MOND-PEČARIĆ METHOD FOR A MEAN-LIKE TRANSFORMATION OF OPERATOR FUNCTIONS

Akemi Matsumoto* and Masaru Tominaga**

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Abstract. As a generalization of the quasi-arithmetic mean, we consider a mean-like transformation of operator functions. Let Φ be a unital positive linear map of $B(H)$, the algebra of all bounded linear operators on a Hilbert space H, and $f(t)$ (resp. $g(t)$) a continuous function on an interval $[m, M]$ (resp. $f([m, M])$). Then it is defined by $(g \circ \Phi \circ f)(A)$ for a selfadjoint operator A with $m \le A \le M$. We give a lower bound of the difference between $(g \circ \Phi \circ f)(A)$ and $\Phi(A)$. Precisely we prove that if $f(t)$ is concave on $[m, M]$ and $g(t)$ is increasing and convex on $f([m, M])$, then for each $\lambda \in \mathbb{R}$, $(g \circ \Phi \circ f)(A) - \lambda\Phi(A) \ge \min_{t\in[m,M]} \{g(\alpha_f t + \beta_f) - \lambda t\}$ where $\alpha_f := \frac{f(M)-f(m)}{M-m}$ and $\beta_f := \frac{Mf(m)-mf(M)}{M-m}$. It is an extension of our previous estimation for $\Phi = \omega_x$, the vector state for a unit vector $x \in H$.

1 Introduction Let $f(t)$ be a strictly monotone, continuous function on an interval $[m, M]$ and $w = (w_1, \dots , w_n)$ a weight, i.e., $\sum_{i=1}^n w_i = 1$ and $w_i \ge 0$. Then the quasi-arithmetic mean is defined by $f^{-1}(\sum_{i=1}^n w_i f(t_i))$ for $t_1, \dots , t_n \in [m, M]$, cf.[2], [1]. Moreover if $f(t)$ is concave on $[m, M]$, then the quasi-arithmetic mean and arithmetic mean inequality

$$f^{-1}\left(\sum_{i=1}^n w_i f(t_i)\right) \le \sum_{i=1}^n w_i t_i$$

follows from the classical Jensen inequality (see (7)).

It can be expressed as

$$f^{-1}(\langle f(A)x, x\rangle) \le \langle Ax, x\rangle \tag{1}$$

for a selfadjoint operator A on H with $m \le A \le M$ and a unit vector $x \in H$. By the way, replacing f^{-1} in (1) to an increasing function g on $f([m, M])$, we considered a low bound of $g(\langle f(A)x, x\rangle)$ by the arithmetic mean $\langle Ax, x\rangle$ in our previous notes [6], [7] and [8]. As a matter of fact, we showed that for every real number $\lambda > 0$

$$g(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle \ge \min_{t\in[m,M]} \{g(\alpha_f t + \beta_f) - \lambda t\} \tag{2}$$

holds for all unit vectors $x \in H$ where

$$\alpha_f := \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad \beta_f := \frac{Mf(m) - mf(M)}{M - m}.$$

It suggests us that a unital positive linear map Φ on $B(H)$, the algebra of all bounded linear operators on H, is regarded as a mean-like transformation of operator functions.

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Namely we can define a new operator function $g \circ \Phi \circ f$ for given f and g. For example, if we take $\Phi(A) = \langle Ax, x\rangle$ for some unit vector $x \in H$, then (1) is rephrased by $(f^{-1} \circ \Phi \circ f)(A) \le \Phi(A)$.

As a continuation of our previous notes, we estimate a lower bound of the difference between $(g \circ \Phi \circ f)(A)$ and $\Phi(A)$ in this note. As an application of Mond-Pečarić method [5], we prove: Let A be a selfadjoint operator on a Hilbert space H with $m \le A \le M$. Let $f(t)$ be a concave function on $[m, M]$ and $g(t)$ an increasing convex function on $f([m, M])$. Then for every real number λ

$$g(\Phi(f(A))) - \lambda\Phi(A) \ge \min_{t\in[m,M]} \{g(\alpha_f t + \beta_f) - \lambda t\}.$$

2 Estimations of $g(\Phi(f(A))) - \lambda\Phi(A)$ We give a lower bound of $g(\Phi(f(A)))$ by $\Phi(A)$ without the convexity of $g(t)$.

