# OPTIMAL SEQUENTIAL DECISIONS WITH INFORMATION INVARIANCE

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Received December 12, 2003

Abstract. $N$ independent and identically distributed random variables are sampled sequentially from a known continuous distribution. The optimal policy for maximizing the probability of obtaining the maximum of the sequence is formulated generally for any information source and illustrated with several examples. When following the optimal policy, it is shown in each case considered that the expected number of random variables sampled is simply related to the probability of selecting the maximum of the sequence. This relationship is posed as a conjecture for all information sources.

**1 Introduction** $N$ independent and identically distributed random variables are sampled sequentially from a known continuous distribution. At each sampling an irrevocable decision for acceptance or rejection must be made. Exactly one of the $N$ random variables sampled is to be accepted at which time the sampling procedure terminates. The problem is to find the optimal strategy which maximizes the probability of selecting the largest in the sequence when various types of information are available from the sample. Under this strategy we also find the expected number of observations made until a choice is made.

The types of information considered in this article are:

A. **No information**: In this case there is no information about the value the random variable assumes when it is sampled.

B. **Complete information**: In this case the value a random variable assumes when sampled is known exactly.

C. **Sample extremum information**: In this case the information available is whether the random variable sampled is greater or less than the maximum of the random variables previously sampled.

D. **Single level information**: In this example the information available is whether the random variable sampled is greater or less than some fixed number.

Examples B and D have been considered by Gilbert and Mosteller [1]. D is also a special case of Enns [2] in which case he considers an arbitrary number of sampling levels. Example C is equivalent to the complete information case B with the exception that no information about the distribution of the random variables sampled is known. This has been considered by Morgenstern [3]. This article combines the varied approaches in a single formulation.

In each of the above cases, the distribution and expected value of the number of random variables sampled has been obtained.

2000 *Mathematics Subject Classification.* Primary 90D35, Secondary 62L15.
*Key words and phrases.* Optimal stopping, Information Invariance, Best choice problem.

**Conjecture 1** *If $M_N$ is the event that the maximum in the sequence of size $N$ is selected and $K_N$ is the position in the sequence of the random variable selected, then we show that under the optimal strategy:*

$$P(M_N) = (E(K_N) - \beta)/N \tag{1.1}$$

*$\beta$ is the number of random variables at the beginning of the sequence that are not available for selection under the optimal policy.*

All examples in this article confirm this conjecture.

**2 Formulation** Let $(Y_1, Y_2, \ldots, Y_N)$ be $N$ independent and identically distributed random variables with known distribution $F(y) = P\{Y_i \leq y\}$. As the distribution $F(y)$ is known and the problem is to select the maximum of the sequence, then this problem is equivalent to selecting the maximum of the sequence $(X_1, X_2, \ldots, X_N)$ where the transformation from $Y \to X$ preserves the ordering of the sequence. In this paper the transformation $X_i = F(Y_i)$, $i = 1, 2, \ldots, N$ will be used. Hence $X_i$ has a uniform distribution on $[0, 1]$.

Let $\underline{z}_n = (x_1, x_2, \ldots, x_n)$ for $n \leq N$ denote the sequential sample obtained to the $n$th sampling from $(X_1, X_2, \ldots, X_N)$. Let $\omega_n$ be the information available ¿from $\underline{z}_n$; this will vary in each of the cases mentioned. Then define:

$M_N$ : the event that the maximum of the sample of size $N$ is selected.
$K_N$ : the number of random variables sampled, including the one selected.
$A_i$ : the event that $X_i$, $i = 1, 2, \ldots, N$, is selected in the sequential procedure.

