ALMOST POINTWISE ESTIMATE AND EXTRAPOLATION THEOREM

TAKUYA SOBUKAWA

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Dedicated to Professor Kôzô YABUTA on his Sixtieth birthday

Abstract. We shall give a useful decomposition of function related to its non-increasing rearrangement in order to get some extrapolation estimates "nearer" to L^1 which contain Yano's classical work.

1. Introduction and Result

Let (Ω, μ) be a σ-finite measure space. In extrapolation theory on L^p-spaces, we treat the operator which satisfies the following assumptions, so called "Yano's condition":

Condition . Let $1 < p_1 < \infty$ and fix it.

1. T is a sub-additive operator on $L^p(\Omega, \mu)$ for any p, $1 < p \leq p_1$, *i.e.* $|T(f+g)| \leq |Tf| + |Tg|$ *a.e.* for any $f, g \in L^p(\Omega, \mu)$.
2. For any $f \in L^p(\Omega, \mu)$, $1 < p \leq p_1$,

$$\|Tf\|_{L^p(\Omega)} \leq \frac{A}{(p-1)^\alpha} \|f\|_{L^p(\Omega)} \tag{1.1}$$

Here, positive constants A and α are independent of p and f.

We can find many operators satisfying such conditions: Hilbert transform, Riesz transform, Calderon-Zygmund operators, multiple Wiener integral operators, Hardy-Littlewood many maximal operator, etc. For such operators, we cannot get L^1 boundedness but, instead of it, S.Yano proved that such T is bounded from $L^1 \log^\alpha L$ to L^1 in the case $\mu(\Omega) < \infty$(Yano's extrapolation theorem [8]).

In general case, $\mu(\Omega) \leq \infty$, the author proved the following extrapolation estimates between some Orlicz spaces which include Yano's result([6],[7]): For $1 < q \leq p_1$,

$$\|Tf\|_{L^1 + L^q(\Omega)} \leq \frac{C}{(q-1)^\alpha} \|f\|_{L^1 \log^\alpha L + L^q(\Omega)} \tag{1.2}$$

and, as its consequece,

$$\|Tf\|_{L^1 + L^1 \log^{-\alpha-\varepsilon} L(\Omega)} \leq C_\varepsilon \|f\|_{L^1 \log^\alpha L + L^1 \log^{-\varepsilon} L(\Omega)}, \tag{1.3}$$

for $\varepsilon > 0$. Here we denote $L^{\Phi_0} + L^{\Phi_1}(\Omega) = L^\Phi(\Omega)$, $\Phi = \min\{\Phi_0, \Phi_1\}$ for any two Orlicz classes $L^{\Phi_0}(\Omega)$ and $L^{\Phi_1}(\Omega)$.

Our aim is to get some boundedness between function spaces "near to L^1", however, counter examples are known to (1.3) for $\varepsilon = 0$. In this paper, instead of it, we shall show the following estimate:

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Theorem 1. *Fix $\alpha > 0$ and $1 < p_1 < \infty$. Let T be a sub-additive operator on $L^p(\Omega,\mu)$ for any p, $1 < p \le p_1$, i.e.*

$$|T(f+g)| \le |Tf| + |Tg| \quad a.e.\mu$$

for any simple function f and g and satisfy weak L^p boundedness

$$\|Tf\|_{(p,\infty)} \le \frac{A}{(p-1)^\alpha}\|f\|_{p,1} \tag{1.4}$$

for any p, $1 < p < p_1$. Then, we have

$$\|Tf\|_{(1,\infty;0,-\alpha)} \le C\|f\|_{1,1;\alpha,0} \tag{1.5}$$

for any $f \in L^1 + L^1 \log^\alpha L(\Omega)$. Here,

$$\begin{aligned} \|g\|_{p,1;\alpha_0,\alpha_\infty} &= \int_0^\infty t^{\frac{1}{p}}(1+\log^+\frac{1}{t})^{\alpha_0}(1+\log^+ t)^{\alpha_\infty} g^*(t)\frac{dt}{t}, \\ \|g\|_{(p,\infty;\alpha_0,\alpha_\infty)} &= \sup_{t>0}\left(t^{\frac{1}{p}}(1+\log^+\frac{1}{t})^{\alpha_0}(1+\log^+ t)^{\alpha_\infty} g^{**}(t)\right) \end{aligned} \tag{1.6}$$

for $\alpha_0, \alpha_\infty \in \mathbb{R}$, with

$$g^*(t) = \inf\{\lambda > 0;\ \mu(\{x \in \Omega : |g(x)| > \lambda\}) \le t\} \quad and \quad g^{**}(t) = \frac{1}{t}\int_0^t g^*(s)ds$$

for $t \in (0,\infty)$ and we write $\|g\|_{p,1} = \|g\|_{p,1;0,0}$ and $\|g\|_{(p,\infty)} = \|g\|_{(p,\infty;0,0)}$, simply.

