THE CATEGORIES OF FINITARY BINARY FUNCTIONS AND FINITE AUTOMATA NETWORKS*

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Abstract. Finite Automata Networks (FANs) and Finite Binary Functions (FBFs) are two kinds of finite dynamical systems. In this paper FAN morphisms and FBF morphisms are introduced. The limit structures of the resulting categories **FAN** and **FBF** are studied. It is shown that they are both finitely complete. Finally, a well-known adjunction from directed graphs into sets is lifted to the finite dynamical system level to obtain an adjunction between **FAN** and **FBF**.

1 Introduction

This paper continues in the tradition of [4, 5, 8, 6] in studying finite dynamical systems from the categorical viewpoint. In [1, 2] sequential dynamical systems (SDSs) were introduced as a means to axiomatize computer simulations. In [5] SDSs were generalized to allow greater flexibilty in building a global dynamics from a given collection of local update functions. The new systems were called generalized sequential dynamical systems (GSDSs). Morphisms between GSDSs were subsequently introduced and some properties of the resulting category **GSDS** were investigated. Along similar lines, in [8], threshold agent networks (TANs), forming a subclass of neural networks [3], were introduced (see also [7] for motivation) together with morphisms between them and the resulting category **TAN** was shown to possess finite products. With the aim of enriching this categorical structure, a supercategory **GTAN** of **TAN**, with objects the, so-called, generalized threshold agent networks was also studied in [8] and was shown to be finitely complete. These two papers, revealing aspects of the structures of **GSDS** and **GTAN**, resulted in a further investigation of the categorical relationship between sequential and parallel finite dynamical systems and several functorial transformations between them [6].

In this paper, this tradition of investigating the categorical properties of special classes of finite dynamical systems and their interconnections is continued. Two such classes, the class of finitary binary functions and the class of finite automata networks, are studied with respect to categorical completeness and an adjointness between them is also established.

More precisely, a *finitary binary function* (FBF) is a pair $\langle X, f\rangle$, where X is a finite set and $f : k^X \to k^X$ is a function, where $k = \{0, 1\}$. Given two FBFs $\langle X, f\rangle$ and $\langle Y, g\rangle$, an *FBF morphism* $h : \langle X, f\rangle \to \langle Y, g\rangle$ *from* $\langle X, f\rangle$ *to* $\langle Y, g\rangle$ is a set mapping $h : Y \to X$ that makes the following rectangle commute

$$\begin{array}{ccc} k^X & \xrightarrow{\;f\;} & k^X \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ k^Y & \xrightarrow[\;g\;]{} & k^Y \end{array}$$

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where by $h^* : k^X \to k^Y$ is denoted the function

$$h^*(\vec{x})(y) = \vec{x}(h(y)), \quad \text{for all } \vec{x} \in k^X, y \in Y.$$

FBFs with FBF morphisms between them form a category **FBF**. This category is the object of study in Section 2, where it is shown that it has finite limits.

A *finite automata network* (FAN) is a pair $\mathcal{A} = \langle F, (f_i)_{i \in I} \rangle$, where $F = \langle I, E \rangle$ is a finite digraph with vertex set I and $f_i : k^I \to k$ is a function that depends only on those $j \in I$, such that $\langle j, i \rangle \in E$. Given two FANs $\mathcal{F} = \langle F, (f_i)_{i \in I_F} \rangle$ and $\mathcal{G} = \langle G, (g_i)_{i \in I_G} \rangle$, a *FAN morphism* $h : \mathcal{F} \to \mathcal{G}$ is a digraph morphism $h : G \to F$ that makes the following rectangle commute

$$\begin{array}{ccc} k^{I_F} & \xrightarrow{\langle f_i : i \in I_F \rangle} & k^{I_F} \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ k^{I_G} & \xrightarrow[\langle g_i : i \in I_G \rangle]{} & k^{I_G} \end{array}$$

where, again, by $h^* : k^{I_F} \to k^{I_G}$ is denoted the function defined by

$$h^*(\vec{x})(j) = \vec{x}(h(j)), \quad \text{for all } \vec{x} \in k^{I_F}, j \in I_G.$$

FANs with FAN morphisms between them form a category **FAN**. This is the category that is explored in Section 3. It is shown that this category also possesses finite limits.

