EQUILIBRIUM IN THE THREE-PLAYER GAME OF "RISKY EXCHANGE"

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Abstract. There are some games widely played in the routine world of gambles, roulette, quiz show and the sports exercises. The object of the game is to get the highest score among all of the players, from one or two chances of sampling. A three-player game of "Risky Exchange", which reduces to a three-player continuous game on the unit cube, having symmetry between players' roles, is investigated, and the optimal strategies for the three players and the winning probabilities they can get in the optimal play are derived.

1 Three-Player Games of "Score Showdown" (Simultaneous-Move Version). Consider the three players I, II and III (sometimes they are denoted by 1, 2 and 3). Let $X_{ij}(i = 1, 2, 3; j = 1, 2)$ be the random variable (r.v.) observed by player i at the j-th observation. We assume that X_{ij}'s are *i.i.d.*, each with uniform distribution in $[0, 1]$.

Each player i privately observes X_{i1} and chooses either one of A_i (*i.e.*, he accepts the observed value of X_{i1}) or R_i (*i.e.*, he rejects the observed value of X_{i1} and resamples a new r.v. X_{i2}). The observed value X_{i1} (and X_{i2} also, when R_i is chosen) and the choices of either A_i or R_i are, of course, unknown to his (or her) opponents. Let

$$(1.1) \qquad S_i(X_{i1}, X_{i2}) = \begin{cases} X_{i1} \\ \varphi(X_{i1}, X_{i2}), \end{cases} \text{if } X_{i1} \text{ is} \begin{cases} \text{accepted} \\ \text{rejected,} \end{cases} \text{by player } i$$

which we call the *score* for player i.

After the play is over, the scores are compared, and the player with the highest score among the players becomes the *winner*. Each player aims to maximize the probability of his (or her) winning. The object of the present paper is to solve the three-player game of "Risky Exchange" *i.e.* the case where

$$(1.2) \qquad \varphi(X_{i1}, X_{i2}) = X_{i2} I(X_{i1} < X_{i2}), \quad i = 1, 2, 3.$$

The game reduces to a continuous game on the unit cube having a high symmetry between the players' roles, and the solution to the game is derived in Section 2. The result given by Theorem 1 is somewhat surprising, and the point is mentioned in Remark 1 of Section 3. Remark 2 is concerned with the sequential-move version of the games.

The games where the score function is given by (1.1), with

$$(1.3) \qquad \varphi(X_{i1}, X_{i2}) = X_{i2},$$

$$(1.4) \qquad \varphi(X_{i1}, X_{i2}) = \frac{1}{2}(X_{i1} + X_{i2}),$$

or

$$(1.5) \qquad \varphi(X_{i1}, X_{i2}) = (X_{i1} + X_{i2}) I(X_{i1} + X_{i2} < 1).$$

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are called “Keep-or-Exchange”, “Competing Average” and “Showcase Showdown”, respectively. The simultaneous-move two-player versions of these games are solved in Ref.[1, 2, 4], and the solution to the three-player version of the game “Keep-or-Exchange” is given in Ref.[3].

2 Simultaneous-Move Game of “Risky Exchange”. Suppose that players' strategies have the form of

$$\text{I accepts (rejects) } X_{11} = x, \text{ if } x > (<) a,$$
$$\text{II accepts (rejects) } X_{21} = y, \text{ if } y > (<) b,$$
$$\text{III accepts (rejects) } X_{31} = z, \text{ if } z > (<) c.$$

Let $M_i(a, b, c) \equiv P\{W_i|\text{I, II and III choose } a, b \text{ and } c, \text{ respctively}\}, i = 1, 2, 3.$ Also let $P(D)$ be the probability of draw, that is

$$\begin{aligned} P(D) &= P\{X_{12} < X_{11} < a, X_{22} < X_{21} < b, X_{32} < X_{31} < c\} = \frac{1}{8}a^2b^2c^2 \\ &\equiv M_0(a, b, c), \text{ say.} \end{aligned} \tag{2.1}$$

Evidently

$$\sum_{i=0}^{3} M_i(a, b, c) = 1, \ \forall (a, b, c) \in [0, 1]^3, \tag{2.2}$$

and, by symmetry,

$$M_i(a, a, a) = \frac{1}{3}(1 - M_0(a, a, a)) = \frac{1}{3}\left(1 - \frac{1}{8}a^6\right), \ i = 1, 2, 3 \ \forall a \in [0, 1]. \tag{2.3}$$

