DOUBT FUZZY SUBINCLINES(IDEALS) OF INCLINE ALGEBRAS

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Abstract. In this paper, we introduce the concept of doubt fuzzy subinclines(ideals) of incline algebras and some related properties are investigated. We also state the doubt product of doubt fuzzy subinclines(ideals), and the injections of doubt fuzzy subinclines(ideals). Moreover, we discuss the chain condition of subinclines(ideals).

1.Introduction and Preliminaries An incline algebra is a set $\mathcal{H}$ with two binary operations denoted by " $+$ " and " $*$ " satisfies the following axioms for all $x, y, z \in \mathcal{H}$:

(i) $x + y = y + x$
(ii) $x + (y + z) = (x + y) + z$
(iii) $x * (y * z) = (x * y) * z$
(iv) $x * (y + z) = x * y + x * z$
(v) $(y + z) * x = y * x + z * x$
(vi) $x + x = x$
(vii) $x + (x * y) = x$
(viii) $y + (x * y) = y$

For convenience, we pronounce " $+$ "(resp. " $*$ ") as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice(as simiring) if and only if $x * x = x$ for all $x \in \mathcal{H}$. Note that $x \leq y$ if and only if $x + y = y$ for all $x, y \in \mathcal{H}$. A subincline of an incline $\mathcal{H}$ is a subset $\mathcal{M}$ of $\mathcal{H}$ closed under addition and multiplication . An ideal in an incline $\mathcal{H}$ is a subincline $\mathcal{M} \subset \mathcal{H}$ such that if $x \in \mathcal{M}$ and $y \leq x$ then $y \in \mathcal{M}$. By a homomorphism of incline $\mathcal{H}$ into an incline $\mathcal{I}$ such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in \mathcal{H}$.

2. Doubt fuzzy subinclines(ideals) In what follows, $F(\mathcal{H})$ denotes the set of all fuzzy subsets in $\mathcal{H}$, i.e., maps from $\mathcal{H}$ into $([0, 1], \vee, \wedge)$,where $[0, 1]$ is the set of reals between 0 and 1 and $x \vee y = max\{x, y\}, x \wedge y = min\{x, y\}$.

Definition 2.1. $A \in F(\mathcal{H})$ is called a doubt fuzzy subincline of $\mathcal{H}$ if $A(x + y) \vee A(x * y) \leq A(x) \vee A(y)$ for all $x, y \in \mathcal{H}$. A fuzzy subset $A \in F(\mathcal{H})$ is said to be order preserving if $A(x) \leq A(y)$ whenever $x \leq y$.

Definition 2.2. A fuzzy set subincline A is called a doubt fuzzy fuzzy ideal of $\mathcal{H}$ if it is order preserving.

Example 2.3. Note that for any $x \in \mathcal{H}$, the set $\mathcal{M} = \{a | a \leq x\}$ is an ideal of $\mathcal{H}$.

Define $A \in F(\mathcal{H})$ by

$$A(x) = \begin{cases} 0.3 & \text{if } x \in \mathcal{M} \\ 0.8 & \text{otherwise} \end{cases}$$

for all $x \in \mathcal{H}$.

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It's easy to check that A is a doubt fuzzy ideal of $\mathcal{H}$.

Theorem 2.4. If $A \in F(\mathcal{H})$ is a doubt fuzzy ideal, then $A(x+y) = A(x) \vee A(y)$ for all $x, y \in \mathcal{H}$.

Proof. Clearly $A(x+y) \leq A(x) \vee A(y)$ for all $x, y \in \mathcal{H}$. Note that $x+(x+y) = y+(x+y) = (x+y)+y = x+(y+y) = x+y$, so that $x \leq x+y$ and $y \leq x+y$ for all $x, y \in \mathcal{H}$. Since A is order preserving , it follows that $A(x) \leq A(x+y)$ and $A(y) \leq A(x+y)$, which imply that $A(x) \vee A(y) \leq A(x+y)$. This completes the proof.

Theorem 2.5. $A \in F(\mathcal{H})$ is a doubt fuzzy subincline (resp. doubt fuzzy ideal) of $\mathcal{H}$ if and only if the nonempty subset $A^t = \{x \in \mathcal{H} | A(x) \leq t\}, t \in [0,1]$ of A is a subincline (resp. ideal) of $\mathcal{H}$.