Finally, in Section 4, a well-known adjunction $\langle \mathrm{Vrt}, \mathrm{Cmp}, \eta, \epsilon \rangle : \mathbf{Dgr} \to \mathbf{Set}$ from the category of directed graphs **Dgr** to the category **Set** of all small sets is exploited to obtain an adjunction $\langle \mathrm{Fct}, \mathrm{Net}, \zeta, \xi \rangle : \mathbf{FAN} \to \mathbf{FBF}$ from the category of FANs to the category of FBFs.

2 Finitary Binary Functions As before, let k denote the two element set $k = \{0, 1\}$. A *finitary binary function* (FBF) $\langle X, f \rangle$ consists of a finite set X together with a function $f : k^X \to k^X$. An *FBF morphism* $h : \langle X, f \rangle \to \langle Y, g \rangle$ is a set mapping $h : Y \to X$ that makes the following diagram commute

$$\begin{array}{ccc} k^X & \xrightarrow{f} & k^X \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ k^Y & \xrightarrow[g]{} & k^Y \end{array}$$

where, by $h^* : k^X \to k^Y$ is denoted the function defined by

$$h^*(\vec{x})(y) = \vec{x}(h(y)), \quad \text{for all } \vec{x} \in k^X, y \in Y.$$

Given an FBF $\langle X, f \rangle$, the identity map $i_X : X \to X$ is an FBF morphism $i_X : \langle X, f \rangle \to \langle X, f \rangle$ and, given three FBFs $\langle X, f \rangle, \langle Y, g \rangle$ and $\langle Z, e \rangle$ and FBF morphisms

$$h_1 : \langle X, f \rangle \to \langle Y, g \rangle \quad \text{and} \quad h_2 : \langle Y, g \rangle \to \langle Z, e \rangle,$$

the composition $h_1 \circ h_2 : Z \to X$ is an FBF morphism $h_1 \circ h_2 : \langle X, f \rangle \to \langle Z, e \rangle$.

Thus, FBFs together with FBF morphisms between them form a category, called the *category of FBFs* and denoted by **FBF**.

2.1 FBF has Finite Limits In this section, it is shown that **FBF** has finite limits. To this end, it suffices to show that **FBF** has finite products and equalizers.

FBF has Finite Products To show that **FBF** has finite products, it suffices, in view of the existence of the terminal object $\langle \emptyset, i_\emptyset \rangle$, to show that **FBF** has binary products. To this end, let $\langle X, f\rangle$ and $\langle Y, g\rangle$ be two FBFs. By $X \sqcup Y$ will be denoted the coproduct of X, Y in **Set**, i.e., the disjoint union of the sets X, Y. In what follows, for the sake of simplicity, when this coproduct is under consideration, it will be assumed, without loss of generality, that the sets X and Y are disjoint to begin with. If not, disjoint isomorphic copies $\dot{X}$ and $\dot{Y}$ of X and Y, respectively, have to be considered instead. Construct the FBF $\langle X \sqcup Y, f \times g\rangle$ by setting, for all $\vec{u} \in k^{X \sqcup Y}, v \in X \sqcup Y$,

$$(f \times g)(\vec{u})(v) = \begin{cases} f(\vec{u} \upharpoonright_X)(v), & \text{if } v \in X \\ g(\vec{u} \upharpoonright_Y)(v), & \text{if } v \in Y \end{cases}$$

Now construct the two FBF morphisms $\pi_1 : \langle X \sqcup Y, f \times g\rangle \to \langle X, f\rangle$ and $\pi_2 : \langle X \sqcup Y, f \times g\rangle \to \langle Y, g\rangle$ by setting

