# $k$-NUMERICAL RANGE AND THE STRUCTURAL PERFORMANCE OF BUILDINGS

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ABSTRACT. The authors show that the boundary of the polar set $E^{\wedge}$ of the sum $E$ of $k$ elliptical discs $E_1, \dots, E_k$ with center at the origin lies on an algebraic curve $C$ of degree $2^k$. The authors give an algorithm to compute the defining polynomial of $C$ and apply it to $k$-numerical range and a problem in Architecture.

**1 Sum of elliptical discs** Classical numerical ranges or generalized numerical ranges of matrices are related to many geometric problems. In this paper we use a parallel curve of a convex curve and apply it to the study of the $k$-numerical range of a matrix and a structural performance problem of buildings in Architecture.

We consider the following $(2k) \times (2k)$ matrix $A_k$:

$$A_k = \begin{pmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \alpha_2 \\ \beta_2 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \alpha_k \\ \beta_k & 0 \end{pmatrix}, \tag{1.1}$$

where $\alpha_j$, $\beta_j$ are arbitrary complex numbers ( $1 \le j \le k$).

In the below we mention a close relationship between the $k$-numerical range and a problem of Architecture. We take the following fact into account : The safety zone of columns for horizontal loads are represented by elliptical discs, and the direction of the major axis of one of the elliptical discs is same as the direction of the major or minor axis of another elliptical disc. Corresponding to this fact, we mainly consider the case $\alpha_j$, $\beta_j$ are real numbers satisfying $|\beta_j| \le \alpha_j$ $(1 \le j \le k)$.

The $k$-numerical range $W_k(A)$ of $n \times n$ matrix $A$ ( $1 \le k \le n$) is defined as

$$\{\xi_1^* A \xi_1 + \xi_2^* A \xi_2 + \ldots + \xi_k^* A \xi_k : \{\xi_1, \xi_2, \ldots, \xi_k\} \text{ is an orthonormal system in } \mathbf{C}^n\}. \tag{1.2}$$

(cf. [8],[1], [12], [15]). If $k = 1$, then the range $W_k(A)$ is the (classical ) numerical range and denoted by $W(A)$ (cf. [7]). For subsets $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ of $\mathbf{C}$, define the set

$$\Gamma_1 + \Gamma_2 + \ldots + \Gamma_k$$

of $\mathbf{C}$ as

$$\{\gamma_1 + \gamma_2 + \ldots + \gamma_k : \gamma_j \in \Gamma_j \ (1 \le j \le k)\}.$$

First we prove the following theorem.

**Theorem 1.1** Suppose that $A_k$ is the $2k \times 2k$ complex matrix given by the equation (1.1), where $\alpha_j$, $\beta_j$ are arbitrary complex numbers ( $1 \le j \le k$). Then the following equation holds :

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$$W_k(A_k) = W(\begin{pmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{pmatrix}) + W(\begin{pmatrix} 0 & \alpha_2 \\ \beta_2 & 0 \end{pmatrix}) + \ldots + W(\begin{pmatrix} 0 & \alpha_k \\ \beta_k & 0 \end{pmatrix}). \tag{1.3}$$

**Proof** By the definition of the $k$-numerical range we have the following inclusion :

$$W_k(A_k) \supset W(\begin{pmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{pmatrix}) + W(\begin{pmatrix} 0 & \alpha_2 \\ \beta_2 & 0 \end{pmatrix}) + \ldots + W(\begin{pmatrix} 0 & \alpha_k \\ \beta_k & 0 \end{pmatrix}). \tag{1.4}$$

We shall show the inverse inclusion. By the separation theorem of compact convex sets, the inverse inclusion is deduced from the following equation

$$\max\{\Re(z\, e^{-i\,\theta}) : z \in W_k(A_k)\} = \sum_{j=1}^{k} \max\{\Re(z\, e^{-i\,\theta}) : z \in W(\begin{pmatrix} 0 & \alpha_j \\ \beta_j & 0 \end{pmatrix})\}, \tag{1.5}$$

for each $-\pi \leq \theta \leq \pi$. We set

$$\alpha_j = \exp(i\,(\phi_j + \psi_j))\,(a_j + b_j),$$

$$\beta_j = \exp(i\,(\phi_j - \psi_j))\,(a_j - b_j),$$

where $a_j \geq 0$, $b_j \geq 0$ and $\phi_j$, $\psi_j$ are real numbers ( $1 \leq j \leq k$). Since a $k$-numerical range is invariant under unitary similarity, we may assume that $\psi_j = 0$ for $1 \leq j \leq k$.

The Hermitian part of $e^{-i\,\theta}\,A_k$ is given by the following:

$$\oplus_{j=1}^{k} \begin{pmatrix} 0 & a_j \cos(\phi_j - \theta) + i\,b_j \sin(\phi_j - \theta) \\ a_j \cos(\phi_j - \theta) - i\,b_j \sin(\phi_j - \theta) & 0 \end{pmatrix},$$

We compute that

$$\begin{aligned} & \det(t\,I_{2k} - \Re(e^{-i\,\theta}\,A_k)) \\ = & \prod_{j=1}^{k} (t^2 - a_j^2 \cos^2(\phi_j - \theta) - b_j^2 \sin^2(\phi_j - \theta)) \\ = & \prod_{j=1}^{k} (t - \sqrt{a_j^2 \cos^2(\phi_j - \theta) + b_j^2 \sin^2(\phi_j - \theta)}) \\ & \quad (t + \sqrt{a_j^2 \cos^2(\phi_j - \theta) + b_j^2 \sin^2(\phi_j - \theta)})). \end{aligned}$$

By a theorem of Li-Sung-Tsing in [10], we have the following :

$$\begin{aligned}
& \max\{\Re(z\, e^{-i\,\theta}) : z \in W_k(A_k)\} \\
= \ & \max\{\sum_{j=1}^{k}(p_j \sqrt{a_j^2 \cos^2(\phi_j-\theta) + b_j^2 \sin^2(\phi_j-\theta)} \\
& -q_j \sqrt{a_j^2 \cos^2(\phi_j-\theta) + b_j^2 \sin^2(\phi_j-\theta)}\} : \\
& p_j \in \{0,1\}, q_j \in \{0,1\}, p_1+q_1+p_2+q_2+\ldots+p_k+q_k = k\} \\
= \ & \sum_{j=1}^{k} \sqrt{a_j^2 \cos^2(\phi_j-\theta) + b_j^2 \sin^2(\phi_j-\theta)} \\
= \ & \sum_{j=1}^{k} \max\{\Re(z\, e^{-i\,\theta}) : z \in W(\begin{pmatrix} 0 & \alpha_j \\ \beta_j & 0 \end{pmatrix})\}. \qquad (1.6)
\end{aligned}$$

Hence we obtain the equation (1.5) and the proof of the theorem is complete.

