REGULAR CONGRUENCE RELATIONS ON HYPER BCK-ALGEBRAS

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ABSTRACT. In this manuscript first by definition of regular congruence relation on a hyper BCK-algebra, we construct a quotient hyper BCK-algebra. After that, we state and prove the homomorphism and isomorphism theorems for hyper BCK-algebras. Finally, we show that there exists at least one maximal regular congruence relation in a bounded hyper BCK-algebra.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki[5] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras)was introduced in 1934 by F. Marty [10] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing of hyperstructures has been seen. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [8], Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. They also introduced the notions of hyper BCK-ideal, strong and reflexive hyper BCK-ideals. Now we follow [9] and introduce the concept of quotient hyper BCK-algebras. Then we prove homomorphism and isomorphism theorems for hyper BCK-algebras and we get some related results. Finally, we show that there exists at least one maximal regular congruence relation in a bounded hyper BCK-algebra.

2. Preliminaries

Definition 2.1. [8] By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$.
for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

Theorem 2.2. [8] *In any hyper BCK-algebra H, the following hold:*
(i) $0 \circ 0 = \{0\}$,

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(ii) $0 \ll x$,
(iii) $x \ll x$,
(iv) $0 \circ x = \{0\}$,
(v) $x \circ y \ll x$,
(vi) $x \circ 0 = \{x\}$,

for all $x, y \in H$.

Theorem 2.3. [2] *Let* $(H_1, \circ_1, 0)$ *and* $(H_2, \circ_2, 0)$ *are two hyper* BCK*-algebras such that* $H_1 \cap H_2 = \{0\}$ *and* $H = H_1 \cup H_2$. *Then* $(H, \circ, 0)$ *is a hyper* BCK*-algebra, where the hyper operation "*$\circ$*" on* H *is defined by,*

$$x \circ y = \begin{cases} x \circ_1 y & \text{if } x, y \in H_1 \\ x \circ_2 y & \text{if } x, y \in H_2 \\ \{x\} & \text{otherwise,} \end{cases}$$

for all $x, y \in H$, *and we denote it by* $H_1 \oplus H_2$.

Theorem 2.4. [2] *Let* $(H_1, \circ_1, 0_1)$ *and* $(H_2, \circ_2, 0_2)$ *are two hyper* BCK*-algebras and* $H = H_1 \times H_2$. *We define a hyper operation "*$\circ$*" on* H *as follows,*

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 \circ a_2, b_1 \circ b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in H$. *Where for* $A \subseteq H_1$ *and* $B \subseteq H_2$ *by* (A, B) *we mean*

$$(A, B) = \{(a, b) : a \in A, b \in B\}, \ 0 = (0_1, 0_2)$$

and

$$(a_1, b_1) < (a_2, b_2) \Longleftrightarrow a_1 < a_2 \text{ and } b_1 < b_2.$$

Then $(H, \circ, 0)$ *is a hyper* BCK*-algebra, and it is called the hyper product of* H_1 *and* H_2.

Definition 2.5. [7, 8] Let I be a nonempty subset of a hyper BCK-algebra H and $0 \in I$. Then I is said to be a *hyper* BCK*-ideal* of H if $x \circ y \ll I$ and $y \in I$ implies $x \in I$ for all $x, y \in H$, *reflexive* if $x \circ x \subseteq I$ for all $x \in H$, *strong hyper* BCK*-ideal* of H if $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ implies $x \in I$ for all $x, y \in H$, *hyper subalgebra* of H if $x \circ y \subseteq I$ for all $x, y \in I$.

Theorem 2.6. [7, 8] *Let* H *be hyper* BCK*-algebra. Then,*

(i) *any strong hyper* BCK*-ideal of* H *is a hyper* BCK*-ideal of* H,
(ii) *if* I *is a hyper* BCK*-ideal of* H *and* A *is a nonempty subset of* H. *Then* $A \ll I$ *implies* $A \subseteq I$,
(iii) *if* I *is a reflexive hyper* BCK*-ideal of* H *and* $(x \circ y) \cap I \neq \emptyset$, *then* $x \circ y \subseteq I$ *for all* $x, y \in H$,
(iv) H *is a* BCK*-algebra if and only if* $H = \{x \in H : x \circ x = \{0\}\}$.

Note. From now on in this paper we let H denotes a hyper BCK-algebra.

3. Quotient hyper BCK-algebras

In order to give a definition of a quotient hyper BCK-algebra, Author in [9], defined the notion of regular congruence relation on hyper BCK-algebra as follows:

" An equivalence relation φ is called a congruence if it satisfies the following condition: for all $x, x', y, y' \in H$,

$$(x, y), \ (x', y') \in \varphi \Longrightarrow (x \circ x', y \circ y') \in \varphi$$

where $(x \circ x', \ y \circ y') \in \varphi$ is defined by $(a, b) \in \varphi$ for some a $\in x \circ x'$ and $b \in y \circ y'$.

A congruence relation is called regular, if for all $x, y \in H$,

$$(x \circ y, \{0\}), (y \circ x, \{0\}) \in \varphi \implies (x, \ y) \in \varphi \text{ "}$$

Then he defined the hyper operation "$\circ$" on quotient structure $\frac{H}{\varphi}$ for regular congruence relation φ on H, as follows:

$$[x] \circ [y] = \bigcup \{[a] \mid a \in x \circ y\}$$

Certainly, as we observed in the proof of the Proposition 1. of [9], his purpose of the notation "$\bigcup$" in the above definition is as follows,

$$\bigcup \{[a] : a \in x \circ y\} = \{[a] : a \in x \circ y\}$$

Since, otherwise $[x] \circ [y] \notin \frac{H}{\varphi}$.

Now, in the following example we show that the above hyper operation is not well-defined.

Example 3.1. Let $H = \{0, 1, 2, 3\}$. Then the following table shows a hyper BCK-algebra structure on H.

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$
3	$\{3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{0,1,3\}$

Suppose $\varphi = \{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)\}$. We can check that φ is a regular congruence relation on H and $\frac{H}{\varphi} = \{[0],[1],[3]\}$. Moreover, $[1] = [2]$, but $[1] \circ [1] = \{[0],[1]\} \neq \{[1]\} = [2] \circ [1]$. Hence the hyper operation "$\circ$" which is defined as above is not well-defined. Thus, $\frac{H}{\varphi}$ is not a hyper BCK-algebra .

Now, we introduce the another structure for definition of quotient hyper BCK-algebra.

Definition 3.2. Let Θ be an equivalence relation on H and $A, B \subseteq H$. Then,

(i) $A\Theta B$ means that, there exists $a \in A$ and $b \in B$ such that $a\Theta b$,

(ii) $A\overline{\Theta}B$ means that, for all $a \in A$ there exists $b \in B$ such that $a\Theta b$ and for all $b \in B$ there exists $a \in A$ such that $a\Theta b$,

(iii) Θ is called a congruence relation on H, if $x\Theta y$ and $x^{'}\Theta y^{'}$ then $x \circ x^{'}\overline{\Theta} y \circ y^{'}$, for all $x, y, x', y' \in H$,

(iv) Θ is called a regular relation on H, if $x \circ y\Theta\{0\}$ and $y \circ x\Theta\{0\}$, then $x\Theta y$ for all $x, y \in H$.

Lemma 3.3. *Let Θ be an equivalence relation on H and $A, B \subseteq H$. If $A\overline{\Theta}B$ and $B\overline{\Theta}C$, then $A\overline{\Theta}C$.*

Proof. The proof is easy. □

Lemma 3.4. *Let Θ be an equivalence relation on H. Then the following statements are equivalent:*

(i) *Θ is a congruence relation on H,*

(ii) *if $x\Theta y$, then $x \circ a\overline{\Theta} y \circ a$ and $a \circ x\overline{\Theta} a \circ y$, for all $a, x, y \in H$.*

Proof. $(i) \Longrightarrow (ii)$ Let $x\Theta y$ and $a \in H$. Since Θ is a congruence relation on H and $a\Theta a$, then $x \circ a\overline{\Theta} y \circ a$ and $a \circ x\overline{\Theta} a \circ y$.