Decisions at each stage in the procedure will be based on the information available. Hence let

$$D = \{D_1(\omega_1), D_2(\omega_2), \ldots, D_N(\omega_N)\}$$

represent the policy which is to be followed where

$$D_k(\omega_k) = P\left\{A_k \mid A'_1, A'_2, \ldots, A'_{k-1}, \omega_k\right\} \qquad \text{for } k = 1, 2, \ldots, N \tag{2.1}$$

The probability of obtaining the maximum of the sequence when following policy $D$ with information source $\gamma$ is $P_D^{\gamma}\{M_N\}$. The optimal policy $D^*$ is defined as

$$P^{\gamma}\{M_N\} = P_{D^*}^{\gamma}\{M_N\} = \max_D P_D^{\gamma}\{M_N\}. \tag{2.2}$$

The distribution and expected value of $K_N$ will be obtained for each information source mentioned. In each case the expected value of $K_N$ will be related to the corresponding $P\{M_N\}$. Define:

$$P^{\gamma}\{K_N = k\} = P\{K_N = k \mid \text{ information source } \gamma\}$$

$$E^{\gamma}\{K_N\} = \sum_k k P^{\gamma}\{K_N = k\}. \tag{2.3}$$

**3 The probability of obtaining the Maximum of the sequence and the Expected number of variables sampled** One can now write:

$$P_D^\gamma\{M_N\} = \int_0^1 P\{M_N \mid A_1, \underline{z}_1\} D_1(\omega_1)\, dx_1 + \int_0^1 dx_1(1-D_1(\omega_1)) \int_0^1 dx_2 P\{M_N \mid A_2, A_1', \underline{z}_2\} D_2(\omega_2) + \cdots + \int_0^1 dx_1 (1-D_1(\omega_1)) \ldots$$

$$\int_0^1 dx_{N-1}(1-D_{N-1}(\omega_{N-1})) \int_0^1 dx_N P\left\{M_N \mid A_N, A_1' \ldots, A_{N-1}', \underline{z}_N\right\} D_N(\omega_N). \tag{3.1}$$

Exactly one random variable is to be selected, hence:

$$D_N(\omega_N) \equiv 1$$

and

$$P\left\{M_N \mid A_k, A_1', \ldots, A_{k-1}', \underline{z}_k\right\} = 0 \qquad \text{if } x_k \le r_{k-1}$$
$$= x_k^{N-k} \quad \text{if } x_k > r_{k-1}$$

$$\text{where } r_k = \max(x_1, x_2, \ldots x_k) \ \text{ for } k = 1, \ldots, N. \tag{3.2}$$

Substitution of (3.2) into (3.1) yields:

$$P_D^\gamma\{M_N\} = \int_0^1 x_1^{N-1} D_1(\omega_1)\, dx_1 + \sum_{k=2}^N \left[\prod_{j=1}^{k-1} \int_0^1 dx_j (1-D_j(\omega_j))\right] \int_{r_{k-1}}^1 dx_k x_k^{N-k} D_k(\omega_k). \tag{3.3}$$

In order to find $D^*$, the first step is to find $D_{N-1}^*(\omega_{N-1})$.

The last two terms of the summation in (3.3) may be written as:

$$\prod_{j=1}^{N-2} \int_0^1 dx_j (1-D_j(\omega_j))\, \phi(\underline{z}_{N-2})$$

where

$$\phi(\underline{z}_{N-2}) = (1-r_{N-2}^2)/2 + \int_{r_{N-2}}^1 dx_{N-1}(2x_{N-1}-1) D_{N-1}(\omega_{N-1}) - (1-r_{N-2}) \int_0^{N-2} D_{N-1}(\omega_{N-1})\, dx_{N-1}. \tag{3.4}$$

In order to maximize (3.3), one must first maximize $\phi(\underline{z}_{N-2})$. This can only be done by first specifying the information $\omega_{N-1}$.