We identify the Gaussian plane $\mathbf{C}$ with the 2-dimensional Euclidean space $\mathbf{R}^2$. The numerical range of the $2 \times 2$ matrix

$$\begin{pmatrix} 0 & \exp(i\,\phi_j)\,(a_j+b_j) \\ \exp(i\,\phi_j)\,(a_j-b_j) & 0 \end{pmatrix}$$

coincides with the elliptical disc

$$\begin{aligned}
E_j \ = \ & \{r\,(a_j \cos\phi_j \cos\theta - b_j \sin\phi_j \sin\theta, a_j \sin\phi_j \cos\theta + b_j \cos\phi_j \sin\theta) : \\
& -\pi \leq \theta \leq \pi, 0 \leq r \leq 1\}, \qquad (1.7)
\end{aligned}$$

$(\ 1 \leq j \leq k)$ and the range $W_k(A_k)$ coincides with the sum :

$$E = E_1 + E_2 + \ldots + E_k = \{(u_1+u_2+\ldots+u_k, v_1+v_2+\ldots+v_k) :$$
$$(u_j, v_j) \in E_j\ (1 \leq j \leq k)\}. \qquad (1.8)$$

By Proposition 2.2 of [14], the boundary $\partial E$ lies on an algebraic curve. The set $E$ satisfies $-E = E$. We assume that each ellipse $\partial E_j$ is non-degenerate : $a_j > 0$, $b_j > 0$ $(\ 1 \leq j \leq k)$. The boundary of $E_1 + E_2 + \ldots + E_{k+1}$ lies on an algebraic curve and its parametric representation is given by the following recursive formula :

$$x_{k+1}(\theta) = x_k(\theta) + \frac{C_{11}\, x_k'(\theta) + C_{12} y_k'(\theta)}{\sqrt{C_{01}\, x_k'(\theta)^2 + 2C_{02}\, x_k'(\theta) y_k'(\theta) + C_{03} y_k'(\theta)^2}}, \qquad (1.9)$$

$$y_{k+1}(\theta) = y_k(\theta) + \frac{C_{21}\, x_k'(\theta) + C_{22} y_k'(\theta)}{\sqrt{C_{01}\, x_k'(\theta)^2 + 2C_{02}\, x_k'(\theta) y_k'(\theta) + C_{03} y_k'(\theta)^2}}, \qquad (1.10)$$

where

$$\begin{aligned}
C_{11} &= \sin\phi_{k+1}\cos\phi_{k+1}\,(b_{k+1}^2 - a_{k+1}^2),\\
C_{12} &= \cos^2\phi_{k+1}\,a_{k+1}^2 + \sin^2\phi_{k+1}\,b_{k+1}^2,\\
C_{21} &= -\sin^2\phi_{k+1}\,a_{k+1}^2 - \cos^2\phi_{k+1}\,b_{k+1}^2,\\
C_{22} &= \sin\phi_{k+1}\cos\phi_{k+1}\,(a_{k+1}^2 - b_{k+1}^2),\\
C_{01} &= \sin^2\phi_{k+1}\,a_{k+1}^2 + \cos^2\phi_{k+1}\,b_{k+1}^2 = -C_{21},\\
C_{02} &= \sin\phi_{k+1}\cos\phi_{k+1}\,(b_{k+1}^2 - a_{k+1}^2) = C_{11} = -C_{22}\\
C_{03} &= \cos^2\phi_{k+1}\,a_{k+1}^2 + \sin^2\phi_{k+1}\,b_{k+1}^2 = C_{12}.
\end{aligned}$$

This formula is deduced from its special case for $\phi_{k+1} = 0$ by a rotation. If $\phi_{k+1} = 0$, then the recursive formula is given by

$$x_{k+1}(\theta) = x_k(\theta) + \frac{a_{k+1}^2\, y_k'(\theta)}{\sqrt{b_{k+1}^2\, x_k'(\theta)^2 + a_{k+1}^2\, y_k'(\theta)^2}}, \tag{1.11}$$

$$y_{k+1}(\theta) = y_k(\theta) - \frac{b_{k+1}^2\, x_k'(\theta)}{\sqrt{b_{k+1}^2\, x_k'(\theta)^2 + a_{k+1}^2\, y_k'(\theta)^2}}, \tag{1.12}$$

$(-\pi \le \theta \le \pi)$. This formula is deduced from the theory of parallel curves by using an affine transformation. By the equation (1.6) we obtain a characterization of the curve $\partial E$:

$$\max\{x\cos\theta + y\sin\theta : (x,y) \in \partial E\}$$
$$= \sum_{j=1}^{k} \sqrt{a_j^2\cos^2(\theta - \phi_j) + b_j^2\sin^2(\theta - \phi_j)}. \tag{1.13}$$

Denote by $H(\theta)$ the above value. Then $H$ is a positive valued function satisfying $H(\theta+\pi) = H(\theta)$. Define a positive valued function $r$ by

$$r(\theta)\,\exp(i\,\theta) \in \partial E.$$

The following relation is worthwhile for the application to Architecture :

$$M = \max\{r(\theta) : -\pi \le \theta \le \pi\} = \max\{H(\theta) : -\pi \le \theta \le \pi\},$$

and $M = r(\theta_0) = H(\theta_0)$ for some $0 \le \theta_0 \le \pi$. This follows from the equation $r'(\theta_0) = 0$.We consider the polar set $E^\wedge$ of the compact convex set $E$:

$$E^\wedge = \{(a,b) \in \mathbf{R}^2 : -1 \le a\,x + b\,y \le 1 \text{ for every } (x,y) \in E\},$$

its boundary $\partial E^\wedge$ is the dual curve of the curve $\partial E$, that is $(a,b) \in \mathbf{R}^2$ belongs to $\partial E^\wedge$ if and only if the straight line $a\,x + b\,y + 1 = 0$ is a tangent of $\partial E$.

