# FUZZY DEDUCTIVE SYSTEMS OF HS-ALGEBRAS

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Abstract. In this paper, we introduce the concept of fuzzy deductive systems of HS-algebras, i.e. Hilbert algebras with the additional multiplication distributive with respect to the basic operation, and investigate some related properties. Moreover, we provide characterizations of Noetherian HS-algebra in term of fuzzy deductive systems.

**1. Introduction and Preliminaries** In 1966, Diego [10] introduced the notion of Hilbert algebras and deductive systems, and provided various properties. The theory of Hilbert algebras and deductive systems was further developed by Busneag in [1, 2, 3]. Y.B.Jun [13, 14, 15] construct an extension of a fuzzy deductive system in a Hilbert algebra. In this paper, we introduce a new class of algebras related to Hilbert algebras and semigroups, called an HS-algebra. We fuzzify the concept of deductive systems of HS-algebras and investigate some related properties. Moreover, we provide characterizations of Noetherian HS-algebra in term of fuzzy deductive systems.

We now review some concepts and properties that will be useful in our results.

Definition 1.1 ([10]) A Hilbert algebra $H$ is an algebra $(H, \to, 1)$ satisfying the following conditions.

(i) $x \to (y \to x) = 1$,

(ii) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,

(iii) if $x \to y = y \to x = 1$, then $x = y$.

If $A$ is a Hilbert algebra, then the relation $x \leq y$ if and only if $x \to y = 1$ is a partial ordering on $A$; which is called the natural ordering on $A$; with respect to this ordering 1 is the largest element of $A$.

A bounded Hilbert algebra is a Hilbert algebra with a smallest element 1 relative to natural ordering.

A subset $S$ of a bounded Hilbert algebra $H$ is called a Hilbert subalgebra of $H$ if $1 \in S$ and $x, y \in S \Rightarrow x \to y \in S$.

Proposition 1.2 ([6]) A Hilbert algebra $H$ has the following properties for any $x, y, z \in H$:

(1) $x \leq y \to x$

(2) $x \to 1 = 1$

(3) $x \to (y \to z) = (x \to y) \to (x \to z)$

(4) $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$

(5) $x \to (y \to z) = y \to (x \to z)$

(6) $x \leq (x \to y) \to y$

(7) $((x \to y) \to y) \to y = x \to y$

(8) $1 \to x = x$

(9) $x \to y \leq (y \to z) \to (x \to z)$

(10) if $x \leq y$, then $z \to x \leq z \to y$ and $y \to z \leq x \to z$

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Definition 1.3 ([8]) A subset $A$ of a Hilbert algebra $H$ is called a deductive system of $H$ if it satisfies:

(i) $1 \in A$

(ii) if $x, x \to y \in A$, then $y \in A$.

We now review some fuzzy logic concepts. A fuzzy set in a set $X$ is a function $\mu : X \to [0,1]$. For any $t \in [0,1]$, the set $U(\mu; t) = \{x \in X \mid \mu(x) \geq t\}$ is called a level subset of $\mu$.

Definition 1.4 A fuzzy set $\mu$ in a Hilbert algebra $H$ is called a fuzzy deductive system of $H$ if

(i) $\mu(1) \geq \mu(x)$ for $x \in H$

(ii) $\mu(y) \geq \min\{\mu(x), \mu(x \to y)\}$ for all $x, y \in H$.

## 2.Fuzzy deductive systems

Definition 2.1 A nonempty set $X$ with two binary operation " $\to$ " and " $\cdot$ " and constant 1 is called an HS-algebra if $X$ satisfies the axioms:

(I) $H(X) = (X, \to, 1)$ is a Hilbert algebra.

(II) $S(X) = (X, \cdot)$ is a semigroup.

(III) The operation " $\cdot$ " is distribute over the operation " $\to$ ", that is, $x \cdot (y \to z) = (x \cdot y) \to (x \cdot z)$ and $(x \to y) \cdot z = (x \cdot z) \to (y \cdot z)$ for all $x, y, z \in X$.

