, \in\vee\text{q})$-fuzzy subalgebra of L.*

(i) *If there exists $x \in L$ such that $A(x) \geq 0.5$, then $A(0) \geq 0.5$.*
(ii) *If $A(0) < 0.5$, then A is an $(\in, \in)$-fuzzy subalgebra of L.*

Proof. (i) Assume that $A(x) \geq 0.5$ for some $x \in L$. Then $x_{0.5} \in A$ and so

$$1_{0.5} = (x \multimap x)_{0.5} = (x \multimap x)_{M(0.5,0.5)} \in\vee\,\text{q}\, A.$$

It follows that $A(1) \geq 0.5$ or $A(1) + 0.5 > 1$ so that $A(0) = A(1) \geq 0.5$.

(ii) Suppose that $A(0) < 0.5$. Then $A(x) < 0.5$ for all $x \in L$ by (i). Let $x, y \in L$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in A$ and $y_{t_2} \in A$. Then $A(x) \geq t_1$ and $A(y) \geq t_2$. It follows from Lemma 3.11 that

$$A(x \multimap y) \geq M(A(x), A(y), 0.5) = M(A(x), A(y)) \geq M(t_1, t_2)$$

so that $(x \multimap y)_{M(t_1,t_2)} \in A$. Hence A is an $(\in, \in)$-fuzzy subalgebra of L. □

Theorem 3.13. *Let A be a fuzzy set in L. Then $A_t := \{x \in L \mid A(x) \geq t\}$ is a quasi subalgebra of L for all $t \in (0.5, 1]$ if and only if*

$$(\forall x, y \in L)\,(\max\{A(x \multimap y), 0.5\} \geq M(A(x), A(y))). \tag{2}$$

Proof. ($\Rightarrow$) Assume that there exists $x, y \in L$ such that

$$\max\{A(x \multimap y), 0.5\} < M(A(x), A(y)) = t.$$

Then $t \in (0.5, 1]$, $A(x \multimap y) < t$ and $x, y \in A_t$. Since A_t is a quasi subalgebra of L, it follows that $x \multimap y \in A_t$ so that $A(x \multimap y) \geq t$. This is a contradiction. Hence $\max\{A(x \multimap y), 0.5\} \geq M(A(x), A(y))$ for all $x, y \in L$.

($\Leftarrow$) Suppose that the condition (2) is valid. Let $t \in (0.5, 1]$ and $x, y \in A_t$. Then

$$\max\{A(x \multimap y), 0.5\} \geq M(A(x), A(y)) \geq t > 0.5,$$

and so $A(x \multimap y) \geq t$. Hence $x \multimap y \in A_t$, which shows that A_t is a quasi subalgebra of L for all $t \in (0.5, 1]$. □

Lemma 3.14. *Every* $(\in, \in)$*-fuzzy (quasi) subalgebra is an* $(\in, \in\vee\,\mathrm{q})$*-fuzzy (quasi) subalgebra.*

Proof. Let A be an $(\in, \in)$-fuzzy (quasi) subalgebra of L. Let $x, y \in L$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in A$ and $y_{t_2} \in A$. Then $(x \multimap y)_{M(t_1,t_2)} \in A$ and thus $(x \multimap y)_{M(t_1,t_2)} \in\vee\,\mathrm{q}\,A$. Hence A is an $(\in, \in\vee\,\mathrm{q})$-fuzzy (quasi) subalgebra of L. □

The converse of Lemma 3.14 may not be true in general. For example, the $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebra of L in Example 3.2 is not an $(\in, \in)$-fuzzy subalgebra of L because $a_{0.9} \in A$ but $(a \multimap a)_{M(0.9,0.9)} = 1_{0.9} \overline{\in} A$.

Theorem 3.15. *For any subset* H *of* L*,* χ_H *is an* $(\in, \in\vee\,\mathrm{q})$*-fuzzy quasi subalgebra of* L *if and only if* H *is a quasi subalgebra of* L*, where* χ_H *is the characteristic function of* H*.*

Proof. Let χ_H be an $(\in, \in\vee\,\mathrm{q})$-fuzzy quasi subalgebra of L. Then $x, y \in H$ implies $\chi_H(x) = 1$ and $\chi_H(y) = 1$. Hence $x_1, y_1 \in \chi_H$, which implies that $(x \multimap y)_{M(1,1)} \in\vee\,\mathrm{q}\,\chi_H$. This yields $\chi_H(x \multimap y) > 0$, and so $\chi_H(x \multimap y) = 1$. Thus $x \multimap y \in H$. Conversely, if H is a quasi subalgebra of L, then χ_H is an $(\in, \in)$-fuzzy quasi subalgebra of L. Therefore χ_H is an $(\in, \in\vee\,\mathrm{q})$-fuzzy quasi subalgebra of L by Lemma 3.14. □