$(ii) \Longrightarrow (i)$ Let $x\Theta y$ and $x'\Theta y'$. By (ii), $x \circ x'\overline{\Theta} y \circ x'$ and $y \circ x'\overline{\Theta} y \circ y'$. Hence by lemma 3.2, $x \circ x'\overline{\Theta} y \circ y'$. Therefore, Θ is a congruence relation on H. □

Theorem 3.5. *Let Θ and Θ' are two regular congruence relations on H such that $[0]_\Theta = [0]_{\Theta'}$. Then $\Theta = \Theta'$.*

Proof. Let Θ and Θ' are two regular congruence relation on H such that $[0]_\Theta = [0]_{\Theta'}$. It is enough to show that, for all $x, y \in H$

$$x\Theta y \quad \text{if and only if} \quad x\Theta' y$$

Let $x\Theta y$, for $x, y \in H$. Since Θ is a congruence relation on H, then by Lemma 3.3, $x\circ x\overline{\Theta}x\circ y$. Now, since $0 \in x\circ x$ then there exists $t \in x\circ y$ such that $0\Theta t$ and so $t \in [0]_\Theta$. Thus, $t \in [0]_{\Theta'}$ and so $x \circ y\Theta'\{0\}$. By the similar way, we can show that $y \circ x\Theta'\{0\}$. Now, since Θ' is a regular relation, then $x\Theta' y$. Similarly, we can show that if $x\Theta' y$, then $x\Theta y$, for all $x, y \in H$. Therefore, $\Theta = \Theta'$. □

Lemma 3.6. *Let Θ be a regular congruence relation on H. Then $[0]_\Theta$ is a strong hyper BCK-ideal of H.*

Proof. Clear that $0 \in [0]_\Theta$. Now, let $x \circ y \bigcap [0]_\Theta \neq \emptyset$ and $y \in [0]_\Theta$. Then, there exists $a \in x \circ y$ such that $a \in [0]_\Theta$ and so $a\Theta 0$. Hence, $x \circ y\Theta\{0\}$. Moreover, since $y\Theta 0$ and Θ is a congruence relation on H, then by Lemma 3.3, $y \circ x\overline{\Theta}0 \circ x = \{0\}$. Now, since $x \circ y\Theta\{0\}$, $y \circ x\Theta\{0\}$ and Θ is a regular relation, then $x\Theta y$. Since, $y\Theta 0$ then by transitive condition $x\Theta 0$ and so $x \in [0]_\Theta$. Therefore, $[0]_\Theta$ is a strong hyper BCK-ideal of H. □

Note. Let Θ be a regular congruence relation on H. Then by Lemma 3.5, $[0]_\Theta$ is a strong hyper BCK-ideal of H and so by Theorem 2.6(i), $[0]_\Theta$ is a hyper BCK-ideal of H.

Theorem 3.7. *Let Θ be a regular congruence relation on H, $I = [0]_\Theta$ and $\frac{H}{I} = \{I_x : x \in H\}$, where $I_x = [x]_\Theta$ for all $x \in H$. Then $\frac{H}{I}$ with hyperoperation "$\circ$" and hyperorder "$<$" which is defined as follows, is a hyper BCK-algebra which is called quotient hyper BCK-algebra,*

$$I_x \circ I_y = \{I_z : z \in x \circ y\} \quad , \quad I_x < I_y \Longleftrightarrow I \in I_x \circ I_y$$

Proof. First, we show that the hyperoperation "$\circ$" on $\frac{H}{I}$ is well-defined. Let $x, x', y, y' \in H$ such that $I_x = I_{x'}$ and $I_y = I_{y'}$. Let $I_z \in I_x \circ I_y$. Then there is $u \in x \circ y$ such that $I_z = I_u$. Since, $x\Theta x'$, $y\Theta y'$ and Θ is a congruence relation on H, then $x \circ y\overline{\Theta}x' \circ y'$. Hence, there is $z' \in x' \circ y'$ such that $u\Theta z'$ and so $I_u = I_{z'}$. Since $I_{z'} \in I_{x'} \circ I_{y'}$ and $I_z = I_u = I_{z'}$, then $I_z \in I_{x'} \circ I_{y'}$. Therefore, $I_x \circ I_y \subseteq I_{x'} \circ I_{y'}$. By the similar way, we can show that $I_{x'} \circ I_{y'} \subseteq I_x \circ I_y$ and so $I_x \circ I_y = I_{x'} \circ I_{y'}$. Hence, hyperopration "$\circ$" is well-defied. Now we show that $\frac{H}{I}$ satisfies the axioms of a hyper BCK-algebra.

(HK1): Let $I_w \in (I_x \circ I_z) \circ (I_y \circ I_z)$, for $I_x,\ I_y,\ I_z \in \frac{H}{I}$. Then, there are $I_u \in I_x \circ I_z$ and $I_v \in I_y \circ I_z$ such that $I_w \in I_u \circ I_v$. Hence, there are $u' \in x \circ z$, $v' \in y \circ z$ and $w' \in u \circ v$ such that $I_u = I_{u'}$, $I_v = I_{v'}$, $I_w = I_{w'}$, and so $u\Theta u'$, $v\Theta v'$ and $w\Theta w'$. Since Θ is a congruence relation on H, then $u \circ v\overline{\Theta}u' \circ v'$. Since $w' \in u \circ v$, then there is $a \in u' \circ v'$ such that $w'\Theta a$ and so $I_{w'} = I_a$. Thus $I_w = I_{w'} = I_a$. By (HK1) of H, $a \in u' \circ v' \subseteq (x \circ z) \circ (y \circ z) \ll x \circ y$. Then there is $b \in x \circ y$ such that $a \ll b$ and so $0 \in a \circ b$. Hence, $I_b \in I_x \circ I_y$ and $I = I_0 \in I_a \circ I_b$. Since $I_w = I_a$, then $I \in I_w \circ I_b$ and so $I_w \ll I_b$ and this implies that $(I_x \circ I_z) \circ (I_y \circ I_z) \ll I_x \circ I_y$. Therefore, (HK1) hold in $\frac{H}{I}$.

(HK2): Let $w \in (I_x \circ I_y) \circ I_z$, for $I_x, I_y, I_z \in \frac{H}{I}$. Then, there is $u \in x \circ y$ such that $I_w \in I_u \circ I_z$ and so there is $w' \in u \circ z$ such that $I_w = I_{w'}$. Since, by (HK2) of H, $w' \in u \circ z \subseteq (x \circ y) \circ z = (x \circ z) \circ y$. Then $I_w = I_{w'} \in (I_x \circ I_z) \circ I_y$. Hence, $(I_x \circ I_y) \circ I_z \subseteq (I_x \circ I_z) \circ I_y$. By the similar way, we can show that $(I_x \circ I_z) \circ I_y \subseteq (I_x \circ I_y) \circ I_z$. Therefore, $(I_x \circ I_z) \circ I_y = (I_x \circ I_y) \circ I_z$ and so (HK2) hold in $\frac{H}{I}$.

(HK3): Let $I_z \in I_x \circ \frac{H}{I}$, for $I_x \in \frac{H}{I}$. Then, there is $I_y \in \frac{H}{I}$ such that $I_z \in I_x \circ I_y$. Hence, there is $z' \in x \circ y$ such that $I_z = I_{z'}$. By Theorem 2.2(v), $x \circ y \ll x$ and so $z' \ll x$. Now, since $0 \in z' \circ x$ then $I \in I_{z'} \circ I_x$ and so $I \in I_z \circ I_x$. Thus, $I_z \ll I_x$ and this implies that $I_x \circ \frac{H}{I} \ll I_x$. Therefore, (HK3) hold in $\frac{H}{I}$.

(HK4) Let $I_x \ll I_y$ and $I_y \ll I_x$, for $I_x, I_y \in \frac{H}{I}$. Then, $I \in I_x \circ I_y$ and $I \in I_y \circ I_x$. Hence, there are $u \in x \circ y$ and $v \in y \circ x$ such that $I_u = I = I_v$ and so $u\Theta 0$ and $v\Theta 0$. Then we conclude that $x \circ y \Theta \{0\}$ and $y \circ x \Theta \{0\}$. Since Θ is a regular relation on H, then $x\Theta y$ and so $I_x = I_y$. Therefore, (HK4) hold in $\frac{H}{I}$. □

Example 3.8. Let $H = \{0, 1, 2, 3\}$. Then the following table shows the hyper BCK-algebra structure on H.

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

Let $\Theta = \{(0,0), (0,1), (1,0), (1,1), (2,2), (3,3)\}$. It is easy to check that Θ is a regular congruence relation on H. Moreover, $I = [0]_\Theta = \{0, 1\} = I_1$, $I_2 = \{2\}$ and $I_3 = \{3\}$ and so $\frac{H}{I} = \{I, I_2, I_3\}$. Cayley's table of $\frac{H}{I}$ is as follows:

$\circ$	I	I_2	I_3
I	$\{I\}$	$\{I\}$	$\{I\}$
I_2	$\{I_2\}$	$\{I\}$	$\{I_2\}$
I_3	$\{I_3\}$	$\{I_3\}$	$\{I, I_3\}$

We can check that $\frac{H}{I}$ is a hyper BCK-algebra.