For convenience, we use the multiplication $x \cdot y$ by $xy$.

Example 2.2 Let $X = \{1, a, b, c\}$. Define $\to -$ operation and multiplication " $\cdot$ " by the following tables:

| $\to$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | $b$ | $c$ |
| $b$ | 1 | $a$ | 1 | $c$ |
| $c$ | 1 | $a$ | $b$ | 1 |

| $\cdot$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| $a$ | 1 | $a$ | 1 | $a$ |
| $b$ | 1 | 1 | $b$ | $b$ |
| $c$ | 1 | $a$ | $b$ | $c$ |

By routine calculations, we can see that $X$ is an HS-algebra.

Note that in an HS-algebra $X$, we have $1x = x1 = 1$ for any $x \in X$.

Definition 2.3 A nonempty subset $A$ of a semigroup $S(X) = (X, \cdot)$ is said to be left (resp. right) stable if $xa \in A$ (resp. $ax \in A$) , whenever $x \in S(X)$ and $a \in A$.

Definition 2.4 A nonempty subset $A$ of an HS-algebra $X$ is called a left (resp. right) deductive system of $X$ if

($a_1$) $A$ is a left (resp. right ) stable subset of $S(X)$,

($a_2$) for any $x, y \in H(X), x \to y \in A$ and $x \in A$ imply that $y \in A$.

Note that if $A$ is a left (resp. right )deductive system of $X$, then $1 \in A$. Thus $A$ is a deductive system of $H(X)$.

Example 2.5 Let $X = \{1, a, b, c\}$. Define $\to -$ operation and multiplication " $\cdot$ " by the following tables:

| $\to$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | $b$ | $c$ |
| $b$ | 1 | 1 | 1 | $c$ |
| $c$ | 1 | 1 | 1 | 1 |

| $\cdot$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| $a$ | 1 | $a$ | 1 | 1 |
| $b$ | 1 | 1 | $b$ | $c$ |
| $c$ | 1 | 1 | $c$ | $b$ |

Then $X$ is an HS-algebra. It is to check that $A = \{1, a\}$ is a deductive system of $X$.

Definition 2.6 A fuzzy set $\mu$ in a semigroup $X$ is called stable if $\mu(xy) \geq \mu(y)$(resp.$\mu(xy) \geq \mu(x)$) for all $x, y \in X$.

Definition 2.7 A fuzzy set $\mu$ in an HS-algebra $X$ is called a fuzzy left (resp. right ) deductive system of $X$ if

(1) $\mu$ is a fuzzy left (resp. fuzzy right) stable set in $S(X)$;

(2) $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$ for all $x, y \in H(X)$.

In what follows, a(fuzzy) deductive system shall mean a(fuzzy) left deductive system.

Example 2.8 Consider an HS-algebra $X = \{1, a, b, c\}$ with the following Cayley table:

| $\to$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | $b$ | $c$ |
| $b$ | 1 | $a$ | 1 | $c$ |
| $c$ | 1 | $a$ | $b$ | 1 |

| $\cdot$ | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| $a$ | 1 | $a$ | 1 | $a$ |
| $b$ | 1 | 1 | $b$ | $b$ |
| $c$ | 1 | $a$ | $b$ | $c$ |

Define a fuzzy set $\mu$ in $X$ by $\mu(1) = \mu(a) = 0.7$ and $\mu(b) = \mu(c) = 0.5$. Then $\mu$ is a fuzzy deductive system of $X$.

Theorem 2.9 Let $\mu$ be a fuzzy set in an HS-algebra $X$. Then $\mu$ is a fuzzy deductive system of $X$ if and only if the nonempty level set $U(\mu; t)$ of $\mu$ is a deductive system of $X$ for every $t \in [0, 1]$.