Moreover, we can also show the following Koizumi type estimate after (1.2) (also see [4]), similarly and indendently to Theorem 1.

Theorem 2. *Under the same assumption of Theorem 1, we have*

$$\|Tf\|_{m(1,q),1;0,0} \le C\|f\|_{m(1,q),1;\alpha,0} \tag{1.7}$$

for any $f \in L_{m(1,q),1;\alpha,0}$, $1 < q < p_1$ and

$$\|Tf\|_{m(1,p_1),1;0,0} \le C\|f\|_{m(1,p_1),1;\alpha,1} \tag{1.8}$$

Here,

$$\|g\|_{m(p,q),1;\alpha_0,\alpha_\infty} = \int_0^\infty \min(t^{\frac{1}{p}}, t^{\frac{1}{q}})(1+\log^+\frac{1}{t})^{\alpha_0}(1+\log^+ t)^{\alpha_\infty} g^*(t)\frac{dt}{t}.$$

Remark 1. For any $\alpha \ge 0$, we may prove

$$\|f\|_{1,1;\alpha,0} = \int_0^\infty (1+\log^+\frac{1}{t})^\alpha f^*(t)dt \approx \int_\Omega |f(x)|(1+\log^+|f(x)|)^\alpha d\mu(x)$$

and $L^1 + L^1\log^\alpha L(\Omega) = \{f : \|f\|_{1,1;\alpha,0} < \infty\}$ (see [1]).

Remark 2. As is known, the condition (1.4) is weaker than (1.1). On the left hand side of (1.5),

$$\|Tf\|_{(1,\infty;0,-\alpha)} = \sup_{t>0}\frac{\int_0^t (Tf)^*(s)ds}{(1+\log^+ t)^\alpha} = \int_0^1 (Tf)^*(s)ds + \sup_{t>1}\frac{\int_1^t (Tf)^*(s)ds}{(1+\log t)^\alpha}$$

and

$$\int_0^1 (Tf)^*(s)ds \ge \int_{|Tf|\ge M} |Tf(x)|d\mu(x)$$

for some $M > 0$. Therefore, in the case $\mu(\Omega) < \infty$, our result implies Yano's theorem. After Remark 1 above, it is easy to show that (1.7) or (1.8) does so, too.

2. Proof of the theorems

We note that $f^*(t) \to 0$ $(t \to \infty)$ and $|f| < \infty$, μ-*a.e.* for any $f \in L^1 + L^1 \log^\alpha L(\Omega)$. Now, we shall decompose every function $f \in L^1 + L^1 \log^\alpha L$ as follows.

First, we consider a family of pairwise disjoint measurable sets

$$E_n = \{x \in \Omega : f^*(2^{n+1}) < |f(x)| \le f^*(2^n)\}, \qquad n \in \mathbb{Z}. \tag{2.1}$$

Here, if $f^*(2^n) = f^*(2^{n+1})$, we define $E_n = \emptyset$. Now, we put

$$f_n(x) = \begin{cases} f(x) & x \in E_n \\ 0 & \text{otherwise.} \end{cases} \qquad (n \in \mathbb{Z}). \tag{2.2}$$

It is easy to show

1. $f(x) = \sum_{n=-\infty}^{\infty} f_n(x)$ for any $x \in \Omega$,
2. $\mu(E_n) \le 2^{n+1}$,
3. $|f_n(x)|$, $(f_n)^*(t) \le f^*(2^n)$

for any n.

For simplicity, we shall write $\rho(p) = A(p-1)^{-\alpha}$. ¿From the assumption (1.4), we may have

$$\begin{aligned} s^{\frac{1}{p}}(Tf_n)^{**}(s) &\le \rho(p) \int_0^\infty t^{\frac{1}{p}} f_n^*(t) \frac{dt}{t} \\ &\approx \rho(p) \sum_{i=-\infty}^{\infty} (2^i)^{\frac{1}{p}} f_n^*(2^i) = \rho(p) \sum_{i=-\infty}^{n+1} (2^i)^{\frac{1}{p}} f_n^*(2^i) \\ &\le 2\rho(p)(2^{n+1})^{\frac{1}{p}} f^*(2^n) \le 4\rho(p)(2^n)^{\frac{1}{p}} f^*(2^n) \end{aligned}$$

for any $n \in \mathbb{Z}$. Put $s = 2^k$, $k \in \mathbb{Z}$, we have

$$(Tf_n)^{**}(2^k) \le 4\rho(p)(2^{n-k})^{\frac{1}{p}} f^*(2^n)$$

$(-\infty < k < \infty,\ -\infty < n < \infty)$. Taking infimum with respect to p, we may get

$$(Tf_n)^{**}(2^k) \le 4 \inf_p \left(\rho(p)(2^{n-k})^{\frac{1}{p}}\right) f^*(2^n) \tag{2.3}$$