The curve $\partial E^\wedge$ lies on an algebraic curve $C$ of degree $2^k$. By using a general theory of plane algebraic curves, we find that a trivial upper bound of the order of the defining polynomial of $\partial E$ is given by $2^k\,(2^k - 1)$. We give an algorithm to compute the defining polynomial of $C$. We consider the following polynomial

$$\begin{aligned} f(t,x,y) &= \det(t\,I_{2k} + (x/2)(A_k + A_k^*) - i\,(y/2)(A_k - A_k^*)) \\ &= \prod_{j=1}^{k} (t^2 - a_j^2(x\,\cos\phi_j + y\,\sin\phi_j)^2 - b_j^2\,(-x\,\sin\phi_j + y\,\cos\phi_j)^2) \\ &= \sum_{j=0}^{k} c_j(x,y)\,t^{2k-2j}. \end{aligned}$$

We shall use symmetric polynomials. Set

$$\begin{aligned} F(t;t_1,\dots,t_k) &= \prod_{j=1}^{k}(t-t_j)(t+t_j) = \prod_{j=1}^{k}(t^2-t_j^2) \\ &= \sum_{j=0}^{k}(-1)^j\,s_j(t_1^2,t_2^2,\dots,t_k^2)\,t^{2k-2j}, \end{aligned}$$

where $s_j$'s are fundamental symmetric polynomials :

$$s_p(x_1,x_2,\dots,x_k) = \sum_{1\le j_1<j_2<\cdots<j_p\le k} x_{j_1}\,x_{j_2}\cdots x_{j_p}.$$

We consider the following polynomial :

$$\begin{aligned} G(t;t_1,\dots,t_k) &= \prod_{h_1=1}^{2}\prod_{h_2=1}^{2}\cdots\prod_{h_k=1}^{2}(t+(-1)^{h_1}t_1+(-1)^{h_2}t_2+\dots+(-1)^{h_k}t_k) \\ &= \sum_{p=0}^{2^k} G_p(t_1,\dots,t_k)\,t^{2^k-p}. \end{aligned}$$

Then the polynomials $G_p(t_1,\dots,t_k)$ satisfy the symmetry

$$G_p(-t_1,t_2,\dots,t_k) = G_p(t_1,t_2,\dots,t_k).$$

Similar equation holds for $t_2,\dots,t_k$. Hence $G_p = 0$ for odd $p$. Each polynomial

$$G_p(t_1,\dots,t_k) \in \mathbf{Z}[t_1,\dots,t_k]$$

is a polynomial in $t_1^2,\dots,t_k^2$. Since each $G_p$ is symmetric with respect to $t_1,\dots,t_k$, the fundamental theorem of symmetric polynomials implies that $G_p(t_1,\dots,t_k)$ is written as a polynomial of $s_1(t_1^2,\dots,t_k^2),\dots,s_k(t_1^2,\dots,t_k^2)$ :

$$G_{2j}(t_1,\dots,t_k) = h_{2j}(-s_1(t_1^2,\dots,t_k^2), s_2(t_1^2,\dots,t_k^2),\dots,(-1)^k\,s_k(t_1^2,\dots,t_k^2)),$$

where $h_{2j}(x_1,x_2,\dots,x_k) \in \mathbf{Z}[x_1,x_2,\dots,x_k]$ ( $j = 0,1,\dots,2^{k-1}$) (cf. [18], 5.7 symmetric functions).

We give exact form of the polynomials $h_{2j}$ for $k = 2, 3, 4$. We set

$$S_j(x_1, \ldots, x_k) = (-1)^j s_j(x_1, \ldots, x_k).$$

The case $k = 2$. In this case $h_2(S_1, S_2) = 2S_1$, $h_4(S_1, S_2) = S_1^2 - 4S_2$.

The case $k = 3$. In this case we have the following:

$$\begin{aligned}
h_2(S_1, S_2, S_3) &= 4S_1, \\
h_4(S_1, S_2, S_3) &= 6S_1^2 - 8S_2, \\
h_6(S_1, S_2, S_3) &= 4S_1^3 - 16S_1 S_2 + 64S_3, \\
h_8(S_1, S_2, S_3) &= S_1^4 - 8S_1^2 S_2 + 16S_2^2.
\end{aligned}$$

The case $k = 4$. In this case the expressions of $h_j(S_1, S_2, S_3, S_4)$'s are rather complicated.

$$\begin{aligned}
&h_2(S_1, S_2, S_3, S_4) = 8S_1, \\
&h_4(S_1, S_2, S_3, S_4) = 28S_1^2 - 16S_2, \\
&h_6(S_1, S_2, S_3, S_4) = 56S_1^3 - 96S_1 S_2 + 128S_3, \\
&h_8(S_1, S_2, S_3, S_4) = 70S_1^4 - 240S_1^2 S_2 + 96S_2^2 + 512S_1 S_3 - 2176S_4, \\
&h_{10}(S_1, S_2, S_3, S_4) = 56S_1^5 - 320S_1^3 S_2 + 384S_1 S_2^2 + 768S_1^2 S_3 \\
&\quad -1024S_2 S_3 - 2560S_1 S_4, \\
&h_{12}(S_1, S_2, S_3, S_4) = 28S_1^6 - 240S_1^4 S_2 + 576S_1^2 S_2^2 - 256S_2^3 + 512S_1^3 S_3 \\
&\quad -2048S_1 S_2 S_3 + 4096S_3^2 + 1280S_1^2 S_4 - 7168S_2 S_4, \\
&h_{14}(S_1, S_2, S_3, S_4) = 8S_1^7 - 96S_1^5 S_2 + 384S_1^3 S_2^2 - 512S_1 S_2^3 + 128S_1^4 S_3 \\
&\quad -1024S_1^2 S_2 S_3 + 2048S_2^2 S_3 + 1536S_1^3 S_4 - 6144S_1 S_2 S_4 + 8192S_3 S_4, \\
&h_{16}(S_1, S_2, S_3, S_4) = S_1^8 - 16S_1^6 S_2 + 96S_1^4 S_2^2 - 256S_1^2 S_2^3 + 256S_2^4 \\
&\quad -128S_1^4 S_4 + 1024S_1^2 S_2 S_4 - 2048S_2^2 S_4 + 4096S_4^2.
\end{aligned}$$

To illustrate the above formula, we treat the case $k = 3$. Then we have

$$\begin{aligned}
& G(t; t_1, t_2, t_3) \\
= \; & (t + t_1 + t_2 + t_3)\,(t + t_1 + t_2 - t_3)\,(t + t_1 - t_2 + t_3)\,(t + t_1 - t_2 - t_3) \\
& (t - t_1 + t_2 + t_3)\,(t - t_1 + t_2 - t_3)\,(t - t_1 - t_2 + t_3)\,(t - t_1 - t_2 - t_3) \\
= \; & t^8 + 4\,(-t_1^2 - t_2^2 - t_3^2)\,t^6 + \{6\,(-t_1^2 - t_2^2 - t_3^2)^2 - 8\,(t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2)\,\}t^4 \\
& + \{4\,(-t_1^2 - t_2^2 - t_3^2)^3 - 16\,(-t_1^2 - t_2^2 - t_3^2)(t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2) + 64(-t_1^2 t_2^2 t_3^2)\}t^2 \\
& + \{(-t_1^2 - t_2^2 - t_3^2)^4 - 8\,(-t_1^2 - t_2^2 - t_3^2)^2(t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2) + 16\,(t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2)\}
\end{aligned}$$

We shall treat a general case. We set

$$g(t, x, y) = \sum_{j=0}^{2^{k-1}} h_{2j}(c_1(x, y), c_2(x, y), \ldots, c_k(x, y))\, t^{2^k - 2j}. \tag{1.14}$$

Then $g(1, x, y)$ is a polynomial in $x, y$ with degree $2^k$.