*Proof* Suppose that $\mu$ is a fuzzy deductive system of $X$. Let $x \in S(X)$ and $y \in U(\mu; t)$. Then $\mu(y) \geq t$, and that $\mu(xy) \geq \mu(y) \geq t$, which implies that $xy \in U(\mu; t)$. Hence $U(\mu; t)$ is a stable subset of $S(X)$. Let $x, y \in H(X)$ be such that $x \to y \in U(\mu; t)$ and $x \in U(\mu; t)$. Then $\mu(x \to y) \geq t$ and $\mu(x) \geq t$. It follows that $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\} \geq t$, and that $y \in U(\mu; t)$. Hence $U(\mu; t)$ is a deductive system of $X$.

Conversely, assume that the nonempty level set $U(\mu; t)$ of is a deductive system of $X$ for every $t \in [0, 1]$. If there exist $x_0, y_0 \in S(X)$ such that $\mu(x_0y_0) < \mu(y_0)$, then by taking $t_0 = 1/2(\mu(x_0y_0) + \mu(y_0))$, we have $\mu(x_0y_0) < t_0 < \mu(y_0)$. It follows that $y_0 \in U(\mu; t_0)$ and $x_0y_0 \notin U(\mu; t_0)$. This is a contradiction. Therefore $\mu$ is a fuzzy stable set in $S(X)$. Suppose that $\mu(y_0) < \min\{\mu(x_0 \to y_0), \mu(x_0)\}$ for some $x_0, y_0 \in X$. Putting $s_0 = 1/2(\mu(y_0) + \min\{\mu(x_0 \to y_0), \mu(x_0)\})$, then $\mu(y_0) < s_0 < \min\{\mu(x_0 \to y_0), \mu(x_0)\}$, which shows that $x_0 \to y_0 \in U(\mu; s_0)$. $x_0 \in U(\mu; s_0)$ and $y_0 \in U\mu; s_0$.This is impossible. Therefore $\mu$ is a fuzzy deductive system of $X$.

Corollary 2.10 Let $A$ be a deductive system of an HS- algebra $X$, and let $\mu$ be a fuzzy set in $X$ defined by

$$\mu(x) = \begin{cases} t_0 & \text{if} \quad x \in A \\ t_1 & \text{otherwise} \end{cases}$$

where $t_0 > t_1$ in $[0, 1]$. Then $\mu$ is a fuzzy deductive system of $X$, and $U(\mu; t_0) = A$.

*Proof* Notice that

$$U(\mu; t) = \begin{cases} \varnothing & \text{if} \quad t_0 < t_1, \\ A & \text{if} \quad t_1 < t < t_0, \\ X & \text{if} \quad t < t_1, \end{cases}$$

It follows from Theorem 2.9 that $\mu$ is a fuzzy deductive system of $X$. Clearly, we have $U(\mu; t_0) = A$.

Corollary 2.10 suggests that any deductive systems of an HS-algebra $X$ can be realized as a level deductive system of some fuzzy deductive system of $X$. We now consider the converse of Corollary 2.10.

Corollary 2.11 For a nonempty subset $A$ of an HS-algebra $X$. Let $\mu$ be a fuzzy set in $X$ which is given in Corollary 2.10. If $\mu$ is a fuzzy deductive system of $X$, then $A$ is a deductive system of $X$.

*Proof* Assume that $\mu$ is a fuzzy deductive system of $X$ and let $x \in S(X)$ and $y \in A$. Then $\mu(xy) \geq \mu(y) \geq t$ and so $xy \in U(\mu; t_0) = A$. Hence $A$ is a stable subset of $S(X)$, Let $x, y \in H(X)$ be such that $x \to y \in A$ and $x \in A$. It follows that $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\} = t_0$, so that $y \in U(\mu; t_0) = A$. This completes the proof.

The following Theorem shows that the concept of a fuzzy deductive system of an HS-algebra is a generalization of a deductive system. The proof is straightforward by using Corollary 2.10 and 2.11.