Theorem 3.9. *Let Θ be a regular congruence relation on H and $I = [0]_\Theta$. Then*

$$I \text{ is a reflexive hyper } BCK\text{-ideal of } H \Longleftrightarrow \frac{H}{I} \text{ is a } BCK\text{-algebra}$$

Proof. ($\Longleftarrow$) Let $\frac{H}{I}$ be a BCK-algebra on H. Since Θ is a regular congruence relation on H, then by Lemma 3.5, $I = [0]_\Theta$ is a strong hyper BCK-ideal and so by Theorem 2.6(i), it is a hyper BCK-ideal of H. Now, we must show that for all $x \in H$, $x \circ x \subseteq I$. Let $z \in x \circ x$. Then, $I_z \in I_x \circ I_x$. Since $\frac{H}{I}$ is a BCK-algebra and $I \in I_x \circ I_x$, then $I_x \circ I_x = I$. Hence $I_z = I$ and so $z \,\Theta\, 0$ and this implies that $z \in [0]_\Theta = I$. Therefore, for all $x \in H$, $x \circ x \subseteq I$ and this implies that I is a reflexive hyper BCK-ideal of H.

($\Longrightarrow$) Let $I_a \in \frac{H}{I}$. Since $a \circ a \subseteq I$ for all $a \in H$, then

$$I \in I_a \circ I_a = \{I_z : z \in a \circ a\} \subseteq \{I_z : z \in I\} = \{I_z : I_z = I\} = \{I\}$$

Thus, $I_a \circ I_a = I$ and so by Corollary 2.6(iv), $\frac{H}{I}$ is a BCK-algebra. □

Theorem 3.10. *Let I be a reflexive hyper BCK-ideal of H and relation Θ on H is defined as follows:*

$$x\Theta y \iff x \circ y \subseteq I \text{ and } y \circ x \subseteq I$$

for all $x, y \in H$. Then Θ is a regular congruence relation on H and $I = [0]_\Theta$. Moreover, $\frac{H}{I}$ is a BCK-algebra.

Proof. It is clear that Θ is a reflexive and symmetric relation on H. We show that Θ is a transitive relation. Let $x\Theta y$ and $y\Theta z$, for $x, y, z \in H$. Then $x \circ y \subseteq I$ and so by (HK1) of H, $(x \circ z) \circ (y \circ z) \ll x \circ y \subseteq I$. Since $y\Theta z$, then $y \circ z \subseteq I$ and since I is a hyper BCK-ideal of H, then $x \circ z \subseteq I$. Moreover, since $(z \circ x) \circ (y \circ x) \ll z \circ y$, $z \circ y \subseteq I$ and $y \circ x \subseteq I$, then $z \circ x \subseteq I$ and so $z\Theta x$. Therefore, Θ is a transitive relation and so it is an equivalence relation on H. Now, we prove that Θ is a congruence relation on H. By Lemma 3.3, it is enough to show that if $x\Theta y$, then for all $a \in H$, $x \circ a\overline{\Theta} y \circ a$ and $a \circ x\overline{\Theta} a \circ y$. So, let $x\Theta y$. Then $x \circ y \subseteq I$ and $y \circ x \subseteq I$. By (HK1) of H, $(x \circ a) \circ (y \circ a) \ll x \circ y$. Since $x \circ y \subseteq I$ then $(x \circ a) \circ (y \circ a) \ll I$. Since I is a hyper BCK-ideal of H, then by Theorem 2.6(ii), $(x \circ a) \circ (y \circ a) \subseteq I$. Thus $u \circ v \subseteq I$, for all $u \in x \circ a$ and $v \in y \circ a$. Similarly, since $(y \circ a) \circ (x \circ a) \ll y \circ x$ and $y \circ x \subseteq I$, then $(y \circ a) \circ (x \circ a) \subseteq I$ and so $v \circ u \subseteq I$ for all $u \in x \circ a$ and $v \in y \circ a$. Thus, for all $u \in x \circ a$ and $v \in y \circ a$, $u\Theta v$ and this implies that $x \circ a\overline{\Theta} y \circ a$. Now, let $u \in a \circ x$. Since $(a \circ x) \circ (y \circ x) \ll a \circ y$, then there is $t \in y \circ x$ such that $u \circ t \ll a \circ y$ and so there are $w \in u \circ t$ and $v' \in a \circ y$ such that $w \ll v'$. Hence, by (HK2) of H, $0 \in w \circ v' \subseteq (u \circ t) \circ v' = (u \circ v^{'}) \circ t$. Then, there is $c \in u \circ v^{'}$ such that $0 \in c \circ t$ and so $c \circ t \cap I \neq \emptyset$. Hence by Theorem 2.6(iii), $c \circ t \subseteq I$. Since $t \in y \circ x \subseteq I$ and I is a hyper BCK-ideal of H, then $c \in I$. Therefore, $u \circ v^{'} \cap I \neq \emptyset$ and so by Theorem 2.6(iii), $u \circ v^{'} \subseteq I$. Moreover, since $(a \circ y) \circ (x \circ y) \ll a \circ x$ and $v' \in a \circ y$, then similarly, we can show that there is $u^{'} \in a \circ x$ such that $v^{'} \circ u^{'} \subseteq I$. Since $u, u^{'} \in a \circ x$ and I is a reflexive hyper BCK-ideal of H, then $u^{'} \circ u \subseteq (a \circ x) \circ (a \circ x) \ll a \circ a \subseteq I$ and so $u^{'} \circ u \ll I$. Hence by Theorem 2.6(ii), $u' \circ u \subseteq I$. Now, since $(v^{'} \circ u) \circ (u^{'} \circ u) \ll v^{'} \circ u^{'} \subseteq I, u' \circ u \subseteq I$ and I is a hyper BCK-ideal of H, then $v^{'} \circ u \subseteq I$. Now, since $u \circ v' \subseteq I$ and $v' \circ u \subseteq I$ then $u\Theta v^{'}$. By the similar way, we can prove that for all $v \in a \circ y$ there is $u^{'} \in a \circ x$ such that $u^{'}\Theta v$. Hence, $a \circ x\overline{\Theta} a \circ x$, for all $a \in H$. Therefore, Θ is a congruence relation on H. Also, we must prove that Θ is a regular relation on H. Let $x, y \in H$, $x \circ y\ \Theta\{0\}$ and $y \circ x\ \Theta\{0\}$. Then, there exist $a \in x \circ y$ and $b \in y \circ x$ such that $a\Theta 0$ and $b\Theta 0$ and so, $\{a\} = a \circ 0 \subseteq I$ and $\{b\} = b \circ 0 \subseteq I$. Hence $x \circ y \cap I \neq \emptyset$ and $y \circ x \cap I \neq \emptyset$. Then by Theorem 2.6(iii), $x \circ y \subseteq I$ and $y \circ x \subseteq I$ and this implies that $x\Theta y$. Therefore, Θ is a regular relation on H. Moreover, we show that $I = [0]_\Theta$. Let $x \in [0]_\Theta$. Then $x\Theta 0$ and so $\{x\} = x \circ 0 \subseteq I$. Hence, $[0]_\Theta \subseteq I$. Let $x \in I$, then $x \circ 0 = \{x\} \subseteq I$. Moreover, by Theorem 2.2(iv), $0 \circ x = \{0\} \subseteq I$. Thus, $x\Theta 0$ and so $x \in [0]_\Theta$. Therefore, $I \subseteq [0]_\Theta$ and so $I = [0]_\Theta$. Now, since Θ is a regular congruence relation on H and $I = [0]_\Theta$ is a reflexive hyper BCK-ideal relation of H, then by Theorem 3.8, $(\frac{H}{I}, \circ, I)$ is a BCK-algebra. □

4. Isomorphism theorems on hyper BCK-algebras

Definition 4.1. Let H and H' are two hyper BCK-algebras and $f : H \longrightarrow H'$ be a map. Then f is said to be *a homomorphism* of hyper BCK-algebras if $f(x \circ y) = f(x) \circ f(y)$, for all $x, y \in H$. If f is 1-1 (onto) we say that f is a *monomorphism* (*epimorphism*). If f is both 1-1 and onto, we say that f is an *isomorphism*. If $f : H \longrightarrow H'$ is an isomorphism, then we say that H and H' are *isomorphic* and we write $H \cong H'$. Moreover, the Ker$f$ is defined by $\text{Ker}f = \{x \in H : f(x) = 0\}$.