We shall prove the following theorem.

**Theorem 1.2** Suppose that $g(1, x, y)$ is a real polynomial in $x, y$ with degree $2^k$ given by the equation (1.14). Then the algebraic curve $C \supset \partial E^\wedge$ is defined by the polynomial $g(1, x, y)$:

$$C = \{(x, y) \in \mathbf{R}^2 : g(1, x, y) = 0\}.$$

**Proof** We call a general property of a compact convex set $E \subset \mathbf{R}^2$ satisfying $-E = E$. For every $-\pi \leq \theta \leq \pi$, the value

$$H(\theta) = \max\{x \cos\theta + y \sin\theta : (x, y) \in \partial E\}$$

defines two support lines of $E$:

$$x \cos\theta + y \sin\theta = H(\theta), \quad x \cos\theta + y \sin\theta = -H(\theta).$$

These are also tangents of $\partial E$. By the definition of the polar set $E^\wedge$, its boundary consists of the points

$$(\frac{-\cos\theta}{H(\theta)}, \frac{-\sin\theta}{H(\theta)}),$$

( $-\pi \leq \theta \leq \pi$). We apply this characterization of $\partial E^\wedge$ to the case $H(\theta)$ is given by the equation (1.13). By the homogeneity of the form $g$, the relation $g(1, -\cos\theta/H(\theta), -\sin\theta/H(\theta)) = 0$ is equivalent to $g(H(\theta), -\cos\theta, -\sin\theta) = 0$. So it is sufficient to show that the polynomial $g(t, x, y)$ enjoys the equation

$$g(\sum_{j=1}^{k} \sqrt{a_j^2 \cos^2(\phi_j - \theta) + b_j^2 \sin^2(\phi_j - \theta)}, -\cos\theta, -\sin\theta) = 0$$

for $-\pi \leq \theta \leq \pi$. This equation is shown by the following equation :

$$\begin{aligned} & g(t, -\cos\theta, -\sin\theta) \\ = & \prod_{h_1=1}^{2} \prod_{h_2=1}^{2} \cdots \prod_{h_k=1}^{2} (t + \sum_{j=1}^{k} (-1)^{h_j} \sqrt{a_j^2 \cos^2(\phi_j - \theta) + b_j^2 \sin^2(\phi_j - \theta)}) \end{aligned}$$

for $-\pi \leq \theta \leq \pi$. This equation follows from the properties of the polynomials $G$, $G_{2j}$ ( $j = 1, 2, \ldots, 2^{k-1}$) and the equation

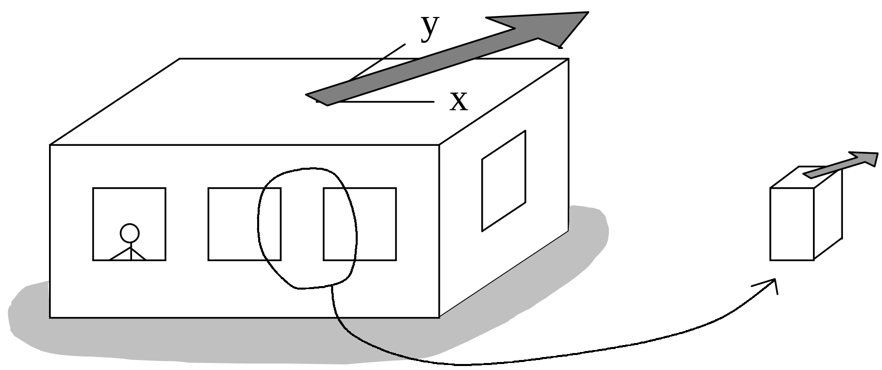
$$\begin{aligned} & f(t, -\cos\theta, -\sin\theta) \\ = & \prod_{j=1}^{k} (t^2 - a_j^2 (\cos\theta \cos\phi_j + \sin\theta \sin\phi_j)^2 - b_j^2 (\cos\theta \sin\phi_j - \sin\theta \cos\phi_j)^2) \\ = & \prod_{j=1}^{k} (t^2 - a_j^2 \cos^2(\phi_j - \theta) - b_j^2 \sin^2(\phi_j - \theta)). \end{aligned}$$

The proof of Theorem 1.2 is complete. □

The function $H$ defined by the equation (1.13) satisfies the equation

$$g(H(\theta), -\cos\theta, -\sin\theta) = g(H(\theta), \cos\theta, \sin\theta) = 0,$$

for every $-\pi \leq \theta \leq \pi$, and $H(\theta)$ is the greatest roots of the equation $g(t, -\cos\theta, -\sin\theta) = 0$ in $t$.




a) Horizontal load acting on a building

b) Horizontal load acting on a column

Fig.1 Horizontal load acting on a building caused by earthquake or wind

**2 Sum of elliptical discs and the structural performance of buildings** In this section we explain the relation between the structural performance of buildings and a problem of sum of elliptical discs.

When an earthquake occurs or wind blows near a building, a large horizontal load may act on the building as shown in Fig.1, which shows the case of single story building. The magnitude and the direction of this load is various under various situations. We want to know whether the building is safe or not against various horizontal loads (cf. [20]). Recent orthodox approach to this kind of problem is using computer simulation methods like finite element methods (cf.[4]) for structural analysis. But it is difficult to get the total picture for the problem we are discussing, by using these methods, because these methods answer only one numerical result for a particular problem. We find the problem is connected with the problem of sum of elliptical discs, as mentioned below. This changing the viewpoint may help us to get the total picture for the problem.

A horizontal load acting on a building or its member column is represented by a point on a plane of horizontal loads ( the end point of the arrow in Fig.1 a)). By using the data of this point and the safety zone for the building or the columns, we can judge safety of them on the plane of horizontal loads. We assume the following things concerned with buildings.

(1) Floor slabs (including roof slabs) are rigid and move parallel.

(2) Safety zones of columns for horizontal loads are represented by elliptical discs.

(3) Mechanical behavior of columns obey the theory of plasticity.

In the above the assumption (1) is set in order to simplify the problem. The assumption (2) is based on experimental results of some reinforced concrete columns (cf. [19]). If the horizontal load acting on the column is on the elliptical disc, then the column is safe, because of the assumption (2) and (3). The column cannot bear holding the horizontal load, which is not on the elliptical disc. Because of plasticity of the materials ( the assumption (3) ), a column does not collapse completely as soon as the horizontal load reaches boundary line of the elliptical disc, but can keep their strength while the horizontal load is on the boundary line ( We refer the reader to Chakrabarty's text book [2] for basic theory for the plastic theory, and W.F.Chen's text book [3]. His other publications [5] help us to know recent applications of the plastic theory to reinforced concrete buildings).