Proof. Let $A \in F(\mathcal{H})$ be a doubt fuzzy subincline and let $x, y \in A^t$ for $t \in [0,1]$. Then $A(x+y) \vee A(x+y) \leq A(x) \vee A(y) \leq t$, which implies that $x+y \in A^t$ and $x * y \in A^t$. Hence A is a doubt fuzzy ideal of $\mathcal{H}$. Let $x \in A^t$ and $y \leq x$. Then $A(y) \leq A(x) \leq t$, since A^t is order preserving , and so $y \in A^t$. This proves that A^t is an ideal of $\mathcal{H}$.

Conversely, assume that the nonempty subset A^t is a subincline of $\mathcal{H}$. We first show that $A(x+y) \leq A(x) \vee A(y)$ for all $x, y \in \mathcal{H}$. If not, then there exist $x_0, y_0 \in \mathcal{H}$ such that $A(x_0) + y_0) > A(x_0) \vee A(y_0)$. Taking $t_0 = \frac{1}{2}\{A(x_0+y_0) + A(x_0) \vee A(y_0)\}$, we have $A(x_0) \vee A(y_0) < t_0 < A(x_0+y_0)$. Thus $x_0 \in A^{t_0}$ and $y_0 \in A^{t_0}$, which imply that $x_0 + y_0 \in A^{t_0}$, and that $A(x_0+y_0) \leq t_0$. This is a contradiction. Secondly , if $A(x_0) \vee A(y_0) < A(x_0 * y_0)$ for some $x_0, y_0 \in \mathcal{H}$, by taking $s_0 = \frac{1}{2}\{A(x_0 * y_0) + A(x_0) \vee A(y_0)\}$, we have $A(x_0) \vee A(y_0) < s_0 < A(x_0 * y_0)$. It follows that $x_0 \in A^{s_0}$ and $y_0 \in A^{s_0}$. Since A^{s_0} is a subincline , we get $x_0 * y_0 \in A^{s_0}$ and that $A(x * y) \leq A(x) \vee A(y)$ for all $x, y \in \mathcal{H}$. By these aspects, we obtain $A(x+y) \vee A(x * y) \leq A(x) \vee A(y)$ for all $x, y \in \mathcal{H}$. Therefore A is a doubt fuzzy subincline of $\mathcal{H}$. Furthermore, assume that every level subset $A^t, t \in [0,1]$ is an ideal of $\mathcal{H}$. If A is not order preserving, then there exist $x_0, y_0 \in \mathcal{H}$ such that $x_0 \leq y_0$ and $A(x_0) > A(y_0)$. Putting $m_0 = \frac{1}{2}\{A(x_0) + A(y_0)\}$, we have $A(y_0) < m_0 < A(x_0)$ and so $y_0 \in A^{m_0}$. Since A^{m_0} is an ideal of $\mathcal{H}$, it follows that $x_0 \in A^{m_0}$ and that $A(x_0) \leq m_0$, which is impossible. Hence A is order preserving , and therefore A is a doubt fuzzy ideal of $\mathcal{H}$. This completes the proof.

Theorem 2.6. Let $\mathcal{M}$ be a subincline(resp. ideal) of $\mathcal{H}$ and let $A \in F(\mathcal{H})$ be defined by

$$A(x) = \begin{cases} t_0 & \text{if } x \in \mathcal{M} \\ t_1 & \text{otherwise} \end{cases}$$

for all $x \in \mathcal{H}$,where $t_0, t_1 \in [0,1], t_0 < t_1$. Then A is a doubt fuzzy subincline (resp. doubt fuzzy ideal) of $\mathcal{H}$, and $A^{t_0} = M$.

Proof. Let $x, y \in \mathcal{M}$.If either x or y does not belong to $\mathcal{M}$, then clearly $A(x+y) \vee A(x * y) \leq t_1 = A(x) \vee A(y)$. If $x, y \in \mathcal{M}$, then $x+y \in \mathcal{M}$ and $x * y \in \mathcal{M}$. It follows that $A(x+y) \vee A(x * y) = A(x) \vee A(y)$. This proves that A is a doubt fuzzy subincline of $\mathcal{H}$. Moreover, assume that $y \leq x$. If $x \in \mathcal{M}$, then $y \in \mathcal{M}$. Since $\mathcal{M}$ is an ideal of $\mathcal{H}, A(x) = A(y)$. If $x \notin \mathcal{M}$, then $A(x) = t_1 \geq A(y)$. Hence A is order preserving , and therefore A is a doubt fuzzy ideal of $\mathcal{H}$. It's clear that $A^{t_0} = \mathcal{M}$. This proves the proof.