Theorem 4.2. *If $f : H \longrightarrow H^{'}$ is a homomorphism of hyper BCK-algebras, then $f(0) = 0$.*

Proof. Let $f(0) = a$. Since by Theorem 2.2(i), $0 \circ 0 = \{0\}$, then

$$0 \in f(0) \circ f(0) = f(0 \circ 0) = f(0) = a$$

Hence, $0 = a$ and so $f(0) = 0$. □

Lemma 4.3. *Let $f : H \longrightarrow H^{'}$ be a homomorphism of hyper BCK-algebras and $A, B \subseteq H$. Then,*

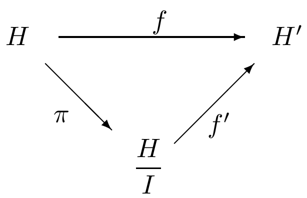
(i) *if $x \ll y$, then $f(x) \ll f(y)$,*
(ii) *if $A \ll B$, then $f(A) \ll f(B)$,*
(iii) *Ker$f$ is a hyper BCK-ideal of H.*

Proof. (i) Let $x, y \in H$ and $x \ll y$. Then $0 \in x \circ y$ and so $0 = f(0) \in f(x \circ y) \subseteq f(x) \circ f(y)$. Hence, $f(x) \ll f(y)$.
(ii) Let $A, B \subseteq H$, $A \ll B$ and $c \in f(A)$. Then, there is $a \in A$ such that $c = f(a)$. Since $A \ll B$, then there is $b \in B$ such that $a \ll b$ and so by (i), $c = f(a) \ll f(b) \in f(B)$. Hence $f(A) \ll f(B)$.
(iii) Let $x \circ y \ll \mathrm{Ker} f$ and $y \in \mathrm{Ker} f$, for $x, y \in H$. Since $\mathrm{Ker} f = f^{-1}(\{0\})$ then by (ii), $f(x) \circ f(y) = f(x \circ y) \ll f(\mathrm{Ker} f) = f(f^{-1}(\{0\})) = \{0\}$. Since $f(y) = 0 \in \{0\}$ and $\{0\}$ is a hyper BCK-ideal of H, then $f(x) \in \{0\}$ and so $x \in \mathrm{Ker} f$. Therefore, $\mathrm{Ker} f$ is a hyper BCK-ideal of H. □

Lemma 4.4. *Let Θ be a regular congruence relation on H and $I = [0]_\Theta$. Then $\pi : H \longrightarrow \frac{H}{I}$ which is defined by $\pi(x) = I_x$, for all $x \in H$, is an epimorphism which is called canonical epimorphism.*

Proof. The proof is straightforward. □

Theorem 4.5. *(Homomorphism theorem) Let Θ be a regular congruence relation on H and $I = [0]_\Theta$. If $f : H \longrightarrow H'$ is a homomorphism of hyper BCK-algebras such that $I \subseteq Kerf$, then $f' : \frac{H}{I} \longrightarrow H'$, which is defined by $f'(I_x) = f(x)$, for all $x \in H$, is an unique homomorphism such that the following diagram is commutative:*



i.e. $f' \circ \pi = f$, where π denotes the canonical epimorphism.

Proof. Since Θ is a regular congruence relation on H, then $\frac{H}{I}$ is a hyper BCK-algebra. Now, let $f' : \frac{H}{I} \longrightarrow H'$ is defined by,

$$f'(I_x) = f(x) \quad , \quad \forall\, x \in H.$$

Let $x, y \in H$, and $I_x = I_y$. Then, $I \in I_x \circ I_y$ and so there exists $z \in x \circ y$ such that $I = I_z$. Hence, $z \in I \subseteq \mathrm{Ker} f$ and so $f(z) = 0$. Since f is a homomorphism, then $0 = f(z) \in f(x \circ y) = f(x) \circ f(y)$ and so $f(x) \ll f(y)$. Similarly, we can prove that $f(y) \ll f(x)$ and so by (HK4) of H, $f(x) = f(y)$. Hence, $f'(I_x) = f'(I_y)$. Therefore, f' is well-defined. Moreover, it is easy to show that

$$f'(I_x \circ I_y) = f'(I_x) \circ f'(I_y)$$

and $f' \circ \pi = f$. Now, we prove that f' is unique. Let $g : \frac{H}{I} \longrightarrow H'$ be a homomorphism such that $g \circ \pi = f$. Then, for all $x \in H$, $g(I_x) = g(\pi(x)) = f(x) = f'(\pi(x)) = f'(I_x)$. □

Example 4.6. Let $f : H \longrightarrow H$ be a homomorphism of hyper BCK-algebras and Θ be a relation on H which is defined as follows:

$$x \Theta y \iff f(x) = f(y)$$

Then Θ is a regular congruence relation on H and $[0]_\Theta = \mathrm{ker} f$. It is easy to check that Θ is an equivalence relation. Let $x, y, a \in H$, $x \Theta y$ and $t \in x \circ a (s \in y \circ a)$. Then $f(x) = f(y)$ and so $f(x \circ a) = f(x) \circ f(a) = f(y) \circ f(a) = f(y \circ a)$. Hence, there exists $s \in y \circ a (t \in x \circ a)$ such that $f(t) = f(s)$. Thus, $t \Theta s$ and so $x \circ a \overline{\Theta} y \circ a$. By the similar way, we can show that $a \circ x \overline{\Theta} a \circ y$. Hence, Θ is a congruence relation on H. Now, let $x \circ y \Theta \{0\}$ and $y \circ x \Theta \{0\}$

for $x, y \in H$. Then, there exist $s \in x \circ y$ and $t \in y \circ x$ such that $s\Theta 0$ and $t\Theta 0$. Hence, $f(s) = f(0) = f(t)$ and so $0 = f(0) \in f(x) \circ f(y) \cap f(y) \circ f(x)$. Now, since $f(x) \ll f(y)$ and $f(y) \ll f(x)$ then by (HK4) of H, $f(x) = f(y)$. Therefore, $x\Theta y$ and so Θ is a regular relation. It is easy to check that $[0]_\Theta = \ker f$.

Theorem 4.7. *(Isomorphism Theorem) Let Θ be a regular congruence relation on H and $I = [0]_\Theta$. If $f : H \longrightarrow H'$ is a homomorphism of hyper BCK-algebras such that $Kerf = I$, then*

$$\frac{H}{I} \cong f(H)$$

Proof. Let $f' : \frac{H}{I} \longrightarrow H'$ is defined by, $f'(I_x) = f(x)$ for all $x \in H$. It is easy to show that f' is a homomorphism. Now, we show that f' is a monomorphism. Let $f'(I_x) = f'(I_y)$, for $x, y \in H$. Then $f(x) = f(y)$ and so

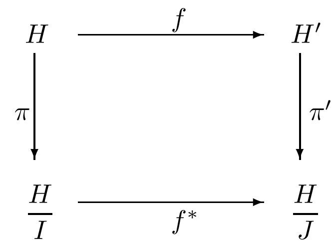
$$0 = f(0) \subseteq f(x \circ x) = f(x) \circ f(x) = f(x) \circ f(y) = f(x \circ y)$$

Hence, there exists $t \in x \circ y$ such that $f(t) = 0$. Then, $t \in \mathrm{Ker} f = I = [0]_\Theta$ and so $t\Theta\, 0$ and this implies that $x \circ y\Theta\{0\}$. Similarly, we can prove that $y \circ x\Theta\{0\}$. Since Θ is a regular relation, then $x\Theta y$ and so $I_x = I_y$. Therefore, f' is a monomorphism and so

$$\frac{H}{I} \cong f(H)$$

□

Theorem 4.8. *Let Θ and Θ' are regular congruence relations on hyper BCK-algebras H and H', respectively, such that $I = [0]_\Theta$ and $J = [0]'_\Theta$. If $f : H \longrightarrow H'$ is a homomorphism of hyper BCK-algebras such that $x\Theta y$ implies $f(x)\Theta' f(y)$, for all $x, y \in H$, then there exists an unique homomorphism $f^* : \frac{H}{I} \longrightarrow \frac{H}{J}$ such that the following diagram is commutative;*



i.e. $\pi' \circ f = f^* \circ \pi$, where π and π' denotes the canonical epimorphisms.

Proof. Let $f^* : \frac{H}{I} \longrightarrow \frac{H}{J}$ is defined by,

$$f^*(I_x) = J_{f(x)},\ \forall x \in H$$

First, we show that f^* is well-defined. Let $x, y \in H$ and $I_x = I_y$. Then, $x\Theta y$ and so $f(x)\Theta' f(y)$. Hence, $J_{f(x)} = J_{f(y)}$. Therefore, f^* is well-defined. Moreover, it is easy to prove that $f^*(I_x \circ I_y) = f^*(I_x) \circ f^*(I_y)$ and $\pi' \circ f = f^* \circ \pi$. Now, we show that f^* is unique. Let $g : \frac{H}{I} \longrightarrow \frac{H}{J}$ be a homomorphism such that $\pi' \circ f = g \circ \pi$. Then, for all $x \in H, g(I_x) = g(\pi(x)) = \pi' \circ f(x) = f^* \circ \pi(x) = f^*(I_x)$. □

Theorem 4.9. *Let $f : H \longrightarrow H'$ be an epimorphism of hyper BCK-algebras, Θ' be a regular congruence relation on H' and $J = [0]_{\Theta'}$. Then, there exists a regular congruence relation Θ on H such that,*

$$\frac{H}{I} \cong \frac{H'}{J}$$

where, $I = [0]_\Theta$.