In addition to the above assumptions, we are interested in the case : the direction of the major axis of one of the elliptical discs is the same as the direction of the major or minor axis of another elliptical disc in a building. This case suits for the building where all of the side faces of the columns and walls make intersect perpendicularly to one of the Cartesian axes in the horizontal plane. We think many buildings are classified into the case, because the external forms and the internal forms of many buildings are composed of cuboids. Remote cause of this character of many buildings may be related with that buildings are usually designed on orthogonal grid sheets.

In the beginning, we discuss about the case, where there are two columns in a single story building. Safety zones of the columns are given by elliptical discs shown in Fig.2.a) and b). We have to note that the sum of the horizontal loads acting on all the columns is equal to the applied horizontal load acting on the floor slab, because of the condition of equilibrium, but the directions of the loads acting on the columns are not always equal to the direction of the applied horizontal loads acting on the floor slab(see the arrows in Fig.1.a) and b)). If we can not find any set of the loads acting on the columns, where the sum of them is equal to the applied loads acting on the floor slab and each of them is on their corresponding elliptical disc, then the building is not safe for the applied horizontal load. If we find them, then the building is safe. Let us discuss about this problem on the figure shown in Fig.2 c) and d). Suppose the horizontal loads acting on the column No.1(see the arrow in Fig.2.c)) is fixed to the boundary line of corresponding elliptical disc as shown in Fig.2 c). Then the safety zone of the building is given by translated elliptical disc of the column No.2, as shown by the shaded area in Fig.2 c), because the column No.2 cannot bear holding the horizontal load, which is not on the disc. In general, the horizontal load acting on the column No.1 is not fixed, and it allowed to be anywhere on the disc. It means we can translate the center of the elliptical disc of the column No.2 to anywhere on the disc of column No.1 on the plane of horizontal loads. The area, which is formed by making parallel translation of the elliptical disc with that center as shown in Fig.2 d), is the safety zone of the building. In the figure, the boundary line of the safety zone is represented by dotted curve, which is envelope curve formed by the discs whose centers are on the boundary line of elliptical disc of the column No.1. Thus, the problem calculating safety zones of buildings are now reduced to the problem of sum of elliptical discs.

In case there are three columns in the building, we shall make parallel translation of the elliptical disc of third column keeping the center on the boundary line of the safety zone made by two columns, then we get the new envelope curve as the boundary line of the safety zone of the building. This procedure is applicable for over three columns cases. In case the building consists of multi stories, we shall calculate safety zones of all the stories using above method, and check the safety of each story for ( the shear force of the story) total applied horizontal loads acting on upper part of the building. And if all the stories are safe, then the building is safe, otherwise the building is not safe. If only one horizontal load is acting on the top of multi story building, then the intersection of all the safety zones

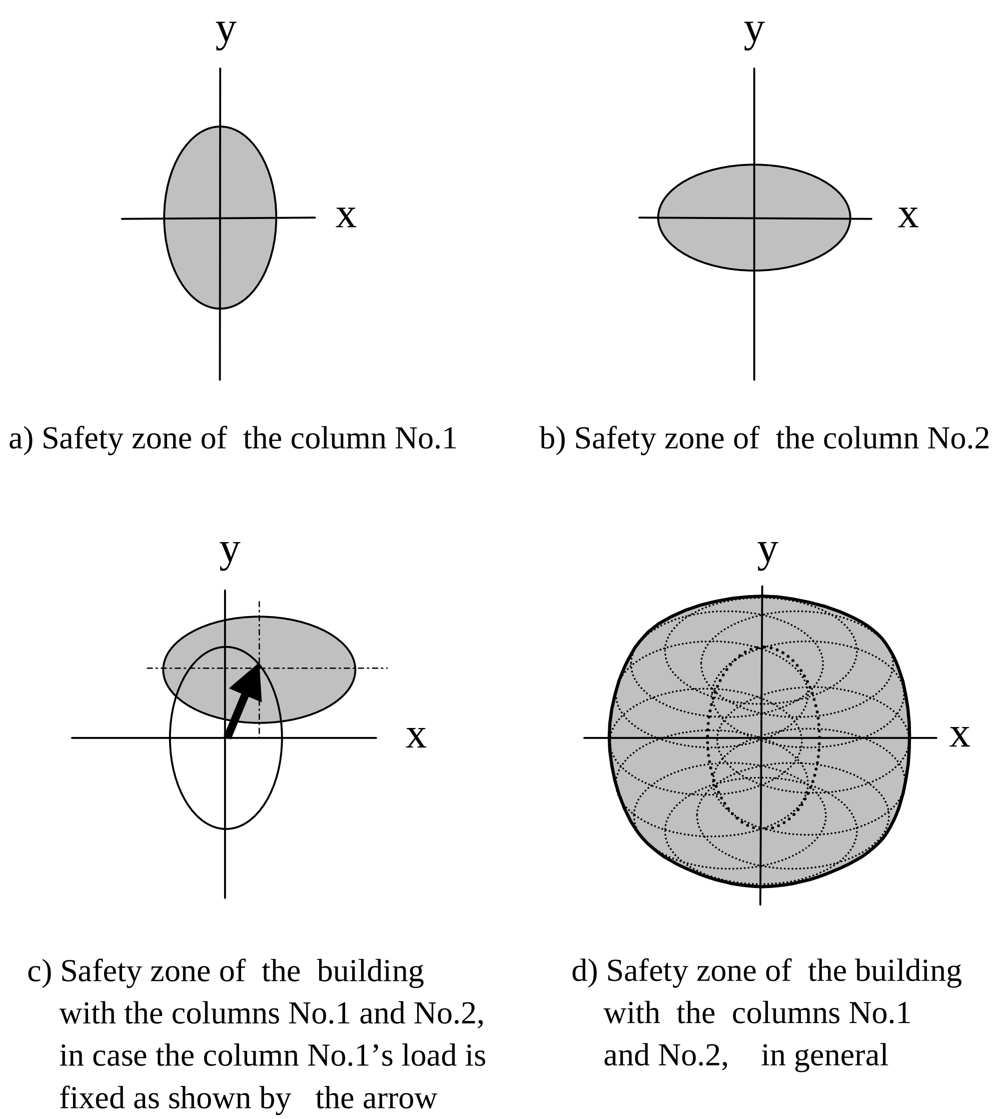



Fig.2 Sum method for safety zones of columns

of the stories is the safety zone of the building.

The following simple property is worthwhile for the application to Architecture.