Theorem 2.12 Let $A$ be a nonempty subset of an HS-algebra and let $\mu$ be a fuzzy set in $X$ such that $\mu$ is into $\{0, 1\}$, so that $\mu$ is the characteristic function A. Then $\mu$ is a fuzzy deductive system of $X$ if and only if $A$ is a deductive system of $X$.

Theorem 2.13 If $\mu$ is a fuzzy deductive system of an HS-algebra $X$, then $\mu(x) = \sup\{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}$ for all $x \in X$.

*Proof* Let $\beta = \sup\{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}$ and Let $\varepsilon > 0$ be given. Then $\beta - \varepsilon < \sup\{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}$, whence $\beta - \varepsilon < \alpha$ for some $\alpha \in [0, 1]$ such that $x \in U(\mu; \alpha)$. Since $\mu(x) \geq \alpha$, it follows that $\beta - \varepsilon < \mu(x)$, so that $\beta \leq \mu(x)$ because $\varepsilon$ is arbitrary. We now show that $\mu(x) \leq \beta$. To do this, assume that $\mu(x) = \gamma$. Then $x \in U(\mu; \gamma)$ and so $\gamma \in \{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}$. Hence $\gamma \leq \sup\{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}$. Whence $\mu(x) \leq \beta$. Therefore $\mu(x) = \beta$, as desired.

We note that the intersection of all deductive systems of an HS-algebra X is also a deductive system of $X$. Let $\Lambda$ be a totally ordered set and let $\{I_\alpha \mid \alpha \in \Lambda\}$ be a family of deductive systems of an HS-algebra $X$ such that for all $\alpha, \beta \in \Lambda, \beta > \alpha$ if and only if $I_\beta \subset I_\alpha$. Then $\bigcup_{\alpha \in \Lambda} I_\alpha$ is a deductive system of $X$.

Now we consider the converse of Theorem 2.13.

Let $\Lambda$ be a nonempty subset of $[0, 1]$. There is no loss of generality in using $\Lambda$ as an index set in the following results.

Theorem 2.14 Let $\{I_\alpha \mid \alpha \in \Lambda\}$ be a collection of deductive systems of an HS-algebra $X$ such that

(i) $X = \bigcup_{\alpha \in \Lambda} I_\alpha$;

(ii) $\beta > \alpha$ if and only if $I_\beta \subset I_\alpha$ for all $\alpha, \beta \in \Lambda$. Define a fuzzy set $\mu$ in $X$ by, for all $x \in X$, $\mu(x) = \sup\{\alpha \in \Lambda \mid x \in I_\alpha\}$. Then $\mu$ is a fuzzy deductive system of $X$.

*Proof* For any $\beta \in [0, 1]$, we consider the following two cases:

(i) $\beta = \sup\{\alpha \in \Lambda \mid \alpha < \beta\}$;

(ii) $\beta \neq \sup\{\alpha \in \Lambda \mid \alpha < \beta\}$;

For the case (i) we know that $x \in U(\mu; \beta) \Leftrightarrow x \in I_\alpha$ for all $\alpha < \beta \Leftrightarrow x \in \bigcap_{\alpha < \beta} I_\alpha$ whence $U(\mu; \beta) = \bigcap_{\alpha < \beta} I_\alpha$, which is a deductive system of $X$. Case (ii) implies that there exists $\varepsilon > 0$ such that $(\beta - \varepsilon, \beta) \cap \Lambda = \varnothing$. We claim that $U(\mu; \beta) = \bigcup_{\alpha \geq \beta} I_\alpha$. If $x \in \bigcup_{\alpha \geq \beta} I_\alpha$, then $x \in I_\alpha$ for some $\alpha \geq \beta$. It follows that $\mu(x) \geq \alpha \geq \beta$, so that $x \in U(\mu; \beta)$. Conversely if $x \in \bigcup_{\alpha \geq \beta} I_\alpha$, then $x \in I_\alpha$ for all $\alpha \geq \beta$, which implies that $x \in I_\alpha$ for all $\alpha \geq \beta - \varepsilon$, that is, if $x \notin I_\alpha$, then $\alpha \leq \beta - \varepsilon$. Thus $\mu(x) \leq \beta - \varepsilon$, and so $x \notin U(\mu; \beta)$. Therefore $U(\mu; \beta) = \bigcup_{\alpha \geq \beta} I_\alpha$, which is a deductive system of $X$. Therefore $\mu$ is a fuzzy deductive system of $X$. Therefore $\mu$ is a fuzzy deductive system of $X$.