Let $p_{AAA}, p_{RRR}, p_{AAR}$, *etc.*, denote the winning probability for I when the players' choice-triple is $A-A-A, R-R-R, A-A-R$, *etc.* Also let $q_{AAA}, q_{RRR}, q_{AAR}(r_{AAA}, r_{RRR}, r_{AAR})$ *etc.*, similarly denote the winning probability for II (III). Then we find that

$$M_1(a, b, c) = p_{AAA} + p_{RRR} + (\text{other six probabilities}),$$
$$M_2(a, b, c) = q_{AAA} + q_{RRR} + (\text{other six probabilities}),$$
$$M_3(a, b, c) = r_{AAA} + r_{RRR} + (\text{other six probabilities}),$$

$$\begin{aligned} p_{AAA} &= P\{X_{11} > a, X_{21} > b, X_{31} > c, X_{11} > X_{22} \vee X_{31}\} \\ &= \int_{a \vee (b \vee c)}^{1} (t - b)(t - c)dt, \end{aligned} \tag{2.4}$$

$$\begin{aligned} p_{RRR} = {} & P[X_{11} < a \wedge X_{12}, X_{22} < X_{21} < b, \\ & \quad \{X_{32} < X_{31} < c\} \cup \{X_{31} < c \wedge X_{32}, X_{12} > X_{32}\}] \\ & + P[X_{11} < a \wedge X_{12}, X_{21} < b \wedge X_{22}, \\ & \quad \{X_{32} < X_{31} < c, X_{12} > X_{22}\} \cup \{X_{31} < c \wedge X_{32}, X_{12} > X_{22} \vee X_{32}\}] \\ = {} & \frac{1}{8}(2a - a^2)b^2c^2 + \frac{1}{2}b^2 \iint_{1 > s_1 > s_3 > 0} (a \wedge s_1)(c \wedge s_3)ds_1ds_3 \\ & + \frac{1}{2}c^2 \iint_{1 > s_1 > s_2 > 0} (a \wedge s_1)(b \wedge s_2)ds_1ds_2 \\ & + \iiint_{0 < s_2 \vee s_3 < s_1 < 1} (a \wedge s_1)(b \wedge s_2)(c \wedge s_3)ds_1ds_2ds_3, \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
(2.6) \qquad p_{ARA} &= P[X_{11} > a, X_{31} > c, \{X_{22} < X_{21} < b, X_{11} > X_{31}\} \\
&\qquad\qquad \cup\{X_{21} < b \wedge X_{22}, X_{11} > X_{22} \vee X_{31}\}] \\
&= \frac{1}{2}b^2 P\{X_{11} > a, X_{31} > c, X_{11} > X_{31}\} \\
&\qquad + \int_{a\vee c}^{1} (s_1 - c) ds_1 \int_0^{s_1} (b \wedge s_2) ds_2,
\end{aligned}
$$

$$
\begin{aligned}
(2.7) \qquad p_{RAA} &= P\{X_{11} < a \wedge X_{12}, b < X_{21} < X_{12}, c < X_{31} < X_{12}\} \\
&= \int_{b\vee c}^{1} (a \wedge t)(t-b)(t-c) dt,
\end{aligned}
$$

$$
\begin{aligned}
(2.8) \qquad p_{ARR} = P[&X_{11} > a, X_{22} < X_{21} < b, \\
&\{X_{32} < X_{31} < c\} \cup \{X_{31} < c \wedge X_{32}, X_{11} > X_{32}\}] \\
+P[&X_{11} > a, X_{21} < b \wedge X_{22}, \\
&\{X_{32} < X_{31} < c, X_{11} > X_{22}\} \cup \{X_{31} < c \wedge X_{32}, X_{11} > X_{22} \vee X_{32}\}] \\
= \frac{1}{4}&\bar{a}b^2c^2 + \frac{1}{2}b^2 \int_a^1 ds_1 \int_0^{s_1} (c \wedge s_2) ds_2 + \frac{1}{2}c^2 \int_a^1 ds_1 \int_0^{s_1} (b \wedge s_2) ds_2 \\
&+ \int_a^1 ds_1 \iint_{s_2 \vee s_3 < s_1} (b \wedge s_2)(c \wedge s_3) ds_2 ds_3,
\end{aligned}
$$