Using Theorem 2.6, the following theorem is straightforward.

Corollary 2.7. Let $\mathcal{M}$ be a nonempty subset of $\mathcal{H}$ and let $A \in F(\mathcal{H})$ such that A is into $\{0,1\}$, so that A is the characteristic function of $\mathcal{M}$. Then A is a doubt subincline(resp. doubt fuzzy ideal) of $\mathcal{H}$ if and only if $\mathcal{M}$ is a subincline(resp. ideal) of $\mathcal{H}$.

The following lemma is obvious and we omit the proof.

Lemma 2.8. Let Λ be a totally ordered set and let $\{\mathcal{M}_t | t \in \Lambda\}$ be a family of subinclines(resp. ideals) of $\mathcal{H}$ such that for all $s, t \in \Lambda, s > t$ if and only if $\mathcal{M}_t \subset \mathcal{M}_s$. Then $\bigcup_{t\in\Lambda} \mathcal{M}_t$ and $\bigcap_{t\in\Lambda} \mathcal{M}_t$ are subinclines (resp. ideals) of $\mathcal{H}$.

Let Λ be a nonempty subset of $[0,1]$.

Theorem 2.9. Let $\{\mathcal{M}_t | t \in \Lambda\}$ be a collection of subinclines(resp. ideals) of $\mathcal{H}$ such that $\mathcal{H} = \bigcap \mathcal{M}_t$ and for all $s, t \in \Lambda, s > t$ if and only if $\mathcal{M}_t \subset \mathcal{M}_t$. Then $A \in F(\mathcal{H})$ defined by $A(x) = inf\{t | x \in \mathcal{M}_t\}$ for all $x \in \mathcal{H}$ is a doubt fuzzy subincline(resp. doubt fuzzy ideal) of $\mathcal{H}$.

Proof. Following Theorem 2.5, it is sufficient to show that A_s is a subincline(resp. ideal) of $\mathcal{H}$ for every $s \in [0,1]$. To do this,we divide into the following two cases: (i)$s = inf\{t \in \Lambda | t > s\}$ and (ii)$s \neq inf\{t \in \Lambda | t > s\}$.

Case (i) implies that

$x \in A_s \Leftrightarrow \mathcal{M}_t$ for all $t > s \Leftrightarrow x \in \bigcap_{t>s} \mathcal{M}_t$,

so that $A_s = \bigcap_{t>s} \mathcal{M}_t$, which is a subincline (resp. ideal) of $\mathcal{H}$ by Lemma 2.8. For the case (ii), we claim that $A_s = \bigcup_{t\leq s} \mathcal{M}_t$. If $x \in \bigcup_{t\leq s} \mathcal{M}_t$, then $x \in \mathcal{M}_t$ for some $t \leq s$. It follows that $A(x) \leq t \leq s$, so that $x \in A_s$. This proves that $\bigcup_{t\leq s} \mathcal{M}_t \subseteq A_s$. Now assume that $x \notin \bigcup_{t\leq s} \mathcal{M}_t$. Then $x \notin \mathcal{M}_t$ for all $t \leq s$. Since $s \neq inf\{t \in \Lambda | t > s\}$, there exists $\varepsilon > 0$, such that $(s+\varepsilon, s) \cap \Lambda = \varnothing$. Hence $x \notin \mathcal{M}_t$ for all $t < s + \varepsilon$, which means that if $x \in \mathcal{M}_t$, then $t \geqslant s + \varepsilon$. Thus $A(x) \geqslant s + \varepsilon > s$, and so $x \notin A_s$. Therefore $A_s \subseteq \bigcup_{t\leq s} \mathcal{M}_t$. Using Lemma 2.8, $A_s = \bigcup_{t\leq s} \mathcal{M}_t$ is a subincline (resp. ideal) of $\mathcal{H}$. Therefore A is a doubt fuzzy subincline(resp. doubt fuzzy ideal) of $\mathcal{H}$.

3. Doubt product

Definition 3.1. Let $A \in F(\mathcal{H}_1)$ and $B \in F(\mathcal{H}_2)$. Then doubt product $A \times B$ of A and B is the element of $F(\mathcal{H}_1 \times \mathcal{H}_2)$ which is defined by $(A \times B)(x,y) = A(x) \vee B(y)$, for any $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$.