Proof. Let relation Θ on H is defined by $x\Theta y \Longleftrightarrow f(x)\Theta' f(y)$, for all $x, y \in H$. Since Θ' is a regular congruence relation on H', then it is easy to check that Θ is a regular congruence relation on H. Moreover,

$$x \in I = [0]_\Theta \Longleftrightarrow x\Theta 0 \Longleftrightarrow f(x)\Theta' f(0) = f(x)\Theta' 0 \Longleftrightarrow f(x) \in [0]_{\Theta'} = J \Longleftrightarrow x \in f^{-1}(J)$$

Hence $I = f^{-1}(J)$. Now, let $\pi : H' \longrightarrow \frac{H'}{J}$ be canonical epimorphism and $\overline{f} : H \longrightarrow \frac{H'}{J}$ is defined by $\overline{f} = \pi \circ f$. Since π and f are epimorphism, then $\bar{f}$ is an epimorphism. Moreover,

$$\begin{aligned} \text{Ker}\bar{f} &= \{x \in H : \bar{f}(x) = J\} = \{x \in H : \pi(f(x)) = J\} = \{x \in H : J_{f(x)} = J\} \\ &= \{x \in H : f(x) \in J\} = \{x \in H : x \in f^{-1}(J)\} = \{x \in H : x \in I\} = I \end{aligned}$$

Therefore, by the isomorphism theorem $\frac{H}{I} \cong \frac{H'}{J}$. □

Theorem 4.10. *Let Θ and Θ_1 are regular congruence relations on H, $J = [0]_\Theta$ and $I = [0]_{\Theta_1}$. Then,*

$$\frac{\frac{H}{I}}{\frac{J}{I}} \cong \frac{H}{J}$$

where, $\frac{J}{I} = \{I_x \in \frac{H}{I} : x \in J\}$.

Proof. Let relation Θ_2 on $\frac{H}{I}$ is defined by, $I_x\Theta_2\ I_y \Longleftrightarrow x\Theta y$, for all $I_x, I_y \in \frac{H}{I}$. Since Θ is an equivalence relation on H, then it is easy to check that Θ_2 is an equivalence relation on $\frac{H}{I}$. Let $I_x, I_y, I_z \in \frac{H}{I}$ and $I_x\Theta_2 I_y$. Then by the some modifications we can prove that $I_x \circ I_a \overline{\Theta_2} I_y \circ I_a$ and $I_a \circ I_x \overline{\Theta_2} I_a \circ I_y$. Therefore, by Lemma 3.3, Θ_2 is a congruence relation on $\frac{H}{I}$. Now, let $I_x \circ I_y \Theta_2 \{I\}$ and $I_y \circ I_x \Theta_2 \{I\}$. Then there exist $u \in x \circ y$ and $v \in y \circ x$ such that $I_u \Theta_2 I$ and $I_v \Theta_2 I$. Hence, $u\Theta 0$ and $v\Theta 0$ and so $x \circ y\ \Theta\ \{0\}$, $y \circ x\ \Theta\{0\}$. Since Θ is a regular relation on H, then $x\Theta y$ and so $I_x\Theta_2 I_y$. Therefore, Θ_2 is a regular relation on H. Moreover,

$$\begin{aligned} [I]_{\Theta_2} &= \{I_x \in \frac{H}{I} : I_x\ \Theta_2 I\} = \{I_x \in \frac{H}{I} : x\Theta 0\} \\ &= \{I_x \in \frac{H}{I} : x \in [0]_\Theta = J\} = \{I_x \in \frac{H}{I} : x \in J\} = \frac{J}{I} \end{aligned}$$

Now, we define $\varphi : \frac{H}{I} \longrightarrow \frac{H}{J}$ by $\varphi(I_x) = J_x$. If $I_x = I_y$, then $I_x\Theta_2 I_y$ and so $x\Theta y$. Since $J = [0]_\Theta$, then $J_x = J_y$ and so φ is well-defined. Moreover, φ is a homomorphism. Also,

$$\text{Ker}\varphi = \{I_x \in \frac{H}{I} : \varphi(I_x) = J)\} = \{I_x \in \frac{H}{I} : J_x = J\} = \{I_x \in \frac{H}{I} : x \in J\} = \frac{J}{I} = [I]_{\Theta_2}$$

Since φ is onto, then by the isomorphism theorem, $\frac{\frac{H}{I}}{\frac{J}{I}} \cong \frac{H}{J}$. □

Theorem 4.11. *Let Θ and Ω are regular congruence relations on H and K, respectively, such that $I = [0]_\Theta$ and $J = [0]_\Omega$. Then,*

$$\frac{H \times K}{I \times J} \simeq \frac{H}{I} \times \frac{K}{J}$$

Proof. Let Γ be a relation on $H \times K$ which is defined as follows:

$$(a, b)\Gamma(c, d) \quad \text{if and only if} \quad a\Theta c \ \&\ b\Omega d,$$

for all $(a, b), (c, d) \in H \times K$. It is easy to check that Γ is a regular congruence relation on $H \times K$. Now, let $(a, b) \in H \times K$, then

$$(a, b) \in [(0, 0)]_\Gamma \Longleftrightarrow a\ \Theta\ 0 \text{ and } b\ \Omega\ 0 \Longleftrightarrow a \in I \text{ and } b \in J \Longleftrightarrow (a, b) \in I \times J$$

Hence, $[(0,0)]_\Gamma = I \times J$. Now, we define $f : H \times K \longrightarrow \frac{H}{I} \times \frac{K}{J}$ by $f\Big((a,b)\Big) = (I_a, J_b)$, for all $(a,b) \in H \times K$. It is easy to check that f is well-defined. Let $(a,b),(c,d) \in H \times K$. Then,

$$\begin{aligned} f\Big((a,b)\circ(c,d)\Big) &= f\Big((a\circ c, b\circ d)\Big) = \bigcup_{s\in a\circ c, t\in b\circ d} f\Big((s,t)\Big) \\ &= \bigcup_{s\in a\circ c, t\in b\circ d} (I_s, J_t) = (\bigcup_{s\in a\circ c} I_s, \bigcup_{t\in b\circ d} J_t) \\ &= (I_a \circ J_c, I_b \circ J_d) = (I_a, J_b)\circ(I_c, J_d) \\ &= f\Big((a,b)\Big)\circ f\Big((c,d)\Big) \end{aligned}$$

Hence, f is a homomorphism. It is easy to check that Ker $f = [(0,0)]_\Gamma = I \times J$. Moreover, f is onto. Therefore, by isomorphism theorem,

$$\frac{H\times K}{I\times J} \simeq \frac{H}{I}\times\frac{K}{J}$$

□

Lemma 4.12. *Let Γ be a regular congruence relation on $H_1 \times H_2$. Then there are regular congruence relations Θ_1 and Θ_2 on H_1 and H_2, respectively, such that*

$$x\Theta_1 u \iff (x,0)\ \Gamma\ (u,0)$$

$$y\Theta_2 v \iff (0,y)\ \Gamma\ (0,v)$$

for all $x,u \in H_1$ and $y,v \in H_2$.