Definition 2.15 Let $\mu$ and $v$ be fuzzy sets of HS-algebras $X_1$ and $X_2$ respectively. The product $\mu \times v$ of $\mu$ and $v$ is the element of $X_1 \times X_2$ which is defined by

$$(\mu \times v)(x, y) = \min\{\mu(x), v(y)\}, \qquad \forall (x, y) \in X_1 \times X_2$$

Theorem 2.16 If $\mu$ and $v$ are fuzzy deductive systems of HS-algebras $X_1$ and $X_2$ respectively, then $\mu \times v$ is a fuzzy deductive system of $X_1 \times X_2$

Proof. Let $x = (x_1, x_2), y = (y_1, y_2) \in X_1 \times X_2$. Then $(\mu \times v)(xy) = (\mu \times v)((x_1, x_2)(y_1, y_2)) = (\mu \times v)(x_1 y_1, x_2 y_2) = \min\{\mu(x_1 y_1), v(x_2 y_2)\} \geq \min\{\mu(y_{1)}, v(y_2)\} = (\mu \times v)(y_{1,} y_2) =$

$(\mu \times v)(y)$; and $(\mu \times v)(y) = (\mu \times v)(y_1,y_2) = \min\{\mu(y_{1)},v(y_2)\} \geq \min\{\min\{\mu(x_1 \to y_1),\mu(x_1)\},\min\{v(x_2 \to y_2),v(x_2)\}\} = \min\{\min\{\mu(x_1 \to y_1),v(x_2 \to y_2)\},\min(\mu(x_1),v(x_2))\} = \min\{(\mu \times v)(x_1 \to y_1,x_2 \to y_2),(\mu \times v)(x_1,y_1)\} = \min\{(\mu \times v)((x_1,x_2) \to (y_1,y_2)),(\mu \times v)(x_1,x_2)\} = \min\{(\mu \times v)(x \to y),(\mu \times v)(x)\}$

Hence $\mu \times v$ is a fuzzy deductive system of $X_1 \times X_2$.

**3. Noetherian HS-algebras**

Definition 3.1 An HS-algebra $X$ is said to satisfy the ascending (descending) chain condition (brief, ACC (DCC)) if for every ascending (descending) sequence $A_1 \subseteq A_2 \subseteq \ldots (A_1 \supseteq A_2 \supseteq \ldots)$ of deductive systems of $X$ there exists a natural number $n$ such that $A_i = A_n$ for au $i \geq n$.

Definition 3.2 An HS-algebra $X$ is said to be Noetherian if $X$ satisfies the ACC for deductive systems.

Let $\mu$ be a fuzzy set in $X$. We note that Im$\mu$ is a bounded subset of $[0,1]$. Hence we can consider a sequence of elements of Im$\mu$ is either increasing or decreasing.

Theorem 3.3 Let $X$ be an HS-algebra satisfying DCC and let $\mu$ be a fuzzy deductive system of $X$. If a sequence of elements of Im $\mu$ is strictly increasing, then $\mu$ has finite numbers of values.