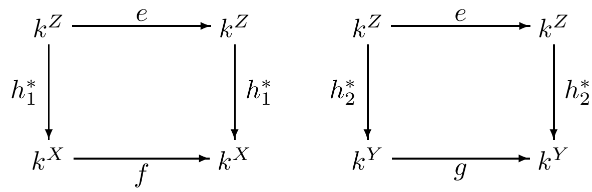
$$\pi_1(x) = x, \quad \text{for all } x \in X,$$

and

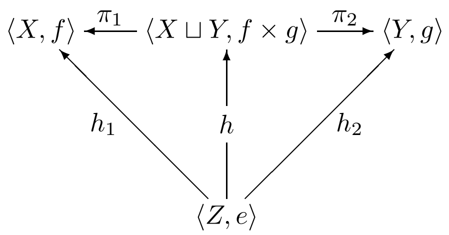
$$\pi_2(y) = y, \quad \text{for all } y \in Y.$$

It is not tough to see that π_1 and π_2, thus defined, are legal FBF morphisms.

It now remains to show that $\langle X \sqcup Y, f \times g\rangle$ with the morphisms π_1 and π_2 is indeed the product of $\langle X, f\rangle, \langle Y, g\rangle$ in **FBF**, i.e., it must be shown that it possesses the universal mapping property of the product in **FBF**. To this end, suppose that $\langle Z, e\rangle$ is an FBF and $h_1 : \langle Z, e\rangle \to \langle X, f\rangle$, $h_2 : \langle Z, e\rangle \to \langle Y, g\rangle$ are FBF morphisms.



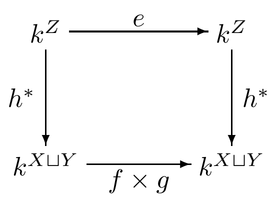
A morphism $h : \langle Z, e\rangle \to \langle X \sqcup Y, f \times g\rangle$ must be constructed that makes the following diagram commute



and its uniqueness must be shown. Define $h : \langle Z, e\rangle \to \langle X \sqcup Y, f \times g\rangle$ to be the set map $h : X \sqcup Y \to Z$ given, for all $v \in X \sqcup Y$, by

$$h(v) = \begin{cases} h_1(v), & \text{if } v \in X \\ h_2(v), & \text{if } v \in Y \end{cases}$$

It is shown that h is indeed an FBF morphism, i.e., that the following diagram commutes:



$$
\begin{aligned}
(f \times g)(h^*(\vec{z}))(v) &= \begin{cases} f(h^*(\vec{z}) \restriction_X)(v), & \text{if } v \in X \\ g(h^*(\vec{z}) \restriction_Y)(v), & \text{if } v \in Y \end{cases} \\
&= \begin{cases} f((h \restriction_X)^*(\vec{z}))(v), & \text{if } v \in X \\ g((h \restriction_Y)^*(\vec{z}))(v), & \text{if } v \in Y \end{cases} \\
&= \begin{cases} f(h_1^*(\vec{z}))(v), & \text{if } v \in X \\ g(h_2^*(\vec{z}))(v), & \text{if } v \in Y \end{cases} \\
&= \begin{cases} h_1^*(e(\vec{z}))(v), & \text{if } v \in X \\ h_2^*(e(\vec{z}))(v), & \text{if } v \in Y \end{cases} \\
&= h^*(e(\vec{z}))(v).
\end{aligned}
$$

Commutativity of the two triangles in the diagram above is straightforward to check and the uniqueness of h follows from the fact that it is uniquely determined by the morphisms h_1 and h_2. Thus $\langle X \sqcup Y, f \times g\rangle$ is in fact the product of $\langle X, f\rangle$ and $\langle Y, g\rangle$ in **FBF**.

FBF has Equalizers It is next shown that **FBF** has equalizers. To this end, let $\langle X, f\rangle$ and $\langle Y, g\rangle$ be two FBFs and $h_1, h_2 : \langle X, f\rangle \to \langle Y, g\rangle$ two FBF morphisms. Construct the FBF $\langle \mathrm{ceq}(h_1, h_2), (f, g)\rangle$ where by $\mathrm{ceq}(h_1, h_2)$ is denoted the coequalizer of h_1, h_2 with accompanying morphism $h : X \to \mathrm{ceq}(h_1, h_2)$

$$
\mathrm{ceq}(h_1, h_2) \xleftarrow{\;h\;} X \underset{h_2}{\overset{h_1}{\leftleftarrows}} Y
$$