Proof. It is easy to check that Θ_1 is an equivalence relation on H_1. Now, let $x\Theta_1 u$ and $a \in H_1$. Then $(x,0)\Gamma(u,0)$. Since Γ is a congruence relation on $H \times K$, then $(x,0)\circ(a,0)\overline{\Gamma}\ (u,0)\circ(a,0)$ and so $(x\circ a,0)\overline{\Gamma}\ (u\circ a,0)$. Hence, for all $s \in x\circ a (t \in u\circ a)$ there exists $t \in u\circ a (s \in x\circ a)$ such that $(s,0)\ \Gamma\ (t,0)$. Thus $s\Theta_1 t$ and this show that $x\circ a\overline{\Theta_1}y\circ a$. By the similar way, we can prove that $a\circ x\overline{\Theta_1}a\circ y$. Therefore, Θ_1 is a congruence relation on H_1. Now, let $x,y \in H_1$, $x\circ y\Theta_1\{0\}$ and $y\circ x\Theta_1\{0\}$. Then, there exist $s \in x\circ y$ and $t \in y\circ x$ such that $s\Theta_1 0$ and $t\Theta_1 0$. Hence $(s,0)\Gamma(0,0)$ and $(t,0)\Gamma(0,0)$. Thus $(x,0)\circ(y,0)\Gamma\{(0,0)\}$ and $(y,0)\circ(x,0)\Gamma\{(0,0)\}$. Since Γ is regular, then $(x,0)\Gamma(y,0)$ and so, $x\Theta_1 y$. Therefore, Θ_1 is a regular congruence relation on H_1. By the similar way, we can prove that Θ_2 is a regular congruence relation on H_2. □

Theorem 4.13. *Let Γ be a regular congruence relation on $H_1 \times H_2$ such that $[0]_\Gamma = L$. Then, there are hyper BCK-ideals I and J of H_1 and H_2, respectively, such that*

$$\frac{H_1\times H_2}{L} \cong \frac{H_1}{I}\times\frac{H_2}{J}$$

Proof. Let relations Θ_1 and Θ_2 on H_1 and H_2 are defined as follow:

$$x\Theta_1 u \iff (x,0)\ \Gamma\ (u,0);$$

$$y\Theta_2 v \iff (0,y)\ \Gamma\ (0,v).$$

Then by Lemma 4.12, Θ_1 and Θ_2 are regular congruence relations on H_1 and H_2, respectively. Let $[0]_{\Theta_1} = I$ and $[0]_{\Theta_2} = J$ and let $f : H_1 \times H_2 \longrightarrow \frac{H_1}{I} \times \frac{H_2}{J}$ is defined by $f(x,y) = (I_x, I_y)$, for all $x \in H_1$ and $y \in H_2$. Then f is a homomorphism of hyper BCK-algebras such that Ker$f = I \times J$. Now, we prove that $L = I \times J$. Let $(x,y) \in L$. Then, $(x,y)\Gamma(0,0)$. Since Γ is a congruence relation on $H_1 \times H_2$, then $(x,y)\circ(0,y)\overline{\Gamma}(0,0)\circ(0,y)$. Hence, $(x\circ 0, y\circ y)\overline{\Gamma}(0,0\circ y)$ and so $(x,y\circ y)\overline{\Gamma}(0,0)$. Since $0 \in y\circ y$, then $(x,0)\Gamma(0,0)$.

Hence by the definition of Θ_1, $x\Theta_1 0$ and so $x \in I$. By the similar way, we can show that $y \in J$. Therefore, $L \subseteq I \times J$. Now, let $(x, y) \in I \times J$. Then, $x \in I$ and $y \in J$ and so $x\Theta_1 0$ and $y\Theta_2 0$. Hence by the definition of Θ_1 and Θ_2, $(x, 0)\Gamma(0, 0)$ and $(0, y)\Gamma(0, 0)$. Since Γ is a congruence relation, then $(x, y) \circ (x, 0)\overline{\Gamma}(x, y) \circ (0, 0)$ and so $(x \circ x, y)\overline{\Gamma}(x, y)$. Since $0 \in x \circ x$, then $(0, y)\Gamma(x, y)$. Since $(0, y)\Gamma(0, 0)$, then $(x, y)\Gamma(0, 0)$. Hence $(x, y) \in L$ and so $I \times J \subseteq L$. Therefore, $L = I \times J$ and so $L =$ker$f$. Since f is onto, then by the isomorphism theorem,

$$\frac{H_1 \times H_2}{L} \cong \frac{H_1}{I} \times \frac{H_2}{J}$$

□

Theorem 4.14. *Let $f : H \longrightarrow H'$ be an epimorphism of hyper BCK-algebras. Then there is a one-to-one correspondence between regular congruence relations on H' and the regular congruence relations on H such that the class of 0 with respect to them is contain $Kerf$.*

Proof. Let $f : H \longrightarrow H'$ be an epimorphism of hyper BCK-algebras and

$$\mathcal{A}=\{\Theta : \Theta \text{ is a regular congruence relation on } H \text{ such that Ker} f \subseteq [0]_\Theta\}$$

$$\mathcal{B}=\{\Omega : \Omega \text{ is a regular congruence relation on } H'\}$$

Let for all $\Theta \in \mathcal{A}$, $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is defined by $\varphi(\Theta) = \Omega$ such that the relation Ω on H' is defined as follows:

$$u\Omega v \Longleftrightarrow \text{there exist } x, y \in H \text{ such that } u = f(x), v = f(y) \text{ and } x\Theta y \quad (1)$$

for all $u, v \in H'$. First, we show that $\Omega \in \mathcal{B}$. Since f is an epimorphism and Θ is reflexive, then Ω is reflexive. It is easy to check that Ω is symmetric. Now, let $u, v, w \in H', u\Omega v$ and $v\Omega w$. Then by (1) there exist $x, y, y', z \in H$ such that $x\Theta y, y'\Theta z, f(x) = u, f(y) = v = f(y')$ and $f(z) = w$. Hence, there exist $s \in y \circ y'$ and $t \in y' \circ y$ such that $f(s) = 0 = f(t)$ and so $s, t \in$ Ker$f$. Since Ker$f \subseteq [0]_\Theta$, then $s\Theta 0$ and $t\Theta 0$. Hence, $y \circ y'\Theta\{0\}$ and $y' \circ y\Theta\{0\}$. Since Θ is a regular relation, then $y\Theta y'$. Hence by the transitive condition of Θ, $x\Theta z$ and so by (1), $u\Theta v$. Therefore, Ω is a transitive relation. Now, let $u, v, b \in H'$ and $u\Omega v$. Then by (1), there exist $x, y \in H$ such that $u = f(x), v = f(y)$ and $x\Theta y$. Since f is an epimorphism, then there exists $c \in H$ such that $f(c) = b$. Since Θ is a congruence relation on H and $x\Theta y$, then $x \circ c\overline{\Theta}y \circ c$. Hence by (1), $f(x \circ c)\overline{\Omega}f(y \circ c)$ and so $f(x) \circ f(c)\overline{\Omega}f(y) \circ f(c)$. Thus, $u \circ b\overline{\Omega}v \circ b$. Similarly, we can show that $b \circ u\overline{\Omega}b \circ v$. Therefore, Ω is a congruence relation on H'. Similar to the proof of congruency of Ω, we can prove that Ω is a regular relation on H'. Therefore, $\Omega \in \mathcal{B}$. Now, we show that φ is injective. Let $\Theta_1, \Theta_2 \in \mathcal{A}$ and $\varphi(\Theta_1) = \varphi(\Theta_2)$. Then there are $\Omega_1, \Omega_2 \in \mathcal{B}$ such that $\Omega_1 = \Omega_2$. Moreover, for all $x, y \in H$,

$$x\Theta_1 y \Longleftrightarrow f(x)\Omega_1 f(y) \Longleftrightarrow f(x)\Omega_2 f(y) \Longleftrightarrow x\Theta_2 y$$

Hence, $\Theta_1 = \Theta_2$ and so φ is injective. Now, let $\Omega \in \mathcal{B}$ and Θ be a relation on H which is defined as follows:

$$x\Theta y \Longleftrightarrow f(x)\Omega f(y)$$

It is easy to check that Θ is a regular congruence relation on H. Let $x \in$ Ker$f$. Then $f(x) = 0 = f(0)$ and so by (1), $x\Theta 0$. Therefore, Ker$f \subseteq [0]_\Theta$ and so $\Theta \in \mathcal{A}$. Now, we claim that $\varphi(\Theta) = \Omega$. Let $\varphi(\Theta) = \Omega'$, for $\Omega' \in \mathcal{B}$. Then by (1) and definition of Θ, for all $u \in H'$

$$u\Omega' 0 \Longleftrightarrow \text{ there exists } x \in H \text{ such that } u = f(x) \text{ and } x\Theta 0 \Longleftrightarrow f(x)\Omega f(0) \Longleftrightarrow u\Omega 0$$

Thus, $[0]'_\Omega = [0]_\Omega$ and so by Lemma 3.4, $\Omega' = \Omega$. Hence, $\varphi(\Theta) = \Omega$ and so φ is onto . Therefore, φ is a bijection. □

Theorem 4.15. *Let H_1 and H_2 are two hyper BCK-algebras. Then there exists a one-to-one correspondence between the set of all regular congruence relations on $H = H_1 \oplus H_2$ and the product of the set of all regular congruence relations on H_1 and the set of all regular congruence relations on H_2. Moreover, if Γ is correspondent to (Θ, Ω) in this correspondence, then, $[0]_\Gamma = [0]_\Theta \cup [0]_\Omega$.*