*Proof* Let $\{t_n\}$ be a strictly increasing sequence of element of Im $\mu$. Then $0 \leq t_1 < t_2 < \ldots \leq 1$. Define $U(\mu;t_r) = \{x \in X \mid \mu(x) \geq t_r\}, r = 2,3\ldots$ Then $U(\mu;t_r)$ is a deductive system of $X$. Let $x \in U(\mu;t_r)$. Then $\mu(x) \geq t_r > t_{r-1}$, which implies that $x \in U(\mu;t_{r-1})$. Hence $U(\mu;t_r) \subseteq U(\mu;t_{r-1})$. Since $t_{r-1} \in Im\mu$, there exists $x_{r-1} \in X$ such that $\mu(x_{r-1}) = t_{r-1}$. It follows that $x_{r-1} \in \mu_{r-1}$, but $x_{r-1} \notin U(\mu;t_r)$. Thus $U(\mu;t_r) \subsetneq U(\mu;t_{r-1})$, and so we obtain a strictly descending sequence $U(\mu;t_1) \supsetneq U(\mu;t_2) \supsetneq U(\mu;t_3)\ldots$ of deductive systems of $X$ which is not terminating. This contradicts the assumption $X$ satisfies DCC, completing the proof.

Now we consider the converse of Theorem 3.3:

Theorem 3.4 Let $X$ be an HS-algebra. If every fuzzy deductive system of $X$ has finite number of values, then $X$ satisfies DCC.

*Proof* Suppose that $X$ does not satisfy DCC. Then there exists a strictly descending chain.

$$A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq \ldots$$

of deductive systems of $X$. Define a fuzzy set $v$ in $X$ by

$$v(x) = \begin{cases} \frac{n}{n+1} & \text{if} \quad x \in A_n - A_{n+1}, n = 0,1,2,\cdots \\ 1 & \text{if} \quad x \in \cap_{n=1}^{\infty} A_n, \end{cases}$$

where $A_0$ stands for $X$. We prove that $v$ is a fuzzy deductive system of $X$. Let $x,y \in X$. If $y \in A_n - A_{n+1}$ then $xy \in A_n$ since $A_n$ is a deductive system of $X$, and that $\mu(xy) \geq \frac{n}{n+1} = \mu(y)$. Now, let $y \in \bigcap_{n=0}^{n}$. Since $A_n$ is a deductive systems for any integer numbed $n$, then $\bigcap_{n=0}^{n} A_n$ is also a deductive system. Hence $xy \in \bigcap_{n=0}^{\infty} A_n$, and that $\mu(xy) = 1 = \mu(y)$. Therefore $\mu$ is a fuzzy stable subset in $S(X)$.

Let $x,y \in X$. Assume that $x \to y \in A_n - A_{n+1}$ and $x \in A_k - A_{k+1}$ for $n = 0,1,2\ldots; k = 0,1,2,\cdots$. Without loss of generality, we may assume that $n \leq k$. Then clearly $x \in A_n$, Since $A_n$ is a deductive system, we have $y \in A_n$. Hence $v(y) \geq \frac{n}{n+1} = \min\{v(x \to y),v(x)\}$. If $x \to y, x \in \bigcap_{n=0}^{\infty} A_n$ and $y \in \bigcap_{n=0}^{\infty} A_n$, Thus $v(y) = 1 = \min\{v(x \to y),v(x)\}$. If $x \to y \notin \bigcap_{n=0}^{\infty} A_n$ and $y \in \bigcap_{n=0}^{\infty} A_n$, then there exists $k \in \mathbb{N}$, such that $x \to y \in A_k \to A_{k+1}$. It follows that $y \in A_k$, so that $v(y) \geq \frac{k}{k+1} = \min\{v(x \to y),v(x)\}$. Finally, assume that $x \to y \in \bigcap_{n=0}^{\infty} A_n$ and $x \notin \bigcap_{n=0}^{\infty} A_n$. Then $x \in A_r - A_{r+1}$ for some $r \in \mathbb{N}$. It follows that $y \in A_r$, and hence $v(y) \geq \frac{k}{k+1} = \min\{v(x \to y),v(x)\}$. Consequently, we find that $v$ is a fuzzy deductive system and $v$ has infinite number of different values. This is a contradiction and the proof is complete.

Theorem 3.5 For any HS-algebra $X$, the following are equivalent:

(i) $X$ is Noetherian;

(ii) The set of vaults of any fuzzy deductive system on $X$ is a well-ordered subset of $[0,1]$.