Theorem 1. *Let A be a selfadjoint operator on a Hilbert space H with $m \le A \le M$ for some $m < M$. Let $f(t)$ be a concave function on $[m, M]$ with $f(m) \ne f(M)$ and $g(t)$ be a continuous function on $f([m, M])$. Let Φ be a unital positive linear map on $B(H)$. Then for every real number λ with $\lambda\alpha_f > 0$*

$$g(\Phi(f(A))) - \lambda\Phi(A) \ge \min_{u\in f([m,M])} \left\{g(u) - \frac{\lambda}{\alpha_f}(u - \beta_f)\right\}. \tag{3}$$

Precisely, if $\lambda > 0$ and $\alpha_f > 0$ (resp. $\lambda < 0$ and $\alpha_f < 0$), then

$$g(\Phi(f(A))) - \lambda\Phi(A) \ge \min_{t\in\left[m, \frac{f_{\max}-\beta_f}{\alpha_f}\right]} \{g(\alpha_f t + \beta_f) - \lambda t\} \tag{4}$$

$$\left(\textit{resp. } g(\Phi(f(A))) - \lambda\Phi(A) \ge \min_{t\in\left[\frac{f_{\max}-\beta_f}{\alpha_f}, M\right]} \{g(\alpha_f t + \beta_f) - \lambda t\}\right) \tag{5}$$

where $f_{\max} := \max_{t\in[m,M]} f(t)$.

Proof. Since $f(t)$ is concave, we have $f(A) \ge \alpha_f A + \beta_f$, and hence $\Phi(f(A)) \ge \alpha_f \Phi(A) + \beta_f$. So it follows from $\lambda\alpha_f > 0$ that

$$\lambda\Phi(A) \le \frac{\lambda}{\alpha_f} (\Phi(f(A)) - \beta_f),$$

so that

$$\begin{aligned} g(\Phi(f(A))) - \lambda\Phi(A) &\ge g(\Phi(f(A))) - \frac{\lambda}{\alpha_f} (\Phi(f(A)) - \beta_f) \\ &\ge \min_{u\in f([m,M])} \left\{g(u) - \frac{\lambda}{\alpha_f}(u - \beta_f)\right\} \end{aligned}$$

by $\sigma(\Phi(f(A))) \subset f([m, M])$.

Moreover if $\lambda > 0$ (and $\alpha_f > 0$), then for every $u \in f([m, M]) = [f(m), f_{\max}]$ the equation $u = \alpha_f t + \beta_f$ has a unique solution $t = t_u = \frac{u-\beta_f}{\alpha_f} \in \left[m, \frac{f_{\max}-\beta_f}{\alpha_f}\right]$. Since

$$g(u) - \frac{\lambda}{\alpha_f}(u - \beta_f) = g(\alpha_f t_u + \beta_f) - \lambda t_u,$$

(3) assures (4).

On the other hand, if $\lambda < 0$ and $\alpha_f < 0$, the proof of (5) is similar to the above. □

We here note that the condition $\lambda\alpha_f > 0$ in Theorem 1 is needed.

Remark. Let $f(t) := 2-(t-1)^2$ on some closed interval I_f and $g(t) := (t-\frac{7}{3})^2$ on $[-2,2]$. Suppose that $\Phi(C) := VCV^*$ for all $C \in M_4(\mathbb{C})$ where $V = \frac{1}{\sqrt{3}}\left[\begin{smallmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{smallmatrix}\right]$. Then (3) does not hold for $\lambda\alpha_f = -\frac{2}{3}$. Indeed,

(i) Let $I_f = [-1,2]$ and $A := \begin{bmatrix} 2 & & & \\ & 0 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$. Then we have $\alpha_f = 1\ (>0)$, $\beta_f = -1$, $f(A) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -2 & \\ & & & 2 \end{bmatrix}$, $\Phi(A) = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\Phi(f(A)) = \frac{1}{3}\begin{bmatrix} 0 & 5 \\ 3 & 1 \end{bmatrix}$ and $g(\Phi(f(A))) = \frac{1}{9}\begin{bmatrix} 64 & -65 \\ -39 & 51 \end{bmatrix}$. This implies that

$$\min_{u\in f([m,M])}\left\{g(u)-\frac{\lambda}{\alpha_f}(u-\beta_f)\right\} = \min_{u\in[-2,2]}\left\{\left(u-\frac{7}{3}\right)^2-\lambda(u+1)\right\}$$
$$= \min_{u\in[-2,2]}\left\{\left(u-\frac{14+3\lambda}{6}\right)^2-\frac{\lambda^2}{4}-\frac{10\lambda}{3}\right\}.$$