Proof. Let $H = H_1 \oplus H_2$ and

$$\mathcal{A}=\{\ \Gamma : \Gamma \text{ is a regular congruence relation on } H\}$$

$$\mathcal{B}=\{\ \Theta : \Theta \text{ is a regular congruence relation on } H_1\}$$

$$\mathcal{C}=\{\ \Omega : \Omega \text{ is a regular congruence relation on } H_2\}$$

and $\varphi : \mathcal{A} \longrightarrow \mathcal{B} \times \mathcal{C}$ is defined by $\varphi(\Gamma) = (\Theta, \Omega)$, where Θ and Ω are defined on H_1 and H_2 as follows:

$$x\Theta y \Longleftrightarrow x\Gamma y \quad , \quad x\Omega y \Longleftrightarrow x\Gamma y \qquad (1)$$

for all $x, y \in H_1$ and for all $x, y \in H_2$. It is easy to check that Θ and Ω are regular congruence relations on H_1 and H_2 and φ is well-defined. Hence, $(\Theta, \Omega) \in \mathcal{B} \times \mathcal{C}$. Now, let $\Gamma, \Gamma' \in \mathcal{A}$ and $\varphi(\Gamma) = \varphi(\Gamma')$. Then $(\Theta, \Omega) = (\Theta', \Omega')$ and so $\Theta = \Theta'$ and $\Omega = \Omega'$. Hence, for all $x \in H$,

$$\begin{aligned} x\Gamma 0 &\Longleftrightarrow (x\Gamma 0,\ x \in H_1) \text{ or } (x\Gamma 0,\ x \in H_2) \\ &\Longleftrightarrow x\Theta 0 \text{ or } x\Omega 0 \\ &\Longleftrightarrow x\Theta' 0 \text{ or } x\Omega' 0 \\ &\Longleftrightarrow (x\Gamma' 0,\ x \in H_1) \text{ or } (x\Gamma' 0,\ x \in H_2) \\ &\Longleftrightarrow x\Gamma' 0 \end{aligned}$$

Hence, $[0]_\Gamma = [0]_{\Gamma'}$ and so by lemma 3.4, $\Gamma = \Gamma'$. Therefore, φ is injective. Now, let $(\Theta, \Omega) \in \mathcal{B}$ and Γ be a relation on H which is defined as follows:

$$x\Gamma y \Longleftrightarrow \begin{cases} x\Theta y & \text{if } x, y \in H_1 \\ x\Omega y & \text{if } x, y \in H_2 \\ x\Theta 0 \text{ and } y\Omega 0 & \text{if } x \in H_1, y \in H_2 \\ x\Omega 0 \text{ and } y\Theta 0 & \text{if } x \in H_2, y \in H_1 \end{cases}$$

It is easy to prove that Γ is a regular congruence relation on H. Now, let $a, x, y \in H$ such that $x\Gamma y$. If $x, y \in H_1$ or $x, y \in H_2$, the proof is clear. Now, without loss of generality, let $x \in H_1$ and $y \in H_2$. Since $x\Gamma y$, then $x\Theta 0$ and $y\Omega 0$. If $a \in H_1$, then by definition of $H_1 \oplus H_2$, $a \circ y = a$. Since Θ is a congruence relation on H, then $a \circ x\overline{\Theta}a \circ 0$=a. Hence, $a \circ x\overline{\Theta}a \circ y$ and so $a \circ x\overline{\Gamma}a \circ y$. By $x\Theta 0$ and Lemma 3.3, $x \circ a\overline{\Theta}0 \circ a = 0$. Since $y \circ a = y$ and $y\Omega 0$ then $y \circ a\Omega 0$ and so $y \circ a\overline{\Omega}0$. Thus by the definition of Γ, $x \circ a\overline{\Gamma}y \circ a$. By the similar way, if $a \in H_2$, then we can show that $a \circ x\overline{\Gamma}a \circ y$ and $x \circ a\overline{\Gamma}y \circ a$. Therefore, Γ is a congruence relation on H.

Now, let $x \circ y\Gamma\{0\}$ and $y \circ x\Gamma\{0\}$ for $x, y \in H$. It $x, y \in H_1$ or $x, y \in H_2$, the proof is clear. Now, without loss of generality, let $x \in H_1$ and $y \in H_2$. Then by definition of $H_1 \oplus H_2$, $x \circ y = x$ and $y \circ x = y$. Hence $x\Gamma 0$ and $y\Gamma 0$. Since Γ is transitive, then $x\Gamma y$ and so Γ is a regular relation on H. Now, we show that $\varphi(\Gamma) = (\Theta, \Omega)$. Let $\varphi(\Gamma) = (\Theta', \Omega')$, for $\Theta' \in \mathcal{B}$ and $\Gamma' \in \mathcal{C}$. Then by (1) and definition of Γ we can check that $[0]_\Theta = [0]'_\Theta$ and $[0]_\Omega = [0]'_\Omega$. Hence by Lemma 3.4, $\Theta = \Theta'$ and $\Omega = \Omega'$. Therefore, $\varphi(\Gamma) = (\Theta, \Omega)$ and so φ is a bijection. Now, let $\varphi(\Gamma) = (\Theta, \Omega)$. Then by definition of Γ,

$$x \in [0]_\Gamma \Longleftrightarrow (x\Theta 0,\ x \in H_1) \text{ or } (x\Omega 0,\ x \in H_2) \Longleftrightarrow x \in [0]_\Theta \text{ or } x \in [0]_\Omega \Longleftrightarrow x \in [0]_\Theta \cup [0]_\Omega$$

Therefore, $[0]_\Gamma = [0]_\Theta \cup [0]_\Omega$. □

Corollary 4.16. Let $H = H_1 \oplus H_2, \Theta$ and Ω are regular congruence relations on H_1 and H_2, respectively, $I = [0]_\Theta$ and $J = [0]_\Omega$. Then,

$$\frac{H}{I \cup J} \cong \frac{H_1}{I} \oplus \frac{H_2}{J}$$

Proof. Let $H = H_1 \oplus H_2, \Theta$ and Ω are regular congruence relations on H_1 and H_2, $I = [0]_\Theta$ and $J = [0]_\Omega$. Then by Theorem 4.15, there exists a regular congruence relation Γ on H such that $[0]_\Gamma = [0]_\Theta \cup [0]_\Omega = I \cup J$. Now, let $f : H \longrightarrow \frac{H_1}{I} \oplus \frac{H_2}{J}$ is defined by,

$$f(x) = \begin{cases} I_x & , \text{ if } x \in H_1 \\ J_x & , \text{ if } x \in H_2 \end{cases}$$

We can check that f is an epimorphism and Ker$f$=$I \cup J$. Hence by the isomorphism theorem

$$\frac{H}{I \cup J} \cong \frac{H_1}{I} \oplus \frac{H_2}{J}$$

□

5. **Maximal regular congruence relation**

Definition 5.1. Let H be a hyper BCK-algebra. If there is an element $e \in H$ such that $x \ll e$ for all $x \in H$, then H is called a *bounded* hyper BCK-algebra and e is said to be the *unit* of H.

Lemma 5.2. *Let Θ be a regular congruence relation on bounded hyper BCK-algebra H and $I = [0]_\Theta$. If $e \in H$ be a unit of H, then $e \in I$ if and only if $I = H$.*

Proof. ($\Rightarrow$) Let $e \in H$ be a unit of H and $e \in I$. Let $x \in H$. Since Θ is a congruence relation and $e\Theta 0$, then by Lemma 3.3, $e \circ x \overline{\Theta} 0 \circ x = \{0\}$ and so $e \circ x \Theta \{0\}$. Since e is unit of H, then $x \ll e$. Hence, $0 \in x \circ e$ and so $x \circ e \Theta \{0\}$. Now, since $e \circ x \Theta \{0\}$, $x \circ e \Theta \{0\}$ and Θ is a regular relation on H, then $x \Theta e$ and so by $e \Theta 0$, we get that $x \Theta 0$. Hence, $x \in [0]_\Theta = I$, for all $x \in H$. Therefore, $I = H$.

($\Leftarrow$) The proof is clear. □

Definition 5.3. Let Θ be a congruence relation on H. Then Θ is called a *maximal congruence relation* on H if $[0]_\Theta \neq H$ and if Θ' is a congruence relation on H such that $\Theta \subset \Theta'$, then $[0]_{\Theta'} = H$.