**Proposition 2.1** *If the elliptical discs in the equation (1.6) satisfy the condition $\phi_j = 0$ ( $1 \le j \le k$), that is, the major axis of each ellipse $\partial E_j$ lies on the real axis or the imaginary axis, then the positive valued function $r(\theta)$ defined by*

$$r(\theta)\, \exp(i\,\theta) \in \partial E = \partial(E_1 + E_2 + \ldots + E_k)$$

*( $0 \le \theta \le \pi/2$) attains its minimum at $\theta = 0$ or $\theta = \pi/2$.*

**Remark** This proposition provides us an information about the weakest direction of the safety zone $E$.

**Proof** The set $E_j$ is contained in the rectangle

$$R_j = \{x + i\,y : (x, y) \in \mathbf{R}^2, -a_j \le x \le a_j,\ -b_j \le y \le b_j\},$$

$(j = 1, 2, \ldots, k)$. Thus the set $E$ is contained in the rectangle

$$R = \{x + i\,y : (x, y) \in \mathbf{R}^2, |x| \le a_1 + a_2 + \ldots, a_k,\ |y| \le b_1 + b_2 + \ldots + b_k\},$$

and hence $r(0) \le a_1 + a_2 + \ldots + a_k$ , $r(\pi/2) \le b_1 + b_2 + \ldots + b_k$. Each set $E_j$ contains a point $a_j\ \cos\theta + i\,b_j\ \sin\theta$ for every $0 \le \theta \le \pi/2$, and hence $E$ contains the arc

$$\{(a_1 + a_2 + \ldots + a_k)\ \cos\theta + i\,(b_1 + b_2 + \ldots + b_k)\ \sin\theta : 0 \le \theta < \pi\},$$

and hence $r(0) = a_1 + a_2 + \ldots + a_k$, $r(\pi/2) = b_1 + b_2 + \ldots + b_k$, and

$$r(\theta) \ge \min\{a_1 + a_2 + \ldots + a_k, b_1 + b_2 + \ldots + b_k\}$$

for $0 < \theta < \pi/2$, which completes the proof of the proposition. □

**3 The case $k = 2$ and an example for $k = 3$.** We shall give an explicit expression of $E$ in the case $k = 2$. We confine ourselves to consider the case $\phi_1 = \phi_2 = 0$, which is satisfied by the situation of safety zones.

In the case $b_1/a_1 = b_2/a_2$, the set $E$ is an elliptical disc. We assume that $b_1/a_1 \ne b_2/a_2$. By using an affine transformation, we may assume that

$$0 < b = b_1 < a_1 = 1, \quad 1 \ge b_2 = \kappa > a_2 = \kappa\, b.$$

In this case we have

$$\begin{aligned} f(t, x, y) &= (t^2 - x^2 - b^2\,y^2)(t^2 - \kappa^2\,b^2\,x^2 - \kappa^2\,y^2) \\ &= t^4 + c_1\,t^2 + c_2, \end{aligned}$$

and hence

$$\begin{aligned} c_1 &= -(1+\kappa^2 b^2)\, x^2 - (b^2+\kappa^2)\, y^2, \\ c_2 &= \kappa^2 b^2 x^4 + \kappa^2 (1+b^4)\, x^2 y^2 + \kappa^2 b^2 y^4. \end{aligned}$$

By Theorem 1.2 , the boundary of the polar of $E$ lies on the curve $g(t,x,y)=0$ where

$$\begin{aligned} g(t,x,y) &= t^4 - 2(1+\kappa^2 b^2)\, t^2 x^2 - 2(b^2+\kappa^2)\, t^2 y^2 \\ &\quad +(1-\kappa^2 b^2)^2 x^4 + 2\,(1-\kappa^2 b^2)\,(b^2-\kappa^2)\, x^2 y^2 + (b^2-\kappa^2)^2 y^4. \end{aligned}$$

The polynomial $K(x,y)=K(1,x,y)$ defining the boundary of $E$ is related with the polynomial $g(t,x,y)$ as the following. The algebraic curve $K(t,x,y)=0$ on the complex projective plane $\mathbf{CP}^2$ is the dual curve of the curve $g(t,x,y)=0$, that is, a generic point $(t_0,x_0,y_0)$ of the curve $K(t,x,y)=0$ represents a tangent of the curve $g(t,x,t)=0$ at some non-singular point $(T_0,X_0,Y_0)$ :

$$\begin{aligned} & g(T_0+T, X_0+X, Y_0+Y) \\ = \;& (t_0\, T + x_0\, X + y_0\, Y) \\ & + (a_{00}\, T^2 + a_{11}\, X^2 + a_{22}\, Y^2 + 2a_{01}\, T\, X + 2a_{02}\, T\, Y + 2a_{12}\, X\, Y) + \ldots. \end{aligned}$$

If a straight line $1+x_0\, X+y_0\, Y=0$ with $y_0 \neq 0$ is the tangent of the curve $g(1,X,Y)=0$, then the equation

$$y_0^n\, g(1, X, -\frac{1}{y_0} - \frac{x_0}{y_0} X) = 0,$$

in $X$ has a multiple root, where $n$ is the degree of $g$ and $n=4$ in the case $k=2$. This property provides us an algorithm to compute the polynomial $K(x,y)=K(1,x,y)$. The polynomial $K(x,y)$ is obtained as a factor of the discriminant of the polynomial $y^n\, g(1,X,-1/y - x\,X/y)$ with respect to $X$. The degree of the polynomial $K$ depends on the singularities of the curve $g(t,x,y)=0$. For its general theory we refer the readers to [13, 21]. To treat the case the degree $n$ of $g$ is 4, we refer [9, 16]. The algebraic curve $g(t,x,y)=0$ on the complex projective plane $\mathbf{CP}^2$ has two double points at $(t,x,y)=(0,\pm\sqrt{(\kappa^2-b^2)/(1-\kappa^2 b^2)},1)$ if $\kappa \neq b$. If $\kappa=b$,then $(t,x,y)=(0,0,1)$ is a tacnode of the curve $g(t,x,y)=0$. Corresponding to these singular points, the degree of the equation of $\partial E$ is $8 = 4\times 3 - 2\times 2$. We have the following equation.

$$\begin{aligned} & K(x,y) \\ = \;& b^4\,(x^8+y^8) + (2b^2+2b^6)\,(x^6 y^2 + x^2 y^6) + (1+4b^4+b^8)\, x^4 y^4 \\ & +(-4b^4+2b^8+2b^2\kappa^2-4b^6\kappa^2)\, x^6 + (-6b^2+2b^6-2b^{10}-2\kappa^2 \\ & +2b^4\kappa^2-6b^8\kappa^2)\, x^4 y^2 \\ & +(-2+2b^4-6b^8-6b^2\kappa^2+2b^6\kappa^2-2b^{10}\kappa^2)\, x^2 y^4 \\ & +(2b^2-4b^6-4b^4\kappa^2+2b^8\kappa^2)\, y^6 \\ & +(6b^4-6b^8+b^{12}-6b^2\kappa^2+10b^6\kappa^2-6b^{10}\kappa^2+\kappa^4-6b^4\kappa^4+6b^8\kappa^4)\, x^4 \\ & +(6b^2-10b^6+6b^{10}+4\kappa^2-6b^4\kappa^2-6b^8\kappa^2+4b^{12}\kappa^2 \end{aligned}$$

$$
\begin{aligned}
&+6b^2\kappa^4 - 10b^6\kappa^4 + 6b^{10}\kappa^4)\,x^2y^2\\
&+(1-6b^4+6b^8-6b^2\kappa^2+10b^6\kappa^2-6b^{10}k^2+6b^4\kappa^4-6b^8\kappa^4+b^{12}\kappa^4)\,y^4\\
&+2(1-b^4)(b^2-\kappa^2)(-2b^2+b^6+\kappa^2+b^8\kappa^2+b^2\kappa^4-2b^6\kappa^4)\,x^2\\
&+2(1-b^4)(b^2\kappa^2-1)(b^2-2b^6+\kappa^2+b^8\kappa^2-2b^2\kappa^4+b^6\kappa^4)\,y^2\\
&+(1-b^4)^2(b^2-\kappa^2)^2(1-b^2\kappa^2)^2.
\end{aligned}
$$