Proposition 3.2. If $A \in F(\mathcal{H}_1)$ and $B \in F(\mathcal{H}_2)$ are order preserving, then so is $A_1 \times A_2 \in F(\mathcal{H}_1 \times \mathcal{H}_2)$.

It's obvious.

Theorem 3.3. If A_i are doubt fuzzy subinclines(resp. doubt fuzzy ideals) of $\mathcal{H}_i (i = 1,2)$, then $A_1 \times A_2$ is a doubt fuzzy ideal) of $\mathcal{H}_1 \times \mathcal{H}_2$.

Proof. Let $(x_1,x_2),(y_1,y_2) \in \mathcal{H}_1 \times \mathcal{H}_2$. Then

$(A_1 \times A_2)((x_1,x_2)+(y_1,y_2)) \vee (A_1 \times A_2)((x_1,x_2)*(y_1,y_2))$

$=(A_1 \times A_2)(x_1+y_1, x_2+y_2) \vee (A_1 \times A_2)(x_1 * y_1, x_2 * y_2)$

$=(A_1(x_1+y_1) \vee A_2(x_2+y_2)) \vee (A_1(x_1*y_1) \vee A_2(x_2*y_2))$

$=(A_1(x_1+y_1) \vee A_1(x_1*y_1)) \vee (A_2(x_2+y_2) \vee A_2(x_2*y_2))$

$\geq (A_1(x_1) \vee A_1(y_1)) \vee (A_2(x_2) \vee A_2(y_2))$

$=(A_1(x_1) \vee A_2(x_2)) \vee (A_1(y_1) \vee A_2(y_2))$

$=(A_1 \times A_2)(x_1,x_2) \vee (A_1 \times)(y_1,y_2)$

Hence $A_1 \times A_2$ is a doubt fuzzy subincline of $\mathcal{H}_1 \times \mathcal{H}_2$. Using Proposition 3.2, we know that $A_1 \times A_2$ is a doubt fuzzy ideal of $\mathcal{H}_1 \times \mathcal{H}_2$ whenever A_1 and A_2 are doubt fuzzy ideals of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Definition 3.4. For any $A \in F(\mathcal{H}_1 \times \mathcal{H}_2)$, the injection of A on $\mathcal{H}_1(resp.\ \mathcal{H}_2)$ is the fuzzy subset $inj_1(A) \in F(\mathcal{H}_1)(resp.\ inj_2(A) \in F(\mathcal{H}_2))$ defined by

$inj_1(A)(x) = inf\{A(x,y)|y \in \mathcal{H}_2\}$, for any $x \in \mathcal{H}_1$, $(resp.\ inj_2(A)(x) = inf\{A(x,y)|y \in \mathcal{H}_1\}$, for any $x \in \mathcal{H}_2)$.

Theorem 3.5. Let $\mathcal{H}_2$ be an idempotent incline. If $A \in F(\mathcal{H}_1 \times \mathcal{H}_2)$ is a doubt fuzzy subincline(doubt fuzzy ideal) , then the injections $inj_i(A)$ are doubt fuzzy subinclines(doubt fuzzy ideals) of $\mathcal{H}_i(i = 1, 2)$, respectively.

Proof. For any $x, y \in \mathcal{H}_1$, we have

$inj_1(A)(x+y) \vee\ inj_1(A)(x*y) = inf\{A(x+y,z)|z \in \mathcal{H}_2\} \vee\ inf\{A(x+y,w)|w \in \mathcal{H}_2\}$

$=inf\{A(x+y,z+z)|z \in \mathcal{H}_2\} \vee\ inf\{A(x+y,w*w)|w \in \mathcal{H}_2\}$

$=inf\{A((x,z)+(y,z))|z \in \mathcal{H}_2\} \vee\ inf\{A((x,w)+(y,w))|w \in \mathcal{H}_2\}$

$\geq\ inf\{A(x,z) \vee A(y,z)|z \in \mathcal{H}_2\} \vee\ inf\{A(x,w) \vee A(y,w)|w \in \mathcal{H}_2\}$

$= (inf\{A(x,z)|z \in \mathcal{H}_2\} \vee inf\{A(y,z)|z \in \mathcal{H}_2\}) \vee (inf\{A(x,w)|w \in \mathcal{H}_2\} \vee inf\{A(y,w)|w \in \mathcal{H}_2\}) = inj_1(A)(x) \vee\ inj_1(A)(y)$

Hence $inj_1(A)$ is a doubt fuzzy subincline of $\mathcal{H}_1$. Similarly, $inj_2(A)$ is a doubt fuzzy subincline of $\mathcal{H}_2$. Assume that $A \in F(\mathcal{H}_1 \times \mathcal{H}_2)$ is order preserving and let $x \leq y$ in $\mathcal{H}_1$. Then $inj_1(A)(x) = inf\{A(x,z)|z \in \mathcal{H}_2\} \leq inf\{A(y,z)|z \in \mathcal{H}_2\} = inj_1(A)(y)$.