*Proof* Suppose that $\mu$ is fuzzy deductive system whose set of values is not a well-ordered subset of $[0,1]$. Then there exists a strictly decreasing sequence $\{t_n\}$ such that $\mu(x_n) = t_n$. Let $B_n = \{x \in X \mid \mu(x) \in t_n\}$. Then $B_1 \subsetneq B_2 \subsetneq B_3 \ldots$ is a strictly ascending chain of seductive systems of $X$, contradicting the assumption that $X$ is Noetherian.

Assume that the condition is satisfied and $X$ is not Noetherian. Then there exists a strictly ascending chain.

$$A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \ldots$$

of deductive system of $X$. Suppose that $A = \bigcup_{n\in\mathbb{N}} A_n$. Then $A$ is a deductive system of $X$.

Define a fuzzy set $v$ in $X$ by.

$$v(x) = \begin{cases} 0 & \text{if} \quad x \notin A_n, \\ \frac{1}{k} & \text{where} \quad k = \min\{n \in \mathbb{N} \mid x \in A_n\} \end{cases}$$

We claim that $v$ is a fuzzy deductive system of $X$. Let $x, y \in X$. If $y \in A_n - A_{n-1}$ for $n = 2, 3 \ldots$, then $xy \in A_n$. It follows that $v(xy) \geqslant \frac{1}{n} = v(y)$. If $y \notin A_n$, then $v(y) = 0$, and that $v(xy) \geqslant v(y)$. Therefore $\mu$ is a fuzzy stance set in $S(X)$. Now, let $x, y \in X$. If $x \to y, x \in A_n - A_{n-1}$ for $n = 2, 3, \ldots$, then $y \in A_n$. It follows that

$$v(y) \geq \frac{1}{n} = \min\{v(x \to y), v(y)\}$$

Assume that $x, y \in A_n$ and $x \in A_{n-}A_m$ for all $m < n$. Since $A_n$ is a deductive system,therefore $y \in A_n$. Hence $v(y) \geq \min\{v(x \to y), v(x)\}$.

Similarly for the case $x \to y \in A_n - A_m$ and $x \in A_n$, we have $v(y) \geq \min\{v(x \to y), v(x)\}$. Thus $v$ is a fuzzy deductive system of $X$. Since the chain $(*)$ is not terminating, $v$ has a strictly descending sequence of values. This contradicts the assumption that the value set of any fuzzy deductive system is well-ordered. Hence $X$ is Noetherian.

We note that a set is well-ordered if and only if it does not contain any infinite descending sequence.

Theorem 3.6 Let $S = \{t_1, t_2, \ldots\} \bigcup \{0\}$, where $\{t_n\}$ is a strictly decreasing sequence in $(0,1)$. Then an HS-algebra $X$ is Noetherian if and only if for each fuzzy deductive system $\mu$ of $X$, $I_m\mu \subseteq S$ implies that there exists a positive integer $n_0$ such that $Im\mu \subseteq \{t_1, t_2, \ldots, t_{n_0}\} \cup \{0\}$.

*Proof* If $X$ is a Noetherian HS-algebra, then we know from Theorem 3.5 that $Im\mu$ is a well-ordered subset of $[0,1]$ and so the condition is necessary.

Conversely, assume that the condition is satisfied. Suppose that $X$ is not Noetherian. Then there exists a strictly ascending chain of deductive systems $A_1 \subsetneq A_2 \subsetneq A_3 \ldots$

Define a fuzzy set $\mu$ in $X$ by

$$\mu(x) = \begin{cases} t_1 & \text{if} \quad x \in A_1, \\ t_n & \text{if} \quad x \in A_n - A_{n-1} \qquad n = 2, 3, \ldots, \\ 0 & \text{if} \quad x \in X - \bigcup_{n=1}^{\infty} A_n, \end{cases}$$

If $y \in X - A_n$, then $\mu(y) = 0$. Hence $\mu(xy) \geq \mu(y)$. If $y \in A_1$, then $xy \in A_1$, and so $\mu(xy) = t_1 = \mu(y)$. If $y \in A_n - A_{n-1}$ for $n = 2, 3 \ldots$. Then $xy \in A_n$, and hence $\mu(xy) \geq t_n = \mu(y)$. Therefore $\mu$ is a fuzzy stable subset in $S(X)$.