We shall consider the case $\kappa = 1$. In this case the convex set $E$ is invariant under the transformation $(x, y) \mapsto (y, x)$. In this case we have the equations

$$
\partial E \cap \{(x,0) : x \geq 0\} = (1+b, 0),
$$
$$
\partial E \cap \{(x,x) : x \geq 0\} = (\sqrt{1+b^2}, \sqrt{1+b^2}).
$$

By the equations (1.11), (1.12) the curve $\partial E$ is parametrized as

$$
\partial E = \{(\cos\theta\,[1+\frac{b^3}{\sqrt{b^4+(1-b^4)\sin^2\theta}}], \sin\theta\,[b+\frac{1}{\sqrt{b^4+(1-b^4)\sin^2\theta}}]) : -\pi \leq \theta \leq \pi\}.
$$

Second we consider the case $k = 3$ and $\phi_1 = \phi_2 = \phi_3 = 0$. By using an affine transformation, we assume that $a_3 = b_3 = 1$. Under these assumptions, the boundary of $E$ is parametrized as

$$
x = x(\theta) = a_1\cos\theta + \frac{a_2^2 b_1 \cos\theta}{\sqrt{R(\theta)}} + \frac{M_1(\theta)\cos\theta}{\sqrt{K(\theta)}},
$$
$$
y = y(\theta) = b_1\sin\theta + \frac{a_1 b_2^2 \sin\theta}{\sqrt{R(\theta)}} + \frac{M_2(\theta)\sin\theta}{\sqrt{K(\theta)}},
$$

where

$$
\begin{aligned}
&R(\theta) = a_2^2 b_1^2\cos^2\theta + a_1^2 b_2^2\sin^2\theta = a_1^2 b_2^2 + (a_2^2 b_1^2 - a_1^2 b_2^2)\cos^2\theta,\\
&M_1(\theta) = a_1 b_2^2 R(\theta) + b_1 R(\theta)^{3/2} + (a_1 a_2^2 b_1^2 b_2^2 - a_1^3 b_2^4)(1-\cos^2\theta),\\
&M_2(\theta) = a_2^2 b_1 R(\theta) + a_1 R(\theta)^{3/2} + (-a_2^4 b_1^3 + a_1^2 a_2^2 b_1 b_2^2)\cos^2\theta,\\
&K(\theta) = \{a_2^2 b_1 R(\theta) + a_1 R(\theta)^{3/2} + (-a_2^4 b_1^3 + a_1^2 a_2^2 b_1 b_2^2)\cos^2\theta\}^2(1-\cos^2\theta)\\
&+\{a_1 b_2^2 R(\theta) + b_1 R(\theta)^{3/2} + (a_1 a_2^2 b_1^2 b_2^2 - a_1^3 b_2^4)(1-\cos^2\theta)\}^2\cos^2\theta\\
&= M_1(\theta)^2\cos^2\theta + M_2(\theta)^2(1-\cos^2\theta)
\end{aligned}
$$

To examine algebraic properties of $\partial E$, we consider an example $a_1 = 5$, $a_2 = 2$, $b_1 = 3$, $b_2 = 1$, $a_3 = b_3 = 1$. In this case, the parametric representation $(x(\theta), y(\theta))$ of $\partial E$ implies the following two equations in $x, y, t$. We set $t = \cos^2\theta$.

$$
A(t,x) = 18496000000000000t^4 + \ldots + 959512576t^8x^8 = 0,
$$
$$
B(t,y) = 308990478515625 + \ldots + 2725888000t^7y^8 = 0,
$$

where $A(t,x)$ has 45 terms and $B(t,y)$ has 55 terms.

The polynomial defining the irreducible algebraic curve $C$ containing the boundary $\partial E$ appears as a factor of the resultant of $A(t,x)$ and $B(t,y)$ with respect to $t$. Roughly speaking, the equation of $\partial E$ is obtained by the elimination of $t = \cos^2\theta$ from two equations $A(t,x) = 0$ and $B(t,y) = 0$. The resultant of $A(t,x)$ and $B(t,y)$ with respect to $t$ is the product of 4 mutually distinct polynomials of degree 24. The polynomial defining $C$ contains 91 terms and given by the following :

$$
\begin{aligned}
& L(x,y) \\
= \; & 24321026304000000 + 6219654271200000x^2 - 9394210810278000x^4 \\
& -119557650310200x^6 + 1148960296277289\ x^8 - 225485918370000x^{10} \\
& +9766119186756x^{12} + 496880820720x^{14} - 41819348898x^{16} \\
& +340997040x^{18} + 33577092x^{20} - 903960x^{22} + 6561x^{24} \\
& -145573057543680000y^2 + 160470758172240000x^2y^2 \\
& -74983506744728160x^4y^2 + 21631932478951920x^6y^2 \\
& -3615852797722368x^8y^2 + 273097884681000x^{10}y^2 \\
& -10116343380240x^{12}y^2 + 459355997880x^{14}y^2 - 33762316896x^{16}y^2 \\
& +1463105880x^{18}y^2 - 28889136x^{20}y^2 + 204120x^{22}y^2 + 276752549006035200y^4 \\
& -327695837918101120x^2y^4 + 129730653917742880x^4y^4 \\
& -29031056926250880x^6y^4 + 3472034123551364x^8y^4 \\
& -262242253173920x^{10}y^4 + 15365609949108x^{12}y^4 \\
& -701349742240x^{14}y^4 + 22038542572x^{16}y^4 - 395922240x^{18}y^4 + 2850876x^{20}y^4 \\
& -160930265588261376y^6 + 210978783393022720x^2y^6 \\
& -87387776403387392x^4y^6 + 15997343953491360x^6y^6 \\
& -1681793837963792x^8y^6 + 117738015146680x^{10}y^6 \\
& -5533530445056x^{12}y^6 + 169468903600x^{14}y^6 - 3082219184x^{16}y^6 \\
& +23594040x^{18}y^6 - 8473879464268544y^8 - 75548840451553280x^2y^8 \\
& +28728792965852896x^4y^8 - 4541575657302320x^6y^8 + 411228018078046x^8y^8 \\
& -22588136336720x^{10}y^8 + 760605714796\ x^{12}y^8 - 15131544080x^{14}y^8 \\
& +128612806\ x^{16}y^8 + 15499748747118592y^{10} + 15200716316517120x^2y^{10} \\
& -5286853460701888x^4y^{10} + 717727895418880x^6y^{10} - 51290469392160x^8y^{10} \\
& +2086231097240x^{10}y^{10} - 48947784944x^{12}y^{10} + 485423960\ x^{14}y^{10} \\
& -1302734568177152y^{12} - 2134564244657920x^2y^{12} + 586973681258240x^4y^{12} \\
& -63047101474880x^6y^12 + 3465268162196x^8y^{12} - 105395060800x^{10}y^{12} \\
& +1297807516x^{12}y^{12} - 345463809903616y^{14} + 193415557928960x^2y^{14} \\
& -36300142805248x^4y^{14} + 3226863390560x^6y^{14} - 148262560336x^8y^{14} \\
& +2470745480x^10y^{14} + 53029187088896y^{16} - 6378626983680x^2y^{16} \\
& +1290332412688x^4y^{16} - 128648994520x^6y^{16} + 3317142241x^8y^{16} \\
& -820479232000y^{18} - 93712764800x^2y^{18} - 59761772000x^4y^{18} \\
& +3057712400x^6y^{18} - 133610720000y^{20} - 8790800000x^2y^{20}
\end{aligned}
$$