Similarly, $inj_2(A) \leq inj_2(A)(x)$ whenever $x \leq y$ in $\mathcal{H}_2$. Hence $inj_i(A)(i = 1, 2)$ are doubt fuzzy ideals of $\mathcal{H}_i(i = 1, 2)$ respectively. This proves the proof.

Definition 3.6. $E \in F(\mathcal{H} \times \mathcal{H})$ is called a doubt fuzzy equivalence relation on $\mathcal{H}$ if (i) $E(x,x) = inf\{E(y,z)|y,z \in \mathcal{H}\}$, (ii) $E(x,y) = E(y,x)$, (iii) $E(x,z) \vee E(x,z) \geq E(x,y)$, for all $x,y,z \in \mathcal{H}$. If, moreover, it satisfies (iv) $E(x_1+x_2, y_1+y_2) \vee E(x_1 * x_2, y_1 * y_2) \leq E(x_1,y_1) \vee E(x_2,y_2)$ for all $x_1,x_2,y_1,y_2 \in \mathcal{H}$, we say that E is a doubt fuzzy congruence relation on $\mathcal{H}$.

Theorem 3.7. Let $A \in F(\mathcal{H})$ be a doubt fuzzy ideal. Define a fuzzy subset $R_A \in F(\mathcal{H} \times \mathcal{H})$ by $R_A(x,y) = inf\{A(a)|x+a = y+a, a \in \mathcal{H}\}$ for all $x,y \in \mathcal{H}$. Then R_A is a doubt fuzzy congruence relation on $\mathcal{H}$.

We call R_A the doubt fuzzy relation induced by A.

Proof. For any $x,y,z \in \mathcal{H}$, we have

$R_A(x,x) = inf\{A(a)|x+a = x+a, a \in \mathcal{H}\} = inf\{A(a)|a \in \mathcal{H}\}$

and $R_A(y,z) = inf\{A(b)|y+b = z+b, b \in \mathcal{H}\} \leq inf\{A(b)|b \in \mathcal{H}\} = R_A(x,x)$.

Hence $R_A(x,x) = inf\{R_A(y,z)|y,z \in \mathcal{H}\}$. Clearly $R_A(x,y) = R_A(y,x)$ for all $x,y \in \mathcal{H}$. If $x+a = z+a$ and $z+b = y+b$, then $x+c = y+c$ where $c = a+b$. Using Theorem 2.4, it follows that

$R_A(x,z) \vee R_A(z,y) = inf\{A(a)|x+a = z+a, a \in \mathcal{H}\} \vee inf\{A(b)|z+b = y+b, b \in \mathcal{H}\}$

$=inf\{A(a+b)|x+a = z+a, z+b = y+b, a,b \in \mathcal{H}\} \leq inf\{A(c)|x+c = y+c, c \in \mathcal{H}\} = R_A(x,y)$.

Let $x_1,x_2,y_1,y_2 \in \mathcal{H}$. Then

$R_A(x_1,y_1) \vee R_A(x_2,y_2) = inf\{A(a)|x_1+a = y_1+a, a \in \mathcal{H}\} \vee inf\{A(b)|x_2+b = y_2, b \in \mathcal{H}\}$

$=inf\{A(a+b)|x_1+a = y_1+a, x_2+b = y_2+b, a,b \in \mathcal{H}\}$

$\leq inf\{A(c)|(x_1 + x_2) + c = (y_1 + y_2) + c, c \in \mathcal{H}\} = R_A(x_1 + x_2)$.
Moreover, if $x_1 + a = y_1 + a$ and $x_2 + b = y_2 + b$, then $(x_1 + a) * (x_2 + b) = (y_1 + a) * (y_2 + b)$, and so

$(x_1 * x_2) + \{(a * x_2) + (x_1 * b) + (a * b)\} = (y_1 * y_2) + \{(a * y_2) + (y_1 * b) + (a * b)\}$

Since $(x_1 + a) * b = (y_1 + a) * b$ and $a * (x_2 + b) = a * (y_2 + b)$, we get

$(a * x_2) + (x_1 * b) + (a * b) = (a * y_2) + (y_1 * b) + (a * b)$.