$$
\begin{aligned}
(2.9) \qquad p_{AAR} &= P\{X_{11} > a, X_{21} > b, X_{32} < X_{31} < c, X_{11} > X_{21}\} \\
&\quad +P\{X_{11} > a, X_{21} > b, X_{31} < c \wedge X_{32}, X_{11} > X_{21} \vee X_{32}\} \\
&= \frac{1}{2}c^2 P\{X_{11} > a, X_{21} > b, X_{11} > X_{21}\} + \int_{a\vee b}^{1} (s_1 - b) ds_1 \int_0^{s_1} (c \wedge s_3) ds_3 \\
&\qquad\qquad (\text{by denoting } s_1 = X_{11} \text{ and } s_3 = X_{32}),
\end{aligned}
$$

$$
\begin{aligned}
(2.10) \qquad p_{RRA} &= P[X_{11} < a \wedge X_{12}, X_{31} > c, \\
&\qquad \{X_{22} < X_{21} < b, X_{12} > X_{31}\} \cup \{X_{21} < b \wedge X_{22}, X_{12} > X_{22} \vee X_{31}\}] \\
&= \frac{1}{2}b^2 \int_c^1 (s_1 - c)(a \wedge s_1) ds_1 + \int_c^1 (s_1 - c)(a \wedge s_1) ds_1 \int_0^{s_1} (b \wedge s_2) ds_2 \\
&\qquad\qquad (\text{by denoting } s_1 = X_{12} \text{ and } s_2 = X_{22}),
\end{aligned}
$$

and finally

$$
\begin{aligned}
(2.11) \qquad p_{RAR} &= P[X_{11} < a \wedge X_{12}, X_{21} > b, \\
&\qquad \{X_{32} < X_{31} < c, X_{12} > X_{21}\} \cup \{X_{31} < c \wedge X_{32}, X_{12} > X_{21} \vee X_{32}\}] \\
&= \frac{1}{2}c^2 \int_b^1 (s_1 - b)(a \wedge s_1) ds_1 + \int_b^1 (s_1 - b)(a \wedge s_1) ds_1 \int_0^{s_1} (c \wedge s_3) ds_3 \\
&\qquad\qquad (\text{by denoting } s_1 = X_{12} \text{ and } s_3 = X_{32}).
\end{aligned}
$$

Equations (2.6) & (2.9) and (2.10) & (2.11) are the two pairs in each of which one becomes the other by interchange of b and c.

First we want to make sure that Eqs(2.4)$\sim$(2.11) do not involve any error, by showing that (2.3) holds true.

Before doing that, we prepare the following lemma.

Lemma 1.1. *The following five identities about multiple integrals take place.*

$$f(a) \equiv \iint_{1>s_1>s_2>0} (a \wedge s_1)(a \wedge s_2)ds_1ds_2 = \frac{1}{8}a^4 - \frac{1}{2}a^3 + \frac{1}{2}a^2, \tag{2.13}$$

$$\begin{aligned} g(s_1|a) &\equiv \iint_{s_2 \vee s_3 < s_1} (a \wedge s_2)(a \wedge s_3)ds_2ds_3 \\ &= \begin{cases} \frac{1}{4}s_1^4, & \text{if } s_1 < a \\ \frac{1}{4}a^4 + a^2(s_1 - a)^2 + a^3(s_1 - a), & \text{if } s_1 > a, \end{cases} \end{aligned} \tag{2.14}$$