Now let $x, y \in X$. If either $x \to y$ or $X$ belong to $X - A_n$, then either $\mu(x \to y)$ or $\mu(x)$ is equal to 0. Hence $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$. If $x \to y, x \in A_1$, then $y \in A_1$ and so $\mu(y) = t_1 = \min\{\mu(x \to y), \mu(x)\}$. If $x \to y, x \in A_n - A_{n-1}$, then $y \in A_n$. Hence $\mu(y) \geq t_n = \min\{\mu(x \to y), \mu(y)\}$. Assume that $x \to y \in A_1$ and $x \in A_n - A_{n-1}$ for $n = 2, 3, 4, \ldots$. Then $y \in A_n$ and hence $\mu(y) \geq t_n = \min\{t_1, t_n\} = \min\{\mu(x \to y), \mu(x)\}$. Similarly for $y \in A_1$ and $x \to y \in A_n - A_{n-1}, n = 2, 3, 4, \ldots$, we obtain $\mu(y) \geq t_n = \min\{\mu(x \to y), \mu(x)\}$. Hence $\mu$ is a fuzzy deductive system of $X$. this contradicts our assumption.

## 4. On homomorphism of HS-algebras

Definition 4.1 A mapping $f : X \to Y$ of HS- algebras is called a homomorphism if

(i) $f(x \to y) = f(x) \to f(y)$ for all $x, y \in H(X)$

(ii) $f(xy) = f(x)f(y)$ for all $x, y \in S(X)$

Note that if $f : X \to Y$ is a homomorphism of HS- algebra, then $f(1) = 1$. Let $f : X \to Y$ be a homomorphism of HS-algebras, For any fuzzy set $\mu$ in $Y$ we define a set $\mu^f$ in $X$ by $\mu^f(x) = \mu(f(x))$ for all $x \in X$.

Theorem 4.2 Let $f : X \to Y$ be a homomorphism of HS-algebra. If $\mu$ is a fuzzy deductive system of $Y$, then $\mu^f$ is a fuzzy deductive system of $X$.

*Proof* Assume that $\mu$ is a fuzzy deductive system of $Y$, then $\mu^f(xy) = \mu(f(xy)) = \mu(f(x)f(y)) \geq \mu(f(y)) = \mu^f(y)$ and $\mu^f(y) = \mu(f(y)) \geq \min\{\mu(f(x) \to f(y)), \mu(f(x))\} = \min\{\mu(f(x \to y)), \mu(f(x))\} = \min\{\mu^f(x \to y), \mu^f(x)\}$. Hence $\mu^f$ is a fuzzy deductive system of $X$.

If we strengthen the condition $f$, then the converse of Theorem 4.2 is obtained as follows.

Theorem 4.3 Let $f : X \to Y$ be an epimorphism of HS-algebras. If $\mu^f$ is a fuzzy deductive system of $X$, then $\mu$ is a fuzzy deductive system of $Y$.

*Proof* For any $x, y \in Y$, there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Then $\mu(xy) = \mu(f(a)f(b)) = \mu(f(ab)) = \mu^f(ab) \geq \mu^f(b) = \mu(f(b)) = \mu(y)$ and $\mu(y) = \mu(f(b)) = \mu^f(b) \geq \min\{\mu^f(a \to b), \mu^f(a)\} = \min\{\mu(f(a \to b)), \mu(f(a))\} = \min\{\mu(f(a) \to f(b)), \mu(f(a))\} = \min\{\mu(x \to y), \mu(x)\}$. Hence $\mu$ is a fuzzy deductive system of $Y$.

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