Theorem 3.16. *If* A *is an* $(\in, \in\vee\,\mathrm{q})$*-fuzzy quasi subalgebra of* L*, then* $A_t := \{x \in L \mid A(x) \geq t\}$ *is a quasi subalgebra of* L *for all* $t \in (0, 0.5]$*.*

Proof. Let $t \in (0, 0.5]$ and $x, y \in A_t$. Then $A(x) \geq t$ and $A(y) \geq t$, which imply that $x_t \in A$ and $y_t \in A$. It follows that $(x \multimap y)_{M(t,t)} \in\vee\,\mathrm{q}\,A$ so from Lemma 3.11 that

$$A(x \multimap y) \geq M(A(x), A(y), 0.5) \geq M(t, 0.5) = t.$$

Hence $x \multimap y \in A_t$, and A_t is a quasi subalgebra of L. □

Theorem 3.17. *If* A *is a fuzzy set in* L *such that* $A_t := \{x \in L \mid A(x) \geq t\}$ *is a quasi subalgebra of* L *for all* $t \in (0, 0.5]$*, then* A *is an* $(\in, \in\vee\,\mathrm{q})$*-fuzzy quasi subalgebra of* L*.*

Proof. It is sufficient to show that $A(x \multimap y) \geq M(A(x), A(y), 0.5)$ for all $x, y \in L$. Assume that $A(x \multimap y) < M(A(x), A(y), 0.5)$ for some $x, y \in L$. Then there exists $t \in (0, 1)$ such that $A(x \multimap y) < t < M(A(x), A(y), 0.5)$. Thus $x, y \in A_t$ and $t < 0.5$ and so $x \multimap y \in A_t$, that is, $A(x \multimap y) \geq t$, a contradiction. Therefore $A(x \multimap y) \geq M(A(x), A(y), 0.5)$ for all $x, y \in L$. Consequently A is an $(\in, \in\vee\,\mathrm{q})$-fuzzy quasi subalgebra of L. □

For any fuzzy set A in L and $t \in (0, 1]$, we denote

$$A_{1-t} := \{x \in L \mid x_t \mathrm{q} A\} = \{x \in L \mid A(x) + t > 1\}$$

and $[A]_t := A_t \cup A_{1-t} = \{x \in L \mid x_t \in\vee\,\mathrm{q}\,A\}$.

Theorem 3.18. *A fuzzy set A in L is an $(\in, \in\vee\mathrm{q})$-fuzzy quasi subalgebra of L if and only if $[A]_t$ is a quasi subalgebra of L for all $t \in (0,1]$, which is called an $(\in\vee\mathrm{q})$-level quasi subalgebra of A.*

Proof. Let A be an $(\in, \in\vee\mathrm{q})$-fuzzy quasi subalgebra of L. Let $x, y \in [A]_t$. Then $x_t \in\vee\mathrm{q}\, A$ and $y_t \in\vee\mathrm{q}\, A$, that is, $A(x) \geq t$ or $A(x) + t > 1$, and $A(y) \geq t$ or $A(y) + t > 1$. Since $A(x \multimap y) \geq M(A(x), A(y), 0.5)$ by Lemma 3.11, it follows that $A(x \multimap y) \geq M(t, 0.5)$. For otherwise, $A(x \multimap y) < M(t, 0.5)$ which implies that $x_t \overline{\in\vee\mathrm{q}}\, A$ or $y_t \overline{\in\vee\mathrm{q}}\, A$, a contradiction. If $M(t, 0.5) = t$, then $A(x \multimap y) \geq M(t, 0.5) = t$ and so $x \multimap y \in A_t \subset A_t \cup A_{1-t} = [A]_t$. If $M(t, 0.5) = 0.5$, then $A(x \multimap y) \geq M(t, 0.5) = 0.5$ which imply that $A(x \multimap y) + t > 1$ so that $x \multimap y \in A_{1-t} \subset [A]_t$. Therefore $[A]_t$ is a quasi subalgebra of L. Conversely, let A be a fuzzy set in L such that $[A]_t$ is a quasi subalgebra of L for all $t \in (0,1]$. If possible, let $A(x \multimap y) < t < M(A(x), A(y), 0.5)$ for some $t \in (0, 0.5)$. Then $x, y \in A_t \subset [A]_t$, and so $x \multimap y \in [A]_t$. Hence $A(x \multimap y) \geq t$ or $A(x \multimap y) + t > 1$, a contradiction. Therefore $A(x \multimap y) \geq M(A(x), A(y), 0.5)$ for all $x, y \in L$. Thus A is an $(\in, \in\vee\mathrm{q})$-fuzzy quasi subalgebra of L. □

A fuzzy set A in L is said to be *proper* if $\mathrm{Im}(A)$ has at least two elements. Two fuzzy sets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent*.