$$+1835260000x^4y^{20} + 1720000000y^{22} + 644000000x^2y^{22} + 100000000y^{24}$$
$$= 0.$$

We can give an alternating expression of $E$ in the case $a_1 = 5$, $a_2 = 2$, $b_1 = 3$, $b_2 = 1$, $a_3 = b_3 = 1$ by using the equation in $\lambda = H(\theta)$, that is, $g(\lambda, -\cos\theta, -\sin\theta) = 0$, where

$$\begin{aligned} & g(\lambda, -\cos\theta, -\sin\theta) \\ = \; & \lambda^8 - (76\cos^2\theta + 44)\,t^6 + (1782\cos^4\theta + 2012\cos^2\theta + 574)\,t^4 \\ & -(12844\cos^6\theta + 23428\cos^4\theta + 13652\cos^2\theta + 2556)\,t^2 \\ & +28561\cos^8 + 57460\cos^6\theta + 44110\cos^4\theta + 15300\cos^2\theta + 2025. \end{aligned}$$

We close this section by posing a question.

**Question 3.1** Is the degree of the defining polynomial of the algebraic curve containing the boundary of $E = E_1 + E_2 + \ldots + E_k$ equal to $k \times 2^k$ in a generic case ?

**4 Numerical methods and graphics** We can apply many numerical methods to estimate the sum $E$ of elliptical discs $E_1, E_2, \ldots, E_k$. We assume that

$$E_j = \{r\,(a_j\cos\theta, b_j\sin\theta) : -\pi \le \theta \le \pi, 0 \le r \le 1\}$$

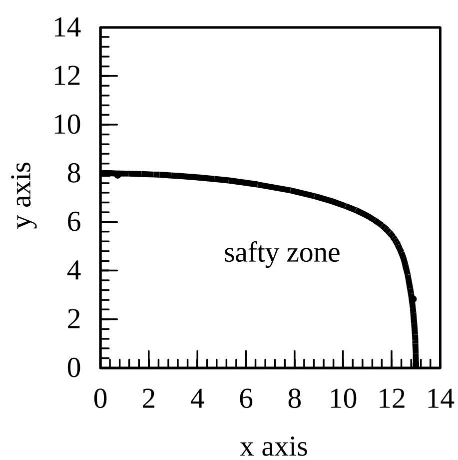



Fig.3 An example of safty zones calculated by the numerical method

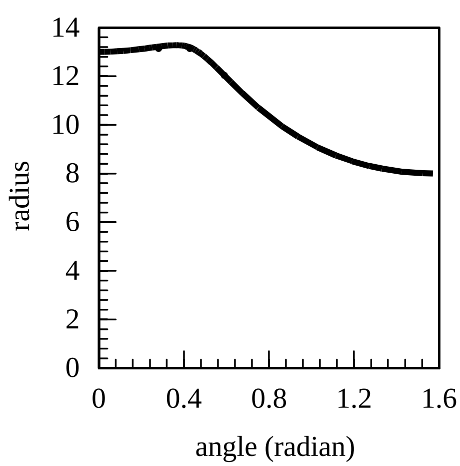



Fig.4 Radius of the safty zone shown in Fig.3

where $a_j > 0, b_j > 0$ ( $1 \leq j \leq k$). By the equation

$$\max\{x\cos\theta + y\sin\theta : (x,y) \in \partial E\}$$
$$= \sum_{j=1}^{k} \sqrt{a_j^2\cos^2\theta + b_j^2\sin^2\theta},$$

an outer approximation of the arc $\partial E \cap \{(x,y) \in \mathbf{R}^2 : x \geq 0, y \geq 0\}$ by the polygon formed by the line segments $\{(t\,x(p-1) + (1-t)\,x(p), t\,y(p-1) + (1-t)\,y(p)) : 0 \leq t \leq 1\}$ is given by

$$a(p) = \sum_{j=1}^{k} \sqrt{a_j^2\cos^2(p\pi/(2L)) + b_j^2\sin^2(p\pi/(2L))},$$

( $p = 0, 1, 2, \ldots, L$)

$$x(p) = \mathrm{cosec}(\pi/(2L))\,\{-a(p+1)\sin(p\pi/(2L)) + a(p)\sin((p+1)\pi/(2L))\},$$
$$y(p) = \mathrm{cosec}(\pi/(2L))\,\{a(p+1)\cos(p\pi/(2L)) - a(p)\cos((p+1)\pi/(2L))\},$$

$(p = 0, 1, \ldots, L-1)$. In the above $L$ is a large natural number. We may describe an approximate graphic of the safety zone by using the above approximation. We may also use some computer programs for $c$-numerical ranges, e. g., [11], [6]. We give the graph of the curve $\partial E$ in Fig.3 and the graph of the radius as a function of the angle in Fig.4 in the case $k = 4$, $a_1 = 5$, $a_2 = 1$, $a_3 = 4$, $a_4 = 3$, $b_1 = 1$, $b_2 = 5$, $b_3 = 1$, $b_4 = 1$. We find that an approximate value of the maximum of radius is 13.2772. The maximum is attained about at $\theta = 0.3636$ (radians). In this case the minimum of the radius is attained at $\theta = \pi/2$.

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