## 4 Optimal policies for various types of information

A. **No information**: In this case the most general decision is:

$$D_{N-1}^{*}(\omega_{N-1}) = p_{N-1}$$

as it cannot be a function of $\underline{z}_{N-2}$. Therefore

$$\phi(\underline{z}_{N-2}) = (1 - r_{N-2}^{2})/2$$

which is not a function of $p_{N-1}$, namely $p_{N-1}$ may be arbitrarily selected. Successively substituting $D_k^{*}(\omega_k) = p_k$, one finally obtains:

$$P^{A}\{M_N\} = \int_0^1 dx_1 x^{N-1} D_1^{*}(\omega_1) + \int_0^1 dx_1 (1 - D_1^{*}(\omega_1))(1 - x_1^{N-1})/(N-1). \tag{4.1}$$

With $D_1^{*}(\omega_1) = p_1$, one finds $P^{A}\{M_N\} = 1/N$ as expected. This illustrates that any policy is as good as any other policy when there is no information available. Hence we may select the policy $D_1^{*}(\omega_1) \equiv 1$. Then

$$\begin{aligned} P^{A}\{K_N = k\} &= 1 \quad \text{if} \quad k = 1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and

$$E^{A}\{K_N\}/N = P^{A}\{M_N\} = 1/N. \tag{4.2}$$

This confirms the Conjecture with $\beta = 0$.

B. **Complete information**: In this case $D_{N-1}(\omega_{N-1})$ can be based on a complete knowledge of $z_{N-2}$. (3.4) is then obviously maximized when:

$$\begin{aligned} D_{N-1}^{*}(\omega_{N-1}) &= 1 \quad \text{if} \quad x_{N-1} > \max(0.5, r_{N-2}) \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{4.3}$$

Hence

$$\begin{aligned} \phi(\underline{z}_{N-2}) &= (1 - r_{N-2}^{2})/2 + 1/4 \qquad \text{if} \quad r_{N-2} \le 1/2 \\ &= (1 - r_{N-2}^{2})/2 + r_{N-2}(1 - r_{N-2}) \quad \text{if} \quad r_{N-2} \ge 1/2 \end{aligned} \tag{4.4}$$

Substitution into (3.3) and evaluation of the next optimal policy step, one obtains:

$$\begin{aligned} D_{N-2}^{*}(\omega_{N-2}) &= 1 \quad \text{if} \quad x_{N-2} > \max(r_{N-3}, (1 + \sqrt{6})/5) \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{4.5}$$

Successive substitution and policy evaluation yields:

$$\begin{aligned} D_k^{*}(\omega_k) &= 1 \quad \text{if} \quad x_k > \max(r_{k-1}, L_{N-k}) \quad \text{for} \quad k = 1, \dots, N-1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $L_r$ is the unique root in $[0, 1)$ of

$$\sum_{j=1}^{r} \left(x^{-j} - 1\right)/j = 1. \tag{4.6}$$

Substitution of (4.6) and (3.3) yields:

$$P^B\{M_N\} = \left[1 + \sum_{r=1}^{N-1}\sum_{s=r}^{N-1} L_{N-r}^s/s\right]/N. \tag{4.7}$$

Table 1 lists values of $L_r$ for $r \leq 100$. Empirically, one also obtains:

$$\lim_{N\to\infty} P^B\{M_N\} = 0.5802. \tag{4.8}$$

The optimal policy in this case is given by (4.6). In order to find $P^B\{K_N = k\}$, consider the following revised policy when the $k$th random variable is sampled:

$$\begin{aligned} \pi_k(\omega_k, z) &= 1 \quad \text{if} \quad x_k > \max\{r_{k-1}, L_{N-k}, z\} \\ &= 0 \quad \text{otherwise for } k = 1, \ldots, N-1 \\ \pi_N(\omega_N, z) &\equiv 1. \end{aligned} \tag{4.9}$$

Then $\pi_k(\omega_k, 0) = D_k^*(\omega_k)$ as defined in (4.6). Let

$$P_z\{K_N = k\} = P\{K_N = k \mid \text{ policy } \pi \text{ is being followed}\}.$$

Then one can write:

$$P_z\{K_N = 1\} = 1 - \max(z, L_{N-1})$$

(4.10)

$$P_z\{K_N = k\} = zP_z\{K_{N-1} = k-1\} + \int_z^{\max(z, L_{N-1})} P_x\{K_{N-1} = k-1\}\,dx, \quad \text{for} \quad k \geq 2.$$