$\left[\begin{array}{l} \textit{c.f., This is piecewise increasing with values from } g(0|a) = 0 \textit{ to} \\ g(1|a) = 2f(a), \textit{ and note that } \int_a^1 g(t|a)dt = -\frac{1}{12}a^5 + \frac{1}{4}a^4 - \frac{1}{2}a^3 + \frac{1}{3}a^2. \end{array} \right]$

$$\begin{aligned} h(a) &\equiv \iiint_{0<s_2 \vee s_3 < s_1 < 1} (a \wedge s_1)(a \wedge s_2)(a \wedge s_3)ds_1ds_2ds_3 \\ &= \int_0^1 (a \wedge s_1)g(s_1|a)ds_1 \\ &= \frac{1}{24}a^6 + \frac{1}{4}a^5\bar{a} + \frac{1}{3}a^3\bar{a}^3 + \frac{1}{2}a^4\bar{a}^2 = -\frac{1}{24}a^6 + \frac{1}{4}a^5 - \frac{1}{2}a^4 + \frac{1}{3}a^3, \end{aligned} \tag{2.15}$$

$$\int_a^1 ds_1 \int_0^{s_1} (a \wedge s_2)ds_2 = \frac{1}{2}a\bar{a}, \tag{2.16}$$

and

$$k(a) \equiv \int_a^1 (s_1 - a)ds_1 \int_0^{s_1} (a \wedge s_2)ds_2 = \frac{1}{4}(a\bar{a})^2 + \frac{1}{3}a\bar{a}^3. \tag{2.17}$$

Proofs are easy.

Applying Lemma 1.1 to Eq.(2.4)∼(2.11), we can compute $M_1(a, a, a)$. We obtain that

$$[p_{AAA} + p_{RAA}]_{a=b=c} = (1 + a) \int_a^1 (t - a)^2dt = \frac{1}{3}(1 + a)\bar{a}^3 \tag{2.18}$$

from (2.4) and (2.7) ;

$$\begin{aligned} [p_{RRR}]_{a=b=c} &= \frac{1}{8}a^4(2a - a^2) + a^2f(a) + h(a) \\ &= -\frac{1}{24}a^6 + \frac{1}{3}a^3, \end{aligned} \tag{2.19}$$

from (2.5), together with (2.13) and (2.15) ;

$$\begin{aligned} [p_{ARA} + p_{AAR} + p_{RRA} + p_{RAR}]_{a=b=c} &= (2 + 2a) \left\{ \frac{1}{4}(a\bar{a})^2 + k(a) \right\} \\ &= (1 + a) \left\{ (a\bar{a})^2 + \frac{2}{3}a\bar{a}^3 \right\}, \end{aligned} \tag{2.20}$$

from (2.6), (2.9), (2.10) and (2.11), together with (2.16) and (2.17) ; and finally

$$\begin{aligned}(2.21)\qquad [p_{ARR}]_{a=b=c} &= \frac{1}{4}\bar{a}a^4 + a^2\int_a^1 ds_1 \int_0^{s_1}(a\wedge s_2)ds_2 + \int_a^1 g(s_1|a)ds_1 \\ &= a^3\bar{a} + \frac{1}{3}a^2\bar{a}^3 = -\frac{1}{3}a^5 + \frac{1}{3}a^2\end{aligned}$$

from (2.8), helped by (2.14) and (2.16).

Adding the four equations (2.18)∼(2.21) we find, after simplification, that

$$\begin{aligned}&[p_{AAA} + p_{RRR} + (\text{other six probabilities})]_{a=b=c} \\ &\quad = \frac{1}{3}(1+a)\bar{a}^3 + \left(-\frac{1}{24}a^6 + \frac{1}{3}a^3\right) + (1+a)\left\{(a\bar{a})^2 + \frac{2}{3}a\bar{a}^3\right\} + \left(-\frac{1}{3}a^5 + \frac{1}{3}a^2\right) \\ &\quad = \frac{1}{3}\left(1 - \frac{1}{8}a^6\right), \quad \forall a \in [0,1],\end{aligned}$$

and hence (2.3) holds true.

Now we want to compute $M_1(a,b,b)$, from Eqs(2.4)∼(2.11). For the subsequent equations we denote $I(a<b), I(a=b)$ and $I(a>b)$, by ξ, η and ζ respectively.