$$
\langle \mathrm{ceq}(h_1, h_2), (f, g)\rangle \xrightarrow{\;h\;} \langle X, f\rangle \underset{h_2}{\overset{h_1}{\rightrightarrows}} \langle Y, g\rangle
$$

and $(f, g) : k^{\mathrm{ceq}(h_1,h_2)} \to k^{\mathrm{ceq}(h_1,h_2)}$ is given by

$$
(f, g)(\vec{x})(y) = f(\pi_\theta^*(\vec{x}))(\tilde{y}), \quad \text{for some } \tilde{y} \in y \in \mathrm{ceq}(h_1, h_2),
$$

for all $\vec{x} \in k^{\mathrm{ceq}(h_1,h_2)}$, where by θ is denoted the equivalence relation on X generated by $\eta = \{\langle h_1(y), h_2(y)\rangle : y \in Y\}$ and by $\pi_\theta : X \to X/\theta$ the natural quotient projection map. (Note that, by the construction of θ, $\pi_\theta = h$, the coequalizer map.)

$$
\begin{array}{ccc}
k^{X/\theta} & \xrightarrow{\;(f,g)\;} & k^{X/\theta} \\
\Big\downarrow{\scriptstyle \pi_\theta^*} & & \Big\downarrow{\scriptstyle \pi_\theta^*} \\
k^X & \xrightarrow[\;f\;]{} & k^X
\end{array}
$$

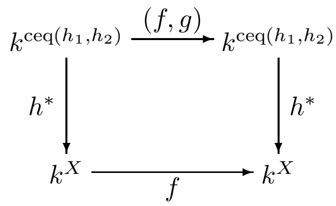
It now remains to show that (f, g) is well-defined, h is a legal FBF morphism and that the pair $\langle\langle \mathrm{ceq}(h_1, h_2), (f, g)\rangle, h\rangle$ possesses the universal mapping property of the equalizer in **FBF**.

First, to show that (f, g) is well defined, it suffices to show that, for all $x_1, x_2 \in X$, if $\langle x_1, x_2\rangle \in \theta$, then $f(h^*(\vec{y}))(x_1) = f(h^*(\vec{y}))(x_2)$, for all $\vec{y} \in k^{\mathrm{ceq}(h_1,h_2)}$. To this end, it suffices, in turn, to show that this is the case for all x_1, x_2, such that $\langle x_1, x_2\rangle \in \eta$. Since h coequalizes h_1 and h_2, it follows that, for all $\vec{y} \in k^{\mathrm{ceq}(h_1,h_2)}$, $g(h_1^*(h^*(\vec{y}))) = g(h_2^*(h^*(\vec{y})))$, whence $h_1^*(f(h^*(\vec{y}))) = h_2^*(f(h^*(\vec{y})))$, i.e.,

$$
f(h^*(\vec{y}))(x_1) = f(h^*(\vec{y}))(x_2), \quad \text{for all } \langle x_1, x_2\rangle \in \eta,
$$

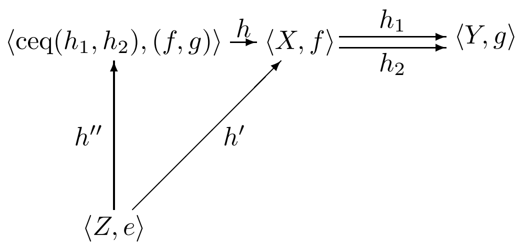
as required.

The fact that h is a legal FBF morphism, i.e., that the following rectangle commutes,



follows immediately by its definition.