When $z = 0$, $P_0\{K_N = k\} = P^B\{K_N = k\}$. From (4.10), the distribution sought is:

$$\begin{aligned} P^B\{K_N = k\} &= 1 - L_{N-1} \quad \text{when} \quad k = 1 \\ &= \sum_{r=1}^{k-1} L_{N-r}^{k-1}/(k-1) - \sum_{r=1}^{k} L_{N-4}^k/k \quad \text{when} \quad 2 \leq k \leq N-1 \\ &= \sum_{r=1}^{N-1} L_{N-r}^{N-1}/(N-1) \quad \text{when} \quad k = N. \end{aligned} \tag{4.11}$$

The first moment is then found to be:

$$E^B\{K_N\}/N = \left(1 + \sum_{r=1}^{N-1}\sum_{s=r}^{N-1} L_{N-r}^s/s\right)/N = P^B\{M_N\}. \tag{4.12}$$

Again this confirms the Conjecture with $\beta = 0$.

C. **Sample extremum information**:

The information that $x_{N-1}$ is greater or less than the previous maximum implies that the most general form of the first policy decision is:

$$\begin{aligned} D_{N-1}(\omega_{N-1}) &= p_{N-1} \quad \text{if} \quad x_{N-1} > r_{N-2} \\ &= q_{N-1} \quad \text{if} \quad x_{N-1} \leq r_{N-2}. \end{aligned} \tag{4.13}$$

This results in:

$$\phi(z_{N-2}) = (1 - r_{N-2}^2)/2 + (p_{N-1} - q_{N-1})r_{N-2}(1 - r_{N-2}) \tag{4.14}$$

which implies the optimal $(N-1)$th decision to be:

$$\begin{aligned} D_{N-1}^*(\omega_{N-1}) &= 1 \quad \text{if} \quad x_{N-1} > r_{N-2} \\ &= 0 \quad \text{otherwise, for } N > 2. \end{aligned} \tag{4.15}$$

Successive integrations in (3.3) finally yield:

$$\begin{aligned} D_k^*(\omega_k) &= 1 \quad \text{if} \quad x_k > r_{k-1} \quad \text{and} \quad k > \xi \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $\xi$ is uniquely determined from:

$$1 \leq \sum_{i=\xi}^{N-1} i^{-1} \leq 1 + 1/\xi. \tag{4.16}$$

(3.3) now becomes:

$$P^c\{M_N\} = \frac{\xi}{N} \sum_{i=\xi}^{N-1} i^{-1}. \tag{4.17}$$

The limiting relation may be obtained by letting:

$$\alpha = \lim_{N\to\infty} \xi/N.$$

From (4.16) one finds $\alpha = e^{-1} = 0.3679$ and hence:

$$\lim_{N\to\infty} P^C\{M_N\} = 0.3679 \tag{4.18}$$

Now

$$\begin{aligned} P^C\{K_N = k\} &= 0 \quad \text{if} \quad k \leq \xi \\ &= \xi/(k(k-1)) \quad \text{if} \quad k = \xi+1, \dots, N-1 \\ &= \xi/(N-1) \quad \text{if} \quad k = N. \end{aligned} \tag{4.19}$$

The first moment is:

$$E^C\{K_N\} = \xi + \xi \sum_{i=\varepsilon}^{N-1} i^{-1}$$

or equivalently:

$$(E^C\{K_N\} - \xi)/N = P^C\{M_N\}. \tag{4.20}$$

Again the Conjecture is satisfied with $\beta = \xi$.

D. **Single level information**:

At each sampling, the information available is whether the random variable sampled is greater or less than some level $l$, see Enns [2]. The most general decision based on this information is:

$$\begin{aligned} D_{N-1}(\omega_{N-1}) &= p_{N-1} \quad \text{if} \quad x_{N-1} > l \\ &= q_{N-1} \quad \text{if} \quad x_{N-1} \le l. \end{aligned} \tag{4.21}$$

One need only consider $l > r_{N-2}$, hence this yields:

$$\phi(\underline{z}_{N-2}) = (1 - r_{N-2}^2)/2 + (p_{N-1} - q_{N-1})l(1-l). \tag{4.22}$$

Thus the optimal $(N-1)$th decision which is the same as all previous optimal decisions is:

$$\begin{aligned} D_k^*(\omega_k) &= 1 \quad \text{if} \quad x_k > l \\ &= 0 \quad \text{if} \quad x_k \le l. \end{aligned} \tag{4.23}$$

In this example (3.3) becomes:

$$P^D\{M_N\} = l^N \left[ 1/N + \sum_{k=1}^{N} (l^{-k} - 1)/k \right] \tag{4.24}$$

where $l$ must be chosen to maximize (4.24).