Lemma 1.2. *The following four identities about multiple integrals take place.*

$$\begin{aligned}(2.13\alpha)\qquad \iint_{1>s_1>s_2>0}(a\wedge s_1)(b\wedge s_2)ds_1ds_2 &= (\xi+\eta)\cdot\left\{-\frac{1}{24}a^4 + \frac{1}{6}ab(b^2-3b+3)\right\} \\ &\quad + \zeta\cdot\left\{\frac{1}{24}b^4 - \frac{1}{12}ab(2a^2-3ab-6\bar{b})\right\},\end{aligned}$$

(See Eq.(2.6) in Ref.[4])

$$\begin{aligned}(2.15\alpha)\qquad &\iiint_{0<s_2\vee s_3<s_1<1}(a\wedge s_1)(b\wedge s_2)(b\wedge s_3)ds_1ds_2ds_3 = \int_0^1 (a\wedge s_1)g(s_1|b)ds_1 \\ &= (\xi+\eta)\cdot\left[\frac{1}{24}a^6 + \frac{a}{20}(b^5-a^5) + a\int_b^1 g(s_1|b)ds_1\right] \\ &\quad + \zeta\cdot\left[\frac{1}{24}b^6 + \int_b^a s_1 g(s_1|b)ds_1 + a\int_a^1 g(s_1|b)ds_1\right],\end{aligned}$$

$$\begin{aligned}(2.16\alpha)\qquad \int_a^1 ds_1\int_0^{s_1}(b\wedge s_2)ds_2 &= (\xi+\eta)\cdot\left[\frac{1}{2}(a^2\bar{a} + \bar{b}^2 b) + \int_a^b s_2\bar{s_2}ds_2\right] \\ &\quad + \zeta\cdot\left[\frac{1}{2}(\bar{a}b^2 - \bar{a}^2 b) + \bar{a}\bar{b}b\right],\end{aligned}$$

$$\begin{aligned}(2.17\alpha)\quad \int_a^1 (s_1-b)ds_1\int_0^{s_1}(b\wedge s_2)ds_2 &= (\xi+\eta)\cdot\left[\frac{1}{2}\int_a^b s_1^2(s_1-b)ds_1 + \frac{1}{4}(b\bar{b})^2 + \frac{1}{3}b\bar{b}^3\right] \\ &\quad + \zeta\cdot\int_a^1 (s_1-b)\left(\frac{b^2}{2} + b(s_1-b)\right)ds_1.\end{aligned}$$

Proofs are easy. The coefficients of η in Eqs$(2.13\alpha), (2.15\alpha) \sim (2.17\alpha)$ are equal to Eqs(2.13), (2.15)$\sim$(2.17) in Lemma 1.1.

Applying Lemma 1.2 to Eqs(2.4)$\sim$(2.11), we obtain

$$\begin{aligned}(2.22)\qquad [p_{AAA}+p_{RAA}]_{b=c} &= \int_{a\vee b}^{1}(t-b)^2dt+\int_b^1(a\wedge t)(t-b)^2dt\\ &= (\xi+\eta)\cdot\frac{1}{3}(1+a)\bar{b}^3+\zeta\cdot\left[(1+a)\int_a^1(t-b)^2dt+\int_b^a t(t-b)^2dt\right]\end{aligned}$$

from (2.4) and (2.7) ;

$$\begin{aligned}(2.23)\qquad [p_{RRR}]_{b=c} &= \frac{1}{8}(2a-a^2)b^4+b^2\iint_{1>s_1>s_2>0}(a\wedge s_1)(b\wedge s_2)ds_1ds_2\\ &\qquad +\int_0^1(a\wedge s_1)g(s_1|b)ds_1\\ &= \frac{1}{8}(2a-a^2)b^4+(\xi+\eta)\cdot\left[-\frac{1}{24}a^4b^2+\frac{1}{6}ab^3(b^2-3b+3)\right.\\ &\qquad \left.+\frac{1}{24}a^6+\frac{a}{20}(b^5-a^5)+a\int_b^1 g(s_1|b)ds_1\right]\\ &\quad +\zeta\cdot\left[\left\{\frac{1}{24}b^6-\frac{1}{12}ab^3(2a^2-3ab-6\bar{b})\right\}\right.\\ &\qquad \left.+\left\{\frac{1}{24}b^6+\int_b^a s_1g(s_1|b)ds_1+a\int_a^1 g(s_1|b)ds_1\right\}\right]\end{aligned}$$

from (2.5), together with (2.13α) and (2.15α) ;