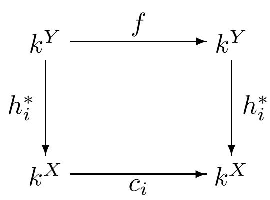
Finally, it remains to check that $\langle \mathrm{ceq}(h_1,h_2),(f,g)\rangle$ possesses the universal mapping property of the equalizer in **FBF**. To this end, suppose that $\langle Z,e\rangle$ is an FBF and $h' : \langle Z,e\rangle \to \langle X,f\rangle$ is an FBF morphism such that $h'h_1 = h'h_2$. Since h coequalizes h_1 and h_2, there exists a unique mapping $h'' : \mathrm{ceq}(h_1,h_2) \to Z$, such that $h''h = h'$. It is not tough to check that this is also a valid FBF morphism $h'' : \langle Z,e\rangle \to \langle \mathrm{ceq}(h_1,h_2),(f,g)\rangle$ and that it is the unique one making the following triangle commute



Denote by $\mathrm{S} : \mathbf{FBF} \to \mathbf{Set}^{\mathrm{op}}$ the contravariant functor that maps an FBF $\langle X,f\rangle$ to the set X and an FBF morphism $h : \langle X,f\rangle \to \langle Y,g\rangle$ to the set mapping $h : Y \to X$. Then, the following theorem has now been proven

Theorem 1 *The category* **FBF** *has finite limits. Moreover, the functor* $\mathrm{S} : \mathbf{FBF} \to \mathbf{Set}^{\mathrm{op}}$ *preserves and creates finite limits.*

2.2 FBF does not have an Initial Object In this section, it will be shown that the category **FBF** does not have an initial object. Suppose to the contrary that $\langle Y,f\rangle$ is an initial object in **FBF**. Let $\langle X,c_i\rangle, 0 \le i \le 2^{|X|}-1$, be the FBFs with constant functions $c_i : k^X \to k^X$, where $\vec{x} \overset{c_i}{\mapsto} b(i)$, for all $\vec{X} \in k^X$, where by $b(i)$ is denoted the binary representation of $i, 0 \le i \le 2^{|X|}-1$, patched with leading zeros so that it has constant length $|X|$. Suppose that $h_i : \langle Y,f\rangle \to \langle X,c_i\rangle, 0 \le i \le 2^{|X|}-1$, is the unique **FBF** morphism from the initial object to $\langle X,c_i\rangle$ in **FBF**, i.e., $h_i : X \to Y, 0 \le i \le 2^{|X|}-1$, is such that the following rectangle commutes



Then, for all $0 \le i \le 2^{|X|}-1$,

$$h_i^*(f(\vec{y}))(x) = f(\vec{y})(h_i(x)) = b(i), \quad \text{for all } \vec{y} \in k^Y, x \in X.$$

This shows that all $f(\vec{y}) \circ h_i, 0 \le i \le 2^{|X|}-1$, are constant functions and that there must be at least $|X|$ in number. Since this holds for all X, it contradicts the existence of a finite Y, such that $\langle Y,f\rangle$ is initial in **FBF**.

3 Finite Automata Networks An *automata network* (AN) [3] $\mathcal{A} = \langle G, Q, (f_i)_{i\in I}\rangle$ on the set I of *agents* consists of

- a digraph $G = \langle I, E\rangle$, with vertex set I,
- a set Q of states and
- a collection $f_i : Q^{E_i} \to Q, i \in I$, where $E_i = \{j \in I : (j, i) \in E\}$. The f_i's are the *local update functions.*

The *global update function* or *dynamics* $F : Q^I \to Q^I$ of such a model is obtained by the local update functions using some updating scheme which, in this paper, will be taken to be parallel or synchronous updating.

A *finite automata network* (FAN) $\mathcal{A} = \langle G, (f_i)_{i\in I}\rangle$ or $\mathcal{A} = \langle I, E, (f_i)_{i\in I}\rangle$ is an automata network $\mathcal{A} = \langle G, Q, (f_i)_{i\in I}\rangle$, where $G = \langle I, E\rangle$, with I finite, and $Q = k = \{0, 1\}$.