Since A is a doubt fuzzy ideal of $\mathcal{H}$, therefore

$A((a * x_2) + (x_1 * b) + (a * b)) = A(a * x_2) \vee A(x_1 * b) \vee A(a * b) \leq A(a) \vee A(b)$.

Hence $R_A(x_1, y_1) \vee R_A(x_2, y_2) = inf\{A(a)|x_1 + a = y_1 + a, a \in \mathcal{H}\} \vee inf\{A(b)|x_2 + b = y_2 + b, b \in \mathcal{H}\} = inf\{A(a) \vee A(b)|x_1 + a = y_1 + a, x_2 + b = y_2 + b, a, b \in \mathcal{H}\}$
$\geq\ inf\{A((a * x_2) + (x_1 * B) + (A * b))|x_1 + a = y_1 + a, x_2 + b = y_2 + b, a, b \in \mathcal{H}\} \geq inf\{A((a * x_2) + (x_1 * b) + (a * b))|(x_1 * x_2) + ((a * x_2) + (x_1 * b) + (a * b))$
$= (y_1 * y_2) + ((a * y_2) + (y_1) + (a * b)), a, b \in \mathcal{H}\} \geq\ inf\{A(c)|(x_1 * x_2) + c = (y_1 * y_2) + c, c \in \mathcal{H}\} = R_A(x_1 * x_2, y_1 * y_2)$.

Combling these aspects, we conclude that

$R_A(x_1 + x_2, y_1 + y_2) \vee R_A(x_1 * x_2, y_1 * y_2) \leq R_A(x_1, y_1) \vee R_A(x_2, y_2)$.

Therefore R_A is a doubt fuzzy congruence relation on $\mathcal{H}$.

4. Chain conditions

Definition 4.1. An incline $\mathcal{H}$ is said to satisfy the ascending (resp.descending) chain condition (briefly,ACC (resp. DCC)) if for every ascending (resp. descending) sequence $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq \cdots (resp. \mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \mathcal{M}_3 \supseteq \cdots)$ of subinclines of $\mathcal{H}$, there exists a natural number n such that $A_n = A_k$ for all $n \geqslant k$. If $\mathcal{H}$ satisfies DCC, we say that $\mathcal{H}$ is an *Artin* incline algebras.

Theorem 4.2. Let $\mathcal{H}$ be an *Artin* incline algebra and let $A \in F(\mathcal{H})$ be a doubt fuzzy subincline of $\mathcal{H}$. If a sequence of elements of $Im(A)$ is strictly decreasing, then $Im(A)$ has finite number of values.

Proof. Let $\{t_n\}$ be a strictly descending sequence of elements of $Im(A)$, then $0 \leqslant \cdots < t_2 < t_1 \leqslant 1$. Define $A^r = \{x \in \mathcal{H}|A(x) \leq t_r\}, r = 1, 2, 3 \cdots$. Then A^r is a subincline by Theorem 2.5. Let $x \in A^r$, then $A(x) \leqslant t_r < t_{r-1}$, which implies that $x \in A^{r-1}$. Hence $A^r \subseteq A^{r-1}$. Since $t_{r-1} \in Im(A)$, there exists $x_{r-1} \in \mathcal{H}$, such that $A(x_{r-1}) = t_{r-1}$. It follows that $x_{r-1} \in A^{r-1}$, but $x_{r-1} \notin A^r$. Thus $A^r \subset A^{r-1}$, and so we obtain a strictly descending sequence $A^1 \supset A^2 \supset A^3 \supset \cdots$ of ideals of $\mathcal{H}$ which is not terminating. This contradicts the assumption that $\mathcal{H}$ is an *Artin* incline algebra. Hence $Im(A)$ has finite number of values.

Now we consider the converse of Theorem 4.2.

Theorem 4.3. Let $\mathcal{H}$ be an incline algebra. If every doubt fuzzy subincline of $\mathcal{H}$ has finite number of values, then $\mathcal{H}$ is an *Artin* incline algebra.