Theorem 3.19. *Let L have proper quasi subalgebras. A proper $(\in, \in)$-fuzzy quasi subalgebra A of L such that $\#\mathrm{Im}(A) \geq 3$ can be expressed as the union of two proper non-equivalent $(\in, \in)$-fuzzy quasi subalgebras of L.*

Proof. Let A be a proper $(\in, \in)$-fuzzy quasi subalgebra of L with $\mathrm{Im}(A) = \{t_0, t_1, \cdots, t_n\}$, where $t_0 > t_1 > \cdots > t_n = L$ and $n \geq 2$. Then

$$A_{t_0} \subseteq A_{t_1} \subseteq \cdots \subseteq A_{t_n} = L$$

is the chain of $\in$-level quasi subalgebras of A. Define fuzzy sets Φ and Ψ in L by

$$\Phi(x) = \begin{cases} r_1 & \text{if } x \in A_{t_1}, \\ t_2 & \text{if } x \in A_{t_2} \setminus A_{t_1}, \\ \cdots & \\ t_n & \text{if } x \in A_{t_n} \setminus A_{t_{n-1}}, \end{cases}$$

and

$$\Psi(x) = \begin{cases} t_0 & \text{if } x \in A_{t_0}, \\ t_1 & \text{if } x \in A_{t_1} \setminus A_{t_0}, \\ r_2 & \text{if } x \in A_{t_3} \setminus A_{t_1}, \\ t_4 & \text{if } x \in A_{t_4} \setminus A_{t_3}, \\ \cdots & \\ t_n & \text{if } x \in A_{t_n} \setminus A_{t_{n-1}}, \end{cases}$$

respectively, where $t_2 < r_1 < t_1$ and $t_4 < r_2 < t_2$. Then Φ and Ψ are $(\in, \in)$-fuzzy quasi subalgebras of L with

$$A_{t_1} \subseteq A_{t_2} \subseteq \cdots \subseteq A_{t_n} = L$$

and

$$A_{t_0} \subseteq A_{t_1} \subseteq A_{t_3} \subseteq \cdots \subseteq A_{t_n} = L$$

as respective chains of $\in$-level quasi subalgebras, and $\Phi, \Psi \leq A$. Thus Φ and Ψ are non-equivalent, and obviously $\Phi \cup \Psi = A$. This completes the proof. □

Theorem 3.20. *Let A be a proper $(\in, \in\vee\mathrm{q})$-fuzzy quasi subalgebra of L such that $\#\{A(x) \mid A(x) < 0.5\} \geq 2$. Then there exist two proper non-equivalent $(\in, \in\vee\mathrm{q})$-fuzzy quasi subalgebra of L such that A can be expressed as the union of them.*

Proof. Let $\{A(x) \mid A(x) < 0.5\} = \{t_1, t_2, \cdots, t_n\}$, where $t_1 > t_2 > \cdots > t_n$ and $n \geq 2$. Then the chain of $(\in\vee\mathrm{q})$-level quasi subalgebras of A is

$$[A]_{0.5} \subseteq [A]_{t_1} \subseteq [A]_{t_2} \subseteq \cdots \subseteq [A]_{t_n} = L.$$

Let Φ and Ψ be fuzzy sets in L defined by

$$\Phi(x) = \begin{cases} t_1 & \text{if } x \in [A]_{t_1}, \\ t_2 & \text{if } x \in [A]_{t_2} \setminus [A]_{t_1}, \\ \cdots & \\ t_n & \text{if } x \in [A]_{t_n} \setminus [A]_{t_{n-1}}, \end{cases}$$

and

$$\Psi(x) = \begin{cases} A(x) & \text{if } x \in [A]_{0.5}, \\ r & \text{if } x \in [A]_{t_2} \setminus [A]_{0.5}, \\ t_3 & \text{if } x \in [A]_{t_3} \setminus [A]_{t_2}, \\ \cdots & \\ t_n & \text{if } x \in [A]_{t_n} \setminus [A]_{t_{n-1}}, \end{cases}$$

respectively, where $t_3 < r < t_2$. Then Φ and Ψ are $(\in, \in\vee\mathrm{q})$-fuzzy quasi subalgebra of L, and $\Phi, \Psi \leq A$. The chains of $(\in\vee\mathrm{q})$-level quasi subalgebras of Φ and Ψ are, respectively, given by

$$[A]_{t_1} \subseteq [A]_{t_2} \subseteq \cdots \subseteq [A]_{t_n} = L$$

and

$$[A]_{0.5} \subseteq [A]_{t_2} \subseteq \cdots \subseteq [A]_{t_n} = L.$$

Therefore Φ and Ψ are non-equivalent and clearly $A = \Phi \cup \Psi$. This completes the proof. □