Summing up with respect to n,

$$\begin{aligned} (Tf)^{**}(2^k) &\le \sum_{n=-\infty}^{\infty} (Tf_n)^{**}(2^k) \\ &\le 4 \sum_{n=-\infty}^{\infty} \inf_p \left(\rho(p)(2^{n-k})^{\frac{1}{p}}\right) f^*(2^n) \end{aligned} \tag{2.4}$$

and we call it "almost pointwise estimate". Note,

$$\inf_{1<p<p_1} \left(\frac{p}{p-1}\right)^\alpha (2^{n-k})^{\frac{1}{p}} \approx \begin{cases} (k-n)^\alpha 2^{n-k} & (n < k) \\ (2^{n-k})^{\frac{1}{p_1}} & (n \ge k), \end{cases} \tag{2.5}$$

we conclude

$$(Tf)^{**}(2^k) \le \sum_{n=-\infty}^{k-1} (k-n)^\alpha 2^{n-k} f^*(2^n) + \sum_{n=k}^{\infty} (2^{n-k})^{\frac{1}{p_1}} f^*(2^n). \tag{2.6}$$

Multiplying 2^k, we have

(2.7)
$$\begin{aligned}\sup_{0<t<1} t(Tf)^{**}(t) &\approx \sup_{k\le 0} 2^k(Tf)^{**}(2^k)\\
&\le \sup_{k\le 0}\left[\sum_{n=-\infty}^{k-1}(k-n)^\alpha 2^n f^*(2^n)+\sum_{n=k}^{\infty}(2^k)^{1-\frac{1}{p_1}}(2^n)^{\frac{1}{p_1}}f^*(2^n)\right]\\
&\le \sup_{k\le 0}\left[\sum_{n=-\infty}^{k-1}(0-n)^\alpha 2^n f^*(2^n)+\sum_{n=k}^{\infty}2^n f^*(2^n)\right]\\
&\le \sum_{n=-\infty}^{0}(1-n)^\alpha 2^n f^*(2^n)+\sum_{n=1}^{\infty}2^n f^*(2^n)\\
&\approx \int_0^1 (1-\log t)^\alpha f^*(t)dt+\int_1^\infty f^*(t)dt.\end{aligned}$$

On the other hand, multiplying $2^k/k^\alpha$, we get

$$\begin{aligned}&\sup_{k\ge 1}\frac{2^k}{k^\alpha}(Tf)^{**}(2^k) \approx \sup_{1\le t<\infty}\frac{t(Tf)^{**}(t)}{(1+\log t)^\alpha}\\
&\le \sup_{k\ge 1}\left[\sum_{n=-\infty}^{k-1}(1-\frac{n}{k})^\alpha 2^n f^*(2^n)+\sum_{n=k}^{\infty}\frac{2^k}{k^\alpha}(2^{n-k})^{\frac{1}{p_1}}f^*(2^n)\right]\\
&\le \sup_{k\ge 1}\left[\sum_{n=-\infty}^{-1}(1-\frac{n}{k})^\alpha 2^n f^*(2^n)+\sum_{n=0}^{k-1}(1-\frac{n}{k})^\alpha 2^n f^*(2^n)+\sum_{n=k}^{\infty}2^n f^*(2^n)\right]\\
&\le \sum_{n=-\infty}^{0}(1-n)^\alpha 2^n f^*(2^n)+\sum_{n=0}^{\infty}2^n f^*(2^n)\\
&\approx \int_0^1 (1-\log t)^\alpha f^*(t)dt+\int_1^\infty f^*(t)dt\end{aligned} \tag{2.8}$$

and Theorem 1 is proved.