$$(2.24)\qquad [p_{ARA}+p_{AAR}+p_{RRA}+p_{RAR}]_{b=c}$$

$$\begin{aligned} =\ & b^2\left\{P(X_{11}>a,X_{21}>b,X_{11}>X_{21})+\int_b^1(a\wedge s_1)(s_1-b)ds_1\right\}\\ &+2\left[\int_{a\vee b}^1(s_1-b)ds_1\int_0^{s_1}(b\wedge s_2)ds_2+\int_b^1(a\wedge s_1)(s_1-b)ds_1\int_0^{s_1}(b\wedge s_2)ds_2\right]\\ =\ & b^2\left[(\xi+\eta)\cdot\frac{1+a}{2}\bar{b}^2+\zeta\cdot\left\{\frac{1}{2}(1-a^2-2\bar{a}b)+\int_b^a s_1(s_1-b)ds_1+a\int_a^1(s_1-b)ds_1\right\}\right]\\ &+2\left[(\xi+\eta)\cdot(1+a)k(b)+\zeta\cdot\left\{(1+a)\int_a^1(s_1-b)ds_1\int_0^{s_1}(b\wedge s_2)ds_2\right.\right.\\ &\qquad\left.\left.+\int_b^a s_1(s_1-b)ds_1\int_0^{s_1}(b\wedge s_2)ds_2\right\}\right]\end{aligned}$$

from (2.6), (2.9), (2.10) and (2.11), helped by (2.17α), and finally

$$\begin{aligned}(2.25)\qquad [p_{ARR}]_{b=c} &= \frac{1}{4}\bar{a}b^4+b^2\int_a^1 ds_1\int_0^{s_1}(b\wedge s_2)ds_2+\int_a^1 g(s_1|b)ds_1\\ &= \frac{1}{4}\bar{a}b^4+(\xi+\eta)\cdot\left[\frac{b^2}{2}(a^2\bar{a}+\bar{b}^2b)+b^2\int_a^b s_2\bar{s_2}ds_2\right]\end{aligned}$$

$$+\zeta\cdot\left\{\frac{1}{2}(\bar{a}b^4-\bar{a}^2b^3)+\bar{a}\bar{b}b^3\right\}+\left[(\xi+\eta)\cdot\left\{\int_a^b\frac{1}{4}s_1^4ds_1+\int_b^1 g(s_1|b)ds_1\right\}\right.$$
$$\left.+\zeta\cdot\int_a^1 g(s_1|b)ds_1\right]$$

from (2.8), helped by (2.16α). It is important to confirm that Eqs(2.22)$\sim$(2.25) becomes Eqs(2.18)$\sim$(2.21) when we make $a=b\pm 0$ (*i.e.*, we compute the sum of the coefficients of η in Eqs(2.18)$\sim$(2.21).)

We now prove the following result.

Theorem 1 *Solution to the simultaneous-move three-player game of "Risky Exchange" is as follows. Let* $b^*(\approx 0.656)$ *be a unique root in* $[0,1]$ *of the equation*

$$2b^4+b^5=1-b+b^2-b^3. \tag{2.26}$$

Then the game has a unique equilibrium point (b^*,b^*,b^*) *and the equilibrium payoffs*

$$M_0(b^*,b^*,b^*)=\frac{1}{8}b^{*6}\approx 0.010 \tag{2.27}$$

$$M_i(b^*,b^*,b^*)=\frac{1}{3}\left(1-\frac{1}{8}b^{*6}\right)\approx 0.330,\quad i=1,2,3. \tag{2.28}$$

Proof is made along the same line as is followed in Ref.[3 : Theorem 3]. We want to prove that

$$\max_{a\in[0,1]}M_1(a,b^*,b^*)=M_1(b^*,b^*,b^*), \tag{2.29}$$

where $b^*\approx 0.656$ is defined by (2.26).