Here we put $\lambda = -\frac{2}{3}\ (<0)$. Then we have $\min_{u\in f([m,M])}\left\{g(u)-\frac{\lambda}{\alpha_f}(u-\beta_f)\right\} = \frac{19}{9}$ and so

$$g(\Phi(f(A))) - \lambda\Phi(A) - \min_{t\in[m,M]}\{g(\alpha_f t+\beta_f)-\lambda t\}$$
$$= \frac{1}{9}\begin{bmatrix} 64 & -65 \\ -39 & 51 \end{bmatrix} - \left(-\frac{2}{3}\right)\times\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \frac{19}{9} = \frac{1}{9}\begin{bmatrix} 47 & -63 \\ -37 & 32 \end{bmatrix} \ngeq 0.$$

(ii) Let $I_f = [0,3]$ and $A := \begin{bmatrix} 0 & & & \\ & 2 & & \\ & & 3 & \\ & & & 1 \end{bmatrix}$. As in the above (i), we have $\alpha_f = -1\ (<0)$, $\beta_f = 1$, $\Phi(A) = \frac{1}{3}\begin{bmatrix} 5 & -1 \\ -1 & 6 \end{bmatrix}$ and $g(\Phi(f(A))) = \frac{1}{9}\begin{bmatrix} 58 & -39 \\ -39 & 45 \end{bmatrix}$. Moreover for $\lambda = \frac{2}{3}\ (>0)$, it follows that

$$\min_{u\in f([m,M])}\left\{g(u)-\frac{\lambda}{\alpha_f}(u-\beta_f)\right\} = \min_{u\in[-2,2]}\left\{\left(u-\frac{14-3\lambda}{6}\right)^2-\frac{\lambda^2}{4}+\frac{4\lambda}{3}\right\} = \frac{7}{9},$$

so that

$$g(\Phi(f(A))) - \lambda\Phi(A) - \min_{t\in[m,M]}\{g(\alpha_f t+\beta_f)-\lambda t\} = \frac{1}{9}\left[\begin{smallmatrix} 41 & -37 \\ -37 & 26 \end{smallmatrix}\right] \ngeq 0.$$

In Theorem 1, if $f(t)$ is increasing (resp. decreasing), then $f_{\max} = f(M)(=\alpha_f M+\beta_f)$ (resp. $f_{\max} = f(m)(=\alpha_f m+\beta_f)$). So the following corollary is easily obtained and corresponds to (2):

Corollary 2. *Let the hypothesis of Theorem 1 be satisfied and $f(t)$ be monotone. Then for every real number λ with $\lambda\alpha_f > 0$*

$$g(\Phi(f(A))) - \lambda\Phi(A) \geq \min_{t\in[m,M]}\{g(\alpha_f t+\beta_f)-\lambda t\}.$$

3 The main result Let $f(t)$ be a concave function on $[m,M]$. In this section, we give a lower bound of $g(\Phi(f(A)))-\lambda\Phi(A)$ assuming the convexity of $g(t)$. First of all, suppose that $g(t)$ is increasing and $g(t)>0$. Since $f(t)$ is concave, it follows that $\Phi(f(A)) \geq \alpha_f\Phi(A)+\beta_f$. By [3, Theorem 2.2], we have $g(\Phi(f(A))) \geq \left\{\min_{r\in I_f}\frac{g(r)}{\alpha_{fg}r+\beta_{fg}}\right\} g(\alpha_f\Phi(A)+\beta_f)$ where $\alpha_{fg} := \frac{g(f_{\max})-g(f_{\min})}{f_{\max}-f_{\min}}$ and $\beta_{fg} := \frac{f_{\max}g(f_{\min})-f_{\min}g(f_{\max})}{f_{\max}-f_{\min}}$ for $f_{\max} := \max_{t\in[m,M]} f(t)$ and $f_{\min} := \min_{t\in[m,M]} f(t)$ and I_f is the closed interval by $f(m)$ and $f(M)$. Hence it follows that for every $\lambda > 0$

$$g(\Phi(f(A)))-\lambda\Phi(A) \geq \min_{t\in[m,M]}\left[\left\{\min_{r\in I_f}\frac{g(r)}{\alpha_{fg}r+\beta_{fg}}\right\} g(\alpha_f t+\beta_f)-\lambda t\right]. \tag{6}$$