Next, we assume $f \in L_{m(1,p_1),1;\alpha,1}(\Omega)$ and use the decomposition (2.6). Putting $k=0$ in (2.6),

$$\begin{aligned}(Tf)^{**}(1) &= \int_0^1 s(Tf)^*(s)\frac{ds}{s}\\
&\le \sum_{n=-\infty}^{-1}(1-n)^\alpha 2^n f^*(2^n)+\sum_{n=0}^{\infty}(2^n)^{\frac{1}{p_1}}f^*(2^n)\\
&\approx \int_0^1 (1-\log t)^\alpha f^*(t)dt+\int_1^\infty t^{\frac{1}{p_1}}f^*(t)\frac{dt}{t}.\end{aligned} \tag{2.9}$$

On the other hand, multiplying $(2^k)^{\frac{1}{p_1}}$ to the left hand side of (2.6) and summing up with respect to k, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}(2^k)^{\frac{1}{p_1}}(Tf)^{**}(2^k)\\
&\approx\sum_{k=0}^{\infty}(2^k)^{\frac{1}{p_1}}\frac{1}{2^k}\sum_{n=-\infty}^{k}2^n(Tf)^*(2^n)\\
&\geq\sum_{k=0}^{\infty}\sum_{n=0}^{k}(2^k)^{\frac{1}{p_1}}2^{-k}2^n(Tf)^*(2^n)=\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}(2^k)^{\frac{1}{p_1}}2^{-k}2^n(Tf)^*(2^n)\\
&\approx\sum_{n=0}^{\infty}(2^n)^{\frac{1}{p_1}}(Tf)^*(2^n)\approx\int_1^{\infty}t^{\frac{1}{p_1}}(Tf)^*(t)\frac{dt}{t}
\end{aligned}\tag{2.10}$$

and from the right handside,

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left[\sum_{n=-\infty}^{k-1}(k-n)^{\alpha}2^n(2^k)^{\frac{1}{p_1}-1}f^*(2^n)+\sum_{n=k}^{\infty}(2^n)^{\frac{1}{p_1}}f^*(2^n)\right]\\
&=\sum_{k=0}^{\infty}\left[\left(\sum_{n=-\infty}^{0}+\sum_{n=1}^{k}\right)(k-n)^{\alpha}2^n(2^k)^{\frac{1}{p_1}-1}f^*(2^n)+\sum_{n=k}^{\infty}(2^n)^{\frac{1}{p_1}}f^*(2^n)\right]\\
&\leq\sum_{n=-\infty}^{0}2^n(1-n)^{\alpha}\sum_{k=1}^{\infty}(1+k)^{\alpha}(2^k)^{\frac{1}{p_1}-1}f^*(2^n)\\
&\qquad+\sum_{n=0}^{\infty}(2^n)^{\frac{1}{p_1}}\sum_{k=0}^{\infty}k^{\alpha}(2^k)^{\frac{1}{p_1}-1}f^*(2^n)+\sum_{n=0}^{\infty}n(2^n)^{\frac{1}{p_1}}f^*(2^n)\\
&\approx\sum_{n=-\infty}^{0}2^n(1-n)^{\alpha}f^*(2^n)+\sum_{n=0}^{\infty}n(2^n)^{\frac{1}{p_1}}f^*(2^n)\\
&\approx\int_0^1(1+\log\frac{1}{t})^{\alpha}f^*(t)dt+\int_1^{\infty}t^{\frac{1}{p_1}}(1+\log t)f^*(t)dt.
\end{aligned}\tag{2.11}$$

Therefore, we conclude

$$\begin{aligned}
&\int_0^1(Tf)^*(t)dt+\int_1^{\infty}t^{\frac{1}{p_1}}(Tf)^*(t)\frac{dt}{t}\\
&\qquad\leq C\left[\int_0^1(1+\log\frac{1}{t})^{\alpha}f^*(t)dt+\int_1^{\infty}t^{\frac{1}{p_1}}(1+\log t)f^*(t)\frac{dt}{t}\right]
\end{aligned}\tag{2.12}$$

and (1.8) is proved.

For $f\in L_{m(1,q),1;\alpha,0}(\Omega)$, $1<q<p_1$, we may get

$$\int_0^1 s(Tf)^*(s)\frac{ds}{s}\leq C\left[\int_0^1(1-\log t)^{\alpha}f^*(t)dt+\int_1^{\infty}t^{\frac{1}{p_1}}f^*(t)\frac{dt}{t}\right]\tag{2.13}$$

similarly to (2.9). Moreover, multiplying $(2^k)^{\frac{1}{q}}$ to (2.6) and summing up with respect to k, we get

$$\sum_{k=0}^{\infty}(2^k)^{\frac{1}{p_1}}(Tf)^{**}(2^k)\leq C\left[\int_0^1(1+\log\frac{1}{t})^{\alpha}f^*(t)dt+\int_1^{\infty}t^{\frac{1}{q}}f^*(t)\frac{dt}{t}\right]\tag{2.14}$$

similarly to (2.10) and (2.11). Therefore (1.7), then, Theorem 2 is proved.

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Department of mathematics education, Okayama University, 3-1-1 Tsushima-naka, Okayama, 700-8530, JAPAN
E-mail address: sobu@cc.okayama-u.ac.jp