Theorem 5.4. *Let $H \neq \{0\}$ be a bounded hyper BCK-algebra. Then there is at least one maximal regular congruence relation on H.*

Proof. Let

$$T = \{\Theta : \Theta \text{ is a regular congruence relation on } H, \text{ and } [0]_\Theta \neq H\}$$

Let ρ be a relation on H which is defined by, $x \rho y \iff x = y$, for all $x, y \in H$. It is easy to check that ρ is a regular congruence relation on H and $[0]_\rho = \{0\} \neq H$. Hence, $\rho \in T$ and so $T \neq \emptyset$. Clear that, $(T, \subseteq)$ is a partially ordered set. Now, let T_0 be a totally ordered subset of T and $\Theta = \bigcup_{\Theta_i \in T_0} \Theta_i$. It is easy to check that Θ is an equivalence relation on H. Now, let $x, y \in H$ such that $x \Theta y$. Then, there is a $\Theta_i \in T$ such that $x \Theta_i y$. Since Θ_i is a congruence relation on H, then by Lemma 3.3, $x \circ a \overline{\Theta_i} y \circ a$ and $a \circ x \overline{\Theta_i} a \circ y$, for all $a \in H$. Since $\Theta_i \subseteq \Theta$, then $x \circ a \overline{\Theta} y \circ a$ and $a \circ x \overline{\Theta} a \circ y$, for all $a \in H$. Therefore, Θ is a congruence relation on H. Now, let $x \circ y \Theta \{0\}$ and $y \circ x \Theta \{0\}$, for $x, y \in H$. Then, there are $\Theta_i, \Theta_j \in T_0$ such that $x \circ y \Theta_i \{0\}$ and $y \circ x \Theta_j \{0\}$. Since T_0 is a totally ordered, then $\Theta_j \subseteq \Theta_i$ or $\Theta_i \subseteq \Theta_j$. Without loss of generality, we assume that $\Theta_j \subseteq \Theta_i$. Then $x \circ y \Theta_i \{0\}$ and $y \circ x \Theta_i \{0\}$ and since Θ_i is a regular relation, then $x \Theta_i y$ and so $x \Theta y$. Hence, Θ is a regular relation on H. Now, let $[0]_\Theta = H$, by contrary. Since H is bounded, then $e \in H$ and so $e \in [0]_\Theta = \bigcup_{\Theta_i \in T0} [0]_{\Theta_i}$. Hence, there is $\Theta_i \in T_0$ such that $e \in [0]_{\Theta_i}$ and so by Theorem 5.2, $[0]_{\Theta_i} = H$, which is a contradiction. Thus $[0]_\Theta \neq H$ and so $\Theta \in T$. Moreover, Θ is a upper bound of T_0. Now, by Zorn's lemma T has at least one maximal element in H.

In the following example, we show that the bounded condition is necessary in Theorem 5.4. □

Example 5.5. Let $N = \{0, 1, 2, 3, ...\}$ and hyper operation "$\circ$" on N is defined as follow:

$$x \circ y = \begin{cases} \{0, x\} & , \quad \text{if } x \leq y \\ \{x\} & , \quad \text{if } x > y \end{cases}$$

for all $x, y \in H$. Then $(N, \circ, 0)$ is a hyper BCK-algebra. It is easy to check that hyperopration "$\circ$" is well-defined. Now we show that N satisfies the axioms of a hyper BCK-algebra.

(HK1): Let $x, y, z \in N$. Then by definition of "$\circ$", $(x \circ z) \circ (y \circ z) \subseteq \{0, x\}$ and $x \in x \circ y$. Since $\{0, x\} \ll x$, then $(x \circ z) \circ (y \circ z) \ll x \circ y$.

(HK2): Let $x, y, z \in N$. Clear that, $x \in (x \circ z) \circ y$ and $x \in (x \circ y) \circ z$. Now, it is enough to show that $0 \notin (x \circ y) \circ z \iff 0 \notin (x \circ z) \circ y$. Let $0 \notin (x \circ z) \circ y$, then $x > z$ and $x > y$ and so by definition of "$\circ$", $(x \circ y) \circ z = \{x\} \circ z = \{x\}$. Thus $0 \notin (x \circ y) \circ z$. The proof of the converse is similar. Therefore, $(x \circ z) \circ y = (x \circ y) \circ z$.

(HK3): Let $x \in N$. Since, $x \circ N = \bigcup_{y \in N} x \circ y \subseteq \{0, x\} \ll x$, then $x \circ N \ll x$.

(HK4) Let $x, y \in N$, $x \ll y$ and $y \ll x$. Then $0 \in x \circ y$ and $0 \in y \circ x$. Hence, $x \leq y$ and $y \leq x$ and so $x = y$.

Therefore, $(N, \circ, 0)$ is a hyper BCK-algebra, which is not bounded. Now, let I be a hyper BCK-ideal of N. Then, we claim that, $I = N$ or $I = \{0, 1, 2, ..., n\}$, for some $n \in N$. Let $I \neq N$. Since $0 \in I$, then there is $0 \neq m \in N$ such that $m \notin I$. Let n be smallest element of N such that $n \in I$ but $n+1 \notin I$. Then by Theorem 2.6(ii), $\{0, 1, 2, ..., n\} \subseteq I$. Now, let $k \in I$ but $k \notin \{0, 1, 2, ..., n\}$, by contrary. Then $n+1 \leq k$ and so $(n+1) \circ n = \{n+1\} \ll \{k\} \subseteq I$. Since I is a hyper BCK-ideal of N and $n \in I$, then $n+1 \in I$, which is a contradiction. Hence, $I \subseteq \{0, 1, 2, ..., n\}$ and so $I = \{0, 1, 2, ..., n\}$. Now, we prove that N has not any maximal regular congruence relation. Let Θ be a maximal regular congruence relation on N, by contrary. Since, by Lemma 3.5 and Theorems 2.6(i), $I = [0]_\Theta$ is a hyper BCK-ideal of N and $[0]_\Theta \neq H$, then by the above comment, there exists $n \in N$ such that $I = \{0, 1, 2, ..., n\}$. Let relation Θ' on N is defined as follow:

$$x\Theta' y \iff (0 \leq x, y \leq n+1) \quad \text{or} \quad (x, y > n+1 \text{ and } x\Theta y)$$

It is easy to check that Θ' is a reflexive and symmetric relation. Now, let $x\Theta' y$ and $y\Theta' z$. If $0 \leq y \leq n+1$, then $0 \leq x, z \leq n+1$ and so $x\Theta' z$. If $y > n+1$, then $x\Theta y$, $y\Theta z$ and $x, z > n+1$ and so $x\Theta z$. Hence, $x\Theta' z$. Therefore, Θ' is transitive. Now, we show that Θ' is a congruence relation on N. First, we claim that if $x, y \in N$ such that $x, y > n+1$ and $x\Theta y$, then $x = y$. Let $x \neq y$, by contrary. Without loss of generality, we assume that $x < y$. Since Θ is a congruence relation and $x\Theta y$, then by Lemma 3.3, $\{y\} = y \circ x \overline{\Theta} y \circ y = \{0, y\}$ and so $0\Theta y$. Since $0 \leq 0 \leq n+1$, then $0 \leq y \leq n$, which is a contradiction. Thus, $x = y$. Let $a \in N$ be an arbitrary element of N and $x, y \in N$ such that $x\Theta y$. Then by definition of Θ', $0 \leq x, y \leq n+1$ or $x, y > n+1$ and $x\Theta y$. If $x, y > n+1$ and $x\Theta y$, then by the above comment, $x = y$ and so $x \circ a \overline{\Theta'} y \circ a$ and $a \circ x \overline{\Theta'} a \circ y$. If $0 \leq x, y \leq n+1$, then $0\Theta' y$ and $0\Theta' x$ and so $x\Theta' y$. Now, since $x \circ a \subseteq \{0, x\}$ and $y \circ a \subseteq \{0, y\}$, then $x \circ a \overline{\Theta'} y \circ a$. For case $a \circ x \bar{\Theta}' a \circ y$, if $a > n+1$, then $a \circ x = \{a\} = a \circ y$ and so $a \circ x \bar{\Theta}' a \circ y$. If $0 \leq a \leq n+1$, then $a\Theta' 0$. Since $a \circ x \subseteq \{0, a\}$ and $a \circ y \subseteq \{0, a\}$, then $a \circ x \bar{\Theta}' a \circ y$. Therefore, by Lemma 3.3,

Θ is a congruence relation on N. Now, let $x, y \in N$ such that $x \circ y \Theta' \{0\}$ and $y \circ x \Theta' \{0\}$. If $x = y$, the proof is clear. Let $x \neq y$. Without loss of generality, we assume that $x < y$. Hence, $y \circ x = \{y\}$ and so $y \Theta' 0$. Thus, $0 \leq x < y \leq n+1$, and so $x \Theta' y$. Hence, Θ' is a regular relation on N. Moreover, $x \in [0]'_\Theta \iff x \Theta' 0 \iff 0 \leq x \leq n+1$ and this implies that $[0]'_\Theta = \{0, 1, 2, ..., n+1\}$. Hence, $\Theta \subset \Theta'$ (since $(n+1, 0) \in \Theta'$ but $(n+1, 0) \notin \Theta$) and $[0]_{\Theta'} \neq H$, which is a contradiction by maximality of Θ. Hence, there is not any maximal regular congruence on N. Therefore, the bounded condition in Theorem 5.4 is necessary.

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