Table 2 lists optimal values of $l$ for $1 \le N \le 50$ after which the asymptotic value of $l$, namely $l = \exp(-a/N)$ with $a = 1.502861$ may be used. For $N > 50$, the accuracy of this estimate is

$$|l - \exp(-a/N| \le 0.00011. \tag{4.25}$$

Asymptotically, (4.24) becomes

$$\lim_{N\to\infty} P^D\{M_N\} = \left(1 - e^{-a}\right)/a = 0.51735. \tag{4.26}$$

Selected values of $P\{M_N\}$ are tabulated in Table 3 for information sources A, B, C and D. Many other information sources could of course have been considered. Another information source could have been whether the random variable sampled is greater or less than some constant times the sample mean of the random variables previously sampled. Other examples could be constructed using mixtures of the above information sources. An example of this is given later in the article.

When following policy (4.23), one obtains:

$$\begin{aligned} P^D\{K_N = k\} &= l^{k-1}(1-l) \quad \text{for} \quad k = 1, \ldots, N-1 \\ &= l^{N-1} \quad\quad\quad\;\; \text{for} \quad k = N. \end{aligned} \tag{4.27}$$

This implies that:

$$E^D\{K_N\} = (1 - l^N)/(1-l). \tag{4.28}$$

Differentiating (4.24), one finds:

$$dP^D\{M_N\}/dl = NP^D\{M_N\}/l - E^D\{K_N\}/l. \tag{4.29}$$

The policy being optimal requires $dP^D\{M_N\}/dl = 0$, hence $E^D\{K_N\}/N = P^D\{M_N\}$.

This confirms the Conjecture with $\beta = 0$.

Variations on the Single-level problem have been published by Sakaguchi and Szajowski [4][5].

## 5 A mixture example

E. **Mixture of information sources B and C:**

When the first random variable is presented, its' value is known, however the distribution from which it was obtained is unknown. For subsequent random variables the distribution is known and complete information about the value a random variable assumes when sampled is known.

The optimal strategy in this case is intuitively obvious based on the previous discussion, namely, $D_1^*(\omega_1) \equiv 0$ and $D_k^*(\omega_k)$ for $k > 1$ is given by (4.6). One can now write:

$$P^E\{M_N\} = P^E\{M_N, X_1 \leq L_{N-1}\} + P^E\{M_N, X_1 > L_{N-1}\} \tag{5.1}$$

where

$$P^E\{M_N, X_1 \leq L_{N-1}\} = P^B\{M_N, X_1 \leq L_{N-1}\} = P^B\{M_N\} - (1 - L_{N-1}^N)/N \tag{5.2}$$

and

$$\begin{aligned} P^E\{M_N, X_1 > L_{N-1}\} &= \sum_{l=2}^{N} \int_{L_{N-1}}^{1} dx x^{l-2} \int_x^1 u^{N-l} du \\ &= \frac{(1 + L_{N-1}^N)}{N} \sum_{i=1}^{N-1} i^{-1} - \sum_{i=1}^{N-1} \frac{L_{N-1}^{-i}}{i(N-i)}. \end{aligned} \tag{5.3}$$

Hence

$$P^E\{M_N\} = P^B\{M_N\} - \frac{(1 - L_{N-1}^N)}{N} + \frac{(1 + L_{N-1}^N)}{N} \sum_{i=1}^{N-1} i^{-1} - \sum_{i=1}^{N-1} L_{N-1}^{-i}/(i(N-i)). \tag{5.4}$$