We note that, for any $b\in[0,1]$,

$$\lim_{a=b-0}M_1(a,b,b)=\lim_{a=b+0}M_1(a,b,b)=\frac{1}{3}(1-\frac{1}{8}b^6),$$

and

$$M_1(a,b,b)=\text{sum of Eqs(2.22)}\sim\text{(2.25)}$$

Denote the partial derivative $\dfrac{\partial}{\partial a}$ of each of (2.22)$\sim$ (2.25) by (2.22α) $\sim$ (2.25α). The partial derivative $\dfrac{\partial}{\partial a}$ is undefined in Case η. Then

$$(2.22\alpha)=\xi\cdot\frac{1}{3}\bar{b}^3+\zeta\cdot\left[\int_a^1(t-b)^2dt-(a-b)^2\right],$$

$$\begin{aligned}(2.23\alpha)\quad=\quad&\frac{1}{4}\bar{a}b^4+\xi\cdot\left[\frac{1}{6}(-a^3b^2+b^5)+\frac{1}{2}\bar{b}b^3+\frac{1}{20}(b^5-a^5)+\int_b^1 g(s_1|b)ds_1\right]\\&+\zeta\cdot\left[\frac{1}{2}(-a^2b^3+ab^4+\bar{b}b^3)+\int_a^1 g(s_1|b)ds_1\right],\end{aligned}$$

$$\begin{aligned}(2.24\alpha)\quad=\quad&\xi\cdot\left[\frac{1}{2}(b\bar{b})^2+2k(b)\right]+\zeta\cdot\left[b^2\left\{-a+b+\int_a^1(s_1-b)ds_1\right\}\right.\\&\left.+2\left\{\int_a^1(s_1-b)ds_1\int_0^{s_1}(b\wedge s_2)ds_2-(a-b)\int_0^a(b\wedge s_2)ds_2\right\}\right],\end{aligned}$$

and

$$(2.25\alpha) = -\frac{1}{4}b^4 + \xi \cdot \left(-\frac{1}{2}a^2b^2 - \frac{1}{4}a^4\right) + \zeta \cdot \left[-\frac{1}{2}b^4 + (b-a)b^3 - g(a|b)\right].$$

The condition that

$$\begin{aligned} (2.30) \qquad 0 &= \left[\frac{\partial}{\partial a}M_1(a,b,b)\right]_{a=b-0} \\ &= [\text{sum of the coefficients of } \xi \text{ in Eqs}(2.22\alpha) \sim (2.25\alpha)]_{a=b-0} \end{aligned}$$

is identical to

$$0 = \frac{1}{3}\bar{b}^3 + \left(\frac{1}{4}\bar{b}b^4 + \frac{1}{2}\bar{b}b^3 + \int_b^1 g(s_1|b)ds_1\right) + \left(\frac{1}{2}(b\bar{b})^2 + 2k(b)\right) - b^4,$$

which, after simplification, becomes

$$-\frac{1}{3}b^5 - \frac{2}{3}b^4 - \frac{1}{3}b^3 + \frac{1}{3}b^2 - \frac{1}{3}b + \frac{1}{3} = 0$$

or, equivalently, Eq.(2.26).

Next we find that, for $0 \le a < b$,

$$\begin{aligned} \frac{\partial^2}{\partial a^2}M_1(a,b,b) &= \frac{\partial}{\partial a}[\text{sum of the coeff. of } \xi \text{ in Eqs}(2.22\alpha) \sim (2.25\alpha)] \\ &= \left(-\frac{1}{4}b^4 - \frac{1}{2}a^2b^2 - \frac{1}{4}a^4\right) + (-ab^2 - a^3) < 0; \end{aligned}$$

and, for $b < a \le 1$,

$$\begin{aligned} \frac{\partial^2}{\partial a^2}M_1(a,b,b) &= \frac{\partial}{\partial a}[\text{sum of the coeff. of } \zeta \text{ in Eqs}(2.22\alpha) \sim (2.25\alpha)] \\ &= \left[-(a-b)^2 - 2(a-b)\right] + \left[-\frac{1}{4}b^4 + \frac{1}{2}(-2ab^3 + b^4) - g(a|b)\right] \\ &\quad + \Bigg[b^2(-1-a+b) + 2\Bigg\{-(a-b)\int_0^a (b \wedge s_2)ds_2 \\ &\quad - \int_0^a (b \wedge s_2)ds_2 - (a-b)(a \wedge b)\Bigg\}\Bigg] \\ &\quad + \left[-b^3 - \frac{\partial}{\partial a}g(a|b)\right] < 0 \end{aligned}$$

(since all four terms of $[\cdots]$ are negative).