Theorem 3.21. *Let A be an $(\in, \in\vee\mathrm{q})$-fuzzy quasi subalgebra of L such that $A(x) > 0.5$ for all $x \in L$. Then A can be expressed as the union of two proper non-equivalent $(\in, \in\vee\mathrm{q})$-fuzzy quasi subalgebras if and only if there are proper quasi subalgebras G and H of L such that $L = G \cup H$.*

Proof. Assume that $L = G \cup H$ for some proper quasi subalgebras G and H of L. Let Φ and Ψ be fuzzy sets in L defined by

$$\Phi(x) = \begin{cases} A(x) & \text{if } x \in G, \\ t_1 < 0.5 & \text{if } x \in L \setminus G, \end{cases} \qquad \Psi(x) = \begin{cases} A(x) & \text{if } x \in H, \\ t_2 < 0.5 & \text{if } x \in L \setminus H. \end{cases}$$

Then $\Phi, \Psi \leq A$, and they are $(\in, \in\vee\mathrm{q})$-fuzzy quasi subalgebras of L. The chains of $(\in\vee\mathrm{q})$-level quasi subalgebras of Φ and Ψ are $G \subseteq L$ and $H \subseteq L$, respectively. Thus Φ and Ψ are non-equivalent and $A = \Phi \cup \Psi$. Conversely, suppose that $A = \Phi \cup \Psi$, where Φ and Ψ are proper non-equivalent $(\in, \in\vee\mathrm{q})$-fuzzy quasi subalgebras of L. Let

$$G = \{x \in L \mid \Phi(x) < 0.5\}, \quad H = \{x \in L \mid \Psi(x) < 0.5\}$$

and $W = \{x \in L \mid (\Phi \cap \Psi)(x) \geq 0.5\}$. Since $\Phi, \Psi \leq A$ and they are non-equivalent, we have $G \neq \emptyset$ and $H \neq \emptyset$. Also, $W \neq \emptyset$ by Theorem 3.12. We now show that $G \cup W$ is a quasi subalgebra of L. Let $x, y \in G \cup W$. If $x \in G$ and $y \in W$, then $\Phi(y) \geq 0.5$, $\Psi(y) \geq 0.5$ and $\Phi(x) < 0.5$. Since $A = \Phi\cup\Psi$ and $A(x) > 0.5$ for all $x \in L$, it follows that $\Psi(x) \geq 0.5$. Assume that $x \multimap y \notin G \cup W$. Then $x \multimap y \notin G$ and $x \multimap y \notin W$, and thus $\Phi(x \multimap y) \geq 0.5$ and $\Psi(x \multimap y) < 0.5$. Using Lemma 3.11, we get $\Psi(x \multimap y) \geq M(\Psi(x), \Psi(y), 0.5)$, which implies that $\Psi(y) < 0.5$ because $\Psi(x) \geq 0.5$ and $\Psi(x \multimap y) < 0.5$. This is a contradiction. Similarly one can obtain a contradiction if $x \in W$ and $y \in G$. Now if $x, y \in G$, then $\Phi(x) < 0.5$ and $\Phi(y) < 0.5$, and so $\Psi(x) \geq 0.5$ and $\Psi(y) \geq 0.5$. Suppose that $x \multimap y \notin G \cup W$. Then $\Phi(x \multimap y) \geq 0.5$ and $\Psi(x \multimap y) < 0.5$. Using Lemma 3.11 again, $\Psi(x \multimap y) < 0.5$ implies $\Psi(x) < 0.5$ or $\Psi(y) < 0.5$. This is a contradiction. Therefore $x \multimap y \in G \cup W$. Finally if $x, y \in W$, then $\Phi(x) \geq 0.5$, $\Phi(y) \geq 0.5$, $\Psi(x) \geq 0.5$, and $\Psi(y) \geq 0.5$. It follows from Lemma 3.11 that

$$\Phi(x \multimap y) \geq M(\Phi(x), \Phi(y), 0.5) \geq 0.5$$

and

$$\Psi(x \multimap y) \geq M(\Psi(x), \Psi(y), 0.5) \geq 0.5$$

so that $x \multimap y \in W \subseteq G \cup W$. Consequently, in any case, $G \cup W$ is a quasi subalgebra of L. Similarly we can show that $H \cup W$ is a quasi subalgebra of L. Also, $L = (G \cup W) \cup (H \cup W)$. This completes the proof. □

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