Similarly:

$$\mathcal{E}^E\{K_N\} = \mathcal{E}^E\{K_N \mid X_1 \leq L_{N-1}\} P\{X_1 \leq L_{N-1}\} + E^E\{K_N \mid X_1 > L_{N-1}\} P\{X_1 > L_{N-1}\} \tag{5.5}$$

where

$$\mathcal{E}^E\{K_N \mid X_1 \leq L_{N-1}\} P\{X_1 \leq L_{N-1}\} = \mathcal{E}^B\{K_N\} - (1 - L_{N-1}) \tag{5.6}$$

and

$$\begin{aligned} P^E\{K_N = k, X_1 > L_{N-1}\} &= (1 - L_{N-1}^{k-1})/(k-1) - (1 - L_{N-1}^k)/k \quad \text{for} \quad k = 2, \ldots, N-1 \\ &= (1 - L_{N-1}^{N-1})/(N-1) \quad \text{for} \quad k = N. \end{aligned} \tag{5.7}$$

Hence

$$\mathcal{E}^E\{K_N \mid X_1 > L_{N-1}\} P\{X_1 > L_{N-1}\} = 1 - L_{N-1} + \sum_{k=1}^{N-1} \left(1 - L_{N-1}^k\right)/k \tag{5.8}$$

from which one finally obtains:

$$\mathcal{E}^E\{K_N\} = \mathcal{E}^B\{K_N\} + \sum_{k=1}^{N-1} (1 - L_{N-1}^k)/k. \tag{5.9}$$

Utilizing (4.12) one obtains the relation:

$$(\mathcal{E}^E\{K_N\}-1)/N = P^E\{M_N\} - \frac{L_{N-1}^N}{N}\left[1-\sum_{r=1}^{N-1}\frac{L_{N-1}^{-r}-1}{r}\right] = P^E\{M_N\} \tag{5.10}$$

from the definition of $L_{N-1}$ in (4.6).

The conjecture is therefore verified in this somewhat more complex example.

## References


[1] Gilbert, J.P. and Mosteller, F., “Recognizing the maximum of a sequence”, J.A.S.A. March 1966, pp. 35–73.

[2] Enns, E.G., “Selecting the maximum of a sequence with imperfect information”, J.A.S.A. September 1975, pp. 640–643.

[3] Morgenstern, D., “Einführung in die Wahrscheinlichkeitsrechnung und Mathematische Statistik”, Berlin 1964, Seite 13.

[4] Sakaguchi, M. and Szajowski, K., “Single-level strategies for Full-information Best-Choice problems I”, Math. Japonica 45, No 3 (1997), 483–495.

[5] Sakaguchi, M. and Szajowski, K., “Single-level strategies for Full-information Best-choice problems II”, Math. Japonica 46, No 2 (1997), 199–208.