Therefore, $\frac{\partial}{\partial a}M_1(a,b^*,b^*)$ is decreasing in $a \in [0,1]$ and $\left[\frac{\partial}{\partial a}M_1(a,b^*,b^*)\right]_{a=b^*} = 0$. So, $M_1(a,b^*,b^*)$ is concave and unimodal with the maximal value $M_1(b^*,b^*,b^*) = \frac{1}{3}(1 - \frac{1}{8}b^{*6})$. Thus Eq.(2.29) is proven.

The rest part in the proof of the theorem proceeds quite analogously as in Theorem 3 of Ref.[3]. It is lengthy and annoying, and we omit the detail. This completes the proof of our Theorem 1. $\square$

3 Remarks.

Remark 1. The present author obtained in Ref.[4 ; Theorem 2] the following result.

Theorem 2 *Solution to the simultaneous-move two-player game of "Risky Exchange" is as follows. Let* $a^*(\approx 0.54386)$ *be a unique root in* $[0,1]$ *of the equation*

$$a(a^2+a+1)=1. \tag{3.1}$$

Then the game has a unique equilibrium point (a^*,a^*) *and the equilibrium payoffs*

$$P(draw)=\frac{1}{4}a^{*4}\approx 0.02184, \tag{3.2}$$

$$M_i(a^*,a^*)=\frac{1}{2}\left(1-\frac{1}{4}a^{*4}\right)\approx 0.48908, \quad i=1,2. \tag{3.3}$$

The optimal thresholds characterizing the solutions of the three-player game in Theorem 1 and the two-player game in Theorem 2, that is, Eqs (2.26) and (3.1) are seemingly unrelated. However, since we can rewrite Eqs (2.26) and (3.1) as

$$b^4(b^2+3b+3)=1, \tag{3.4}$$

and

$$a^2(a^2+2a+2)=1, \tag{3.5}$$

respectively, some connection can be observed.

Furthermore, we already have one more interesting and maybe related result in Ref.[3 ; Theorem 2].

Theorem 3 *Solution to the simultaneous-move three-player game of "Keep-or-Exchange" is as follows. The game has a unique equilibrium point* (a^*,a^*,a^*) *and the common equlibrium payoffs* $1/3$*, where* $a^*\approx 0.691$ *is a unique root in* $[0,1]$ *of the equation*

$$2a^4=1-a+a^2-a^3. \tag{3.6}$$

Note that Eqs (2.26) and (3.6) are very much alike.

Remark2. It is interesting to consider the sequential-move version of the game. There appears the unfair acquisition of information by players. The game is played in three stages.

In the first stage, I observes that $X_{11}=x$ and chooses one of either A_1 (*i.e.*, I accepts x) or R_1 (*i.e.*, I rejects x, and resamples a new r.v. X_{12}). The observed value x and I's choice of either A_1 or R_1 are informed to II and III. But the observed value of X_{12} is not informed to II and III.

In the second stage, after knowing I's choice of $x\&(A_1\cup R_1)$, II observes that $X_{21}=y$ and chooses either one of A_2 (*i.e.*, II accepts y) or R_2 (*i.e.*, II rejects y and resamples a new r.v. X_{22}). The observed value of y and II's choice of either A_2 or R_2 are informed to III. But the observed value of X_{22} is not informed to III.

In the third stage, after knowing I's choice of $x\&(A_1\cup R_1)$ and II's choice of $y\&(A_2\cup R_2)$, III observes that $X_{31}=z$ and chooses either one of A_3 (*i.e.*, III accepts z) or R_3 (*i.e.*, III rejects z and resamples a new r.v. X_{32}).

After the third stage is over, showdown is made, the scores are compared and the player with the highest score among the players becomes the winner. Each player aims to maximize the probability of his (or her) winning. We assume that all players are intelligent, and each player should prepare for that any subsequent player must use their optimal strategies.

The three-player game of 'Keep-or-Exchange' (*i.e.*, the score is defined by (1.1)-(1.3)) is solved in Ref.[3 ; Theorem 2]. The solution is found to be very complicate far more than expected.

The sequential-move three-player games of "Risky Exchange" (*i.e.*, the score is given by (1.1)-(1.2)) remains to be solved. The sequential-move two-player games of "Keep-or-Exchange", "Competing Average" and "Showcase Showdown" are all solved in Ref.[2].

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