Proof. Suppose $\mathcal{H}$ does not satisfy DCC, then there exists a strictly descending chain $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots$ of ideals of $\mathcal{H}$.

Define $A \in F(\mathcal{H})$ by

$$A(x) = \begin{cases} \frac{1}{n+1} & \text{if } x \in \mathcal{M}_n - \mathcal{M}_{n+1} \\ 0 & \text{if } x \in \bigcap_{n=0}^{\infty} \mathcal{M}_n \end{cases}$$

where $\mathcal{M}_0$ stands for $\mathcal{H}$.

We prove that A is a doubt fuzzy subincline of $\mathcal{H}$. Assume that $x \in \mathcal{M}_n - \mathcal{M}_{n+1}$ and $y \in \mathcal{M}_k - \mathcal{M}_{k+1}$ for $n = 0, 1, 2\cdots; k = 0, 1, 2\cdots$. Without loss of generality, we may assume that $n \leq k$. Then clearly $x + y \in \mathcal{M}_n$ and $x * y \in \mathcal{M}_n$. Thus $A(x+y) \vee A(x*y) \leq \frac{1}{n+1} = A(x) \vee A(y)$. If $x, y \in \bigcap_{n=0}^{\infty} \mathcal{M}_n$, then $x+y, x*y \in \bigcap_{n=0}^{\infty} \mathcal{M}_n$. Thus $A(x+y) \vee A(x*y) = A(x) \vee A(y)$. If $x \notin \bigcap_{n=0}^{\infty} \mathcal{M}_n$ and $y \in \bigcap_{n=0}^{\infty} \mathcal{M}_n$, then there exists $k \in \mathbb{N}$, such that $x \in \mathcal{M}_k - \mathcal{M}_{k+1}$. It follows that $A(x+y) \vee A(x*y) \leq \frac{1}{k+1} = A(x) \vee A(y)$. Therefore A is a doubt fuzzy subincline of $\mathcal{H}$. Consequently we find that A is a doubt fuzzy subincline and A has infinite number of different values. This is a contradiction and the proof is complete.

Theorem 4.4. The following are equivalent:
(i) $\mathcal{H}$ is an *Artin* incline algebra;
(ii) The set of values of any fuzzy subincline in $\mathcal{H}$ is a well-ordered subset of $[0, 1]$.

Proof. (i)$\Rightarrow$(ii) Let A be a doubt fuzzy subincline of $\mathcal{H}$. Assume that the set of values of A is not a well-ordered subset of $[0, 1]$, then there exists a strictly infinite ascending sequence $\{t_n\}$ such that $A(x_n) = t_n$. Let $\mathcal{M} = \{x \in \mathcal{H} | A(x) \leq t_n\}$. Then $\mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \cdots$ is a strictly infinite decreasing chain of subinclines of $\mathcal{H}$, a contradiction.

(ii)$\Rightarrow$(i) Assume that there exists a strictly infinite decreasing chain:

$(*)$ $\mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \cdots$

of subinclines of $\mathcal{H}$. Let $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$. Then clearly $\mathcal{M}$ is a subincline of $\mathcal{H}$. Define $A \in F(\mathcal{H})$ by

$$A(x) = \begin{cases} 1 & \text{if } \ x \notin \mathcal{M}_n \\ \frac{1}{k} & where \ k = max\{n \in \mathbb{N} | x \in \mathcal{M}_n\} \end{cases}$$

We claim that A is a doubt fuzzy subincline of $\mathcal{H}$. For any $x, y \in \mathcal{H}$, if any one of x and y does not belong to $\mathcal{M}_n$, then clearly $A(x+y) \vee A(x*y) \leq 1 = A(x) \vee A(y)$. If $x, y \in \mathcal{M}_n - \mathcal{M}_{n+1} \leq \frac{1}{n} = A(x) \vee A(y)$. If $x \in \mathcal{M}_n$ and $y \in \mathcal{M}_n - \mathcal{M}_m (or, y \in \mathcal{M}_n$ and $y \in \mathcal{M}_n - \mathcal{M}_m)$, then $A(x+y) \vee A(x*y) \leq \frac{1}{n} \leq \frac{1}{m+1} \leq A(x) \vee A(y)$.

Hence A is a doubt fuzzy subincline of $\mathcal{H}$. Since the chain (*) is not terminating, A has strictly infinite ascending sequence of values. This contradicts that the value set of any doubt fuzzy subincline is welled-ordered. This completes the proof.

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