As compared with (2), we think about the necessity of the constant $\min_{r\in I_f}\frac{g(r)}{\alpha_{fg}r+\beta_{fg}}$ in (6). We pay our attention to the Jensen inequality in [4]: Let C be a selfadjoint operator on a Hilbert space H. Let $f(t)$ be a concave function on an interval containing $\sigma(C)$. Then for every unit vector $x\in H$

$$\langle f(C)x,x\rangle \leq f(\langle Cx,x\rangle). \tag{7}$$

Theorem 3. *Let A be a selfadjoint operator on a Hilbert space H with $m \leq A \leq M$ for some $m<M$. Let $f(t)$ be a concave function on $[m,M]$ and $g(t)$ be an increasing convex function on $f([m,M])$. Let Φ be a unital positive linear map on $B(H)$. Then for every real number λ*

$$g(\Phi(f(A)))-\lambda\Phi(A) \geq \min_{t\in[m,M]}\{g(\alpha_f t+\beta_f)-\lambda t\}. \tag{8}$$

Proof. Since $f(t)$ is concave, we have $f(A) \geq \alpha_f A+\beta_f$. So the inequality $\Phi(f(A)) \geq \alpha_f\Phi(A)+\beta_f$ holds, i.e.,

$$\langle\Phi(f(A))x,x\rangle \geq \alpha_f\langle\Phi(A)x,x\rangle+\beta_f$$

for all unit vectors $x\in H$. Since $g(t)$ is an increasing convex function on $f([m,M])$, we have

$$\langle g(\Phi(f(A)))x,x\rangle \geq g(\langle\Phi(f(A))x,x\rangle) \geq g(\alpha_f\langle\Phi(A)x,x\rangle+\beta_f)$$

by (7). Hence it follows from $m \leq \Phi(A) \leq M$ that

$$\begin{aligned}\langle g(\Phi(f(A))x,x)\rangle-\lambda\langle\Phi(A)x,x\rangle &\geq g(\alpha_f\langle\Phi(A)x,x\rangle+\beta_f)-\lambda\langle\Phi(A)x,x\rangle\\ &\geq \min_{t\in[m,M]}\{g(\alpha_f t+\beta_f)-\lambda t\},\end{aligned}$$

and the proof is complete. □

Comparing with Remark of Theorem 1, we mention the case $\lambda\alpha_f<0$ in Theorem 3.

Corollary 4. *Let A be a selfadjoint operator on H with $m\leq A\leq M$ for some $m<M$. Let $f(t)$ be a concave function on $[m,M]$ and $g(t)$ be an increasing function on $f([m,M])$. Let Φ be a unital positive linear map on $B(H)$.*
(i) *If $\alpha_f>0$, then for every $\lambda<0$*

$$g(\Phi(f(A)))-\lambda\Phi(A) \geq g(\alpha_f m+\beta_f)-\lambda m.$$

(ii) *If* $\alpha_f < 0$*, then for every* $\lambda > 0$

$$g(\Phi(f(A))) - \lambda\Phi(A) \geq g(\alpha_f M + \beta_f) - \lambda M.$$

Proof. If $\alpha_f > 0$, then $f(t) \geq f(m)$ for all $t \in [m, M]$ by the concavity of $f(t)$. Hence we have $f(A) \geq f(m)$, and so $\Phi(f(A)) \geq f(m)$. Since $g(t)$ is increasing and for $\lambda < 0$,

$$g(\Phi(f(A))) - \lambda\Phi(A) \geq g(f(m)) - \lambda m = g(\alpha_f m + \beta_f) - \lambda m.$$

The latter is shown similarly. □

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*) Higashi Toyonaka Senior Highschool, Toyonaka, Osaka 565-0084, Japan.
Email address : m@higashitoyonaka.osaka-c.ed.jp

**) Ikuei-nishi Senior Highschool, Mimatsu, Nara 631-0074, Japan
Email address : m-tommy@sweet.ocn.ne.jp