TABLE 1

| $r$ | $L_r$ | $r$ | $L_r$ | $r$ | $L_r$ |
|---|---|---|---|---|---|
| 1 | .50000 | 34 | .97674 | 67 | .98810 |
| 2 | .68990 | 35 | .97739 | 68 | .98927 |
| 3 | .77585 | 36 | .97801 | 69 | .98944 |
| 4 | .82459 | 37 | .97859 | 70 | .98860 |
| 5 | .86696 | 38 | .97915 | 71 | .98876 |
| 6 | .87781 | 39 | .97968 | 72 | .98892 |
| 7 | .89391 | 40 | .98018 | 73 | .98907 |
| 8 | .90627 | 41 | .98065 | 74 | .98921 |
| 9 | .91604 | 42 | .98111 | 75 | .98936 |
| 10 | .92398 | 43 | .98154 | 76 | .98950 |
| 11 | .93054 | 44 | .98196 | 77 | .98963 |
| 12 | .93606 | 45 | .98235 | 78 | .98976 |
| 13 | .94077 | 46 | .98273 | 79 | .98989 |
| 14 | .94483 | 47 | .98309 | 80 | .99002 |
| 15 | .94837 | 48 | .98344 | 81 | .99014 |
| 16 | .95148 | 49 | .98378 | 82 | .99026 |
| 17 | .95424 | 50 | .98410 | 83 | .99038 |
| 18 | .95671 | 51 | .98440 | 84 | .99049 |
| 19 | .95892 | 52 | .98470 | 85 | .99060 |
| 20 | .96091 | 53 | .98499 | 86 | .99071 |
| 21 | .96272 | 54 | .98526 | 87 | .99082 |
| 22 | .96437 | 55 | .98553 | 88 | .99092 |
| 23 | .96589 | 56 | .98578 | 89 | .99102 |
| 24 | .96737 | 57 | .98603 | 90 | .99112 |
| 25 | .96855 | 58 | .98627 | 91 | .99122 |
| 26 | .96974 | 59 | .98650 | 92 | .99131 |
| 27 | .97083 | 60 | .98672 | 93 | .99140 |
| 28 | .97185 | 61 | .98694 | 94 | .99150 |
| 29 | .97281 | 62 | .98715 | 95 | .99158 |
| 30 | .97369 | 63 | .98735 | 96 | .99167 |
| 31 | .97453 | 64 | .98754 | 97 | .99176 |
| 32 | .97531 | 65 | .98773 | 98 | .99184 |
| 33 | .97605 | 66 | .98792 | 99 | .99192 |

Optimal values of $L_r$ in the complete information case with $N \leq 100$.

TABLE 2

| $N$ | $l$ | $N$ | $l$ | $N$ | $l$ | $N$ | $l$ | $N$ | $l$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 11 | .87426 | 21 | .93152 | 31 | .95295 | 41 | .96417 |
| 2 | .5 | 12 | .88396 | 22 | .93450 | 32 | .95438 | 42 | .96500 |
| 3 | .62284 | 13 | .89227 | 23 | .93724 | 33 | .95572 | 43 | .96580 |
| 4 | .69784 | 14 | .89947 | 24 | .93975 | 34 | .95699 | 44 | .96656 |
| 5 | .74814 | 15 | .90577 | 25 | .94207 | 35 | .95819 | 45 | .96729 |
| 6 | .78415 | 16 | .91132 | 26 | .94422 | 36 | .95932 | 46 | .96798 |
| 7 | .81118 | 17 | .91626 | 27 | .94622 | 37 | .96039 | 47 | .96865 |
| 8 | .83221 | 18 | .92068 | 28 | .94808 | 38 | .96141 | 48 | .96929 |
| 9 | .84903 | 19 | .92466 | 29 | .94981 | 39 | .96237 | 49 | .96991 |
| 10 | .86279 | 20 | .92825 | 30 | .95143 | 40 | .96329 | 50 | .97050 |

Optimal values of $l$ in the single level information case.

TABLE 3

| $N$ | $P^A\{M_N\}$ | $P^B\{M_N\}$ | $P^C\{M_N\}$ | $P^D\{M_N\}$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 0.5000 | 0.7500 | 0.5000 | 0.7500 |
| 3 | 0.3333 | 0.6843 | 0.5000 | 0.6703 |
| 4 | 0.2500 | 0.6554 | 0.4583 | 0.6312 |
| 5 | 0.2000 | 0.6392 | 0.4333 | 0.6080 |
| 6 | 0.1667 | 0.6288 | 0.4278 | 0.5926 |
| 7 | 0.1429 | 0.6215 | 0.4143 | 0.5817 |
| 8 | 0.1250 | 0.6161 | 0.4098 | 0.5736 |
| 9 | 0.1111 | 0.6120 | 0.4060 | 0.5673 |
| 10 | 0.1000 | 0.6087 | 0.3987 | 0.4522 |
| 20 | 0.0500 | 0.5942 | 0.3842 | 0.5397 |
| 50 | 0.0200 | 0.5857 | 0.3743 | 0.5263 |
| 100 | 0.0100 | 0.5829 | 0.3710 | 0.5218 |
| $\infty$ | 0 | 0.5802 | 0.3679 | 0.5174 |

Selected Values of $P^\gamma\{M_N\}$ for
Various Types of Information

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