Let $\mathcal{A}_1 = \langle G_1, Q_1, (f_i)_{i\in I_1}\rangle$ and $\mathcal{A}_2 = \langle G_2, Q_2, (g_i)_{i\in I_2}\rangle$ be two ANs with dynamics F and G, respectively. An *AN morphism* $h : \mathcal{A}_1 \to \mathcal{A}_2$ is a pair $h = \langle h_1, h_2\rangle$ consisting of

- a directed graph homomorphism $h_1 : G_2 \to G_1$ and
- a mapping $h_2 : Q_1 \to Q_2$,

that make the following rectangle commute

$$\begin{array}{ccc} Q_1^{I_1} & \xrightarrow{F} & Q_1^{I_1} \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ Q_2^{I_2} & \xrightarrow[G]{} & Q_2^{I_2} \end{array}$$

where by $h^* : Q_1^{I_1} \to Q_2^{I_2}$ is denoted the function given by

$$h^*(\langle x_i : i \in I_1\rangle) = \langle h_2(x_{h_1(i)}) : i \in I_2\rangle, \quad \text{for all} \quad \langle x_i : i \in I_1\rangle \in Q^{I_1}.$$

Given an AN $\mathcal{A} = \langle G, Q, (f_i)_{i\in I}\rangle$ the morphism $i_{\mathcal{A}} = \langle i_G, i_Q\rangle : \mathcal{A} \to \mathcal{A}$ acts as an identity morphism between ANs and, given three automata networks $\mathcal{A}_1 = \langle G_1, Q_1, (f_i)_{i\in I_1}\rangle, \mathcal{A}_2 = \langle G_2, Q_2, (g_i)_{i\in I_2}\rangle$ and $\mathcal{A}_3 = \langle G_3, Q_3, (p_i)_{i\in I_3}\rangle$ and two AN morphisms $h : \mathcal{A}_1 \to \mathcal{A}_2$ and $q : \mathcal{A}_2 \to \mathcal{A}_3$ the composition $q \circ h = \langle h_1 \circ q_1, q_2 \circ h_2\rangle$ is also an AN morphism $q \circ h : \mathcal{A}_1 \to \mathcal{A}_3$. Thus, ANs with AN morphisms between them form a category, called the *category of ANs* and denoted by **AN**. AN morphisms $h = \langle h_1, h_2\rangle$, between two ANs with the same set of states, such that h_2 is the identity on the set of states are called *strict.* FANs with strict AN morphisms between them also form a category, the *category of FANs*, denoted **FAN**, which is a subcategory of **AN**. Its morphisms will be called *FAN morphisms.* When a strict AN morphism $h = \langle h_1, h_2\rangle$ is under consideration, h_1 will usually be denoted by h and h_2, which is the identity on states, will usually be omitted from the notation.

Let **Dgr** denote the category of digraphs and $\mathrm{G} : \mathbf{FAN} \to \mathbf{Dgr}^{\mathrm{op}}$ the functor that sends a FAN $\mathcal{A} = \langle G, (f_i)_{i\in I}\rangle$ to the digraph G and a FAN morphism $h : \langle F, (f_i)_{i\in I_F}\rangle \to \langle G, (g_i)_{i\in I_G}\rangle$ to the digraph morphism $h : G \to F$. Then, following similar constructions and techniques as in the previous section, the following theorem may be proved

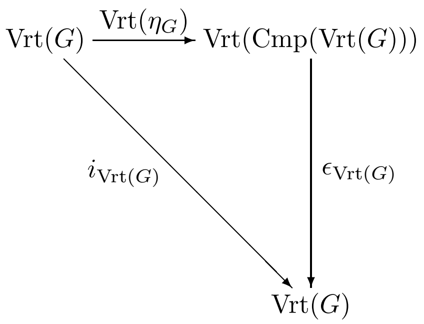
Theorem 2 *The category* **FAN** *has finite limits. Moreover, the functor* $\mathrm{G} : \mathbf{FAN} \to \mathbf{Dgr}^{\mathrm{op}}$ *preserves and creates finite limits.*

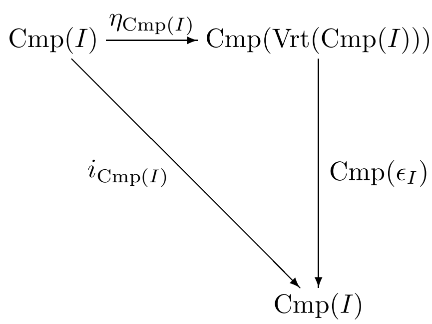
4 Sets, Graphs, FBFs and FANs In this section it is shown that the two categories **FBF** and **FAN**, whose structures were studied in the previous sections, are related via an adjunction that is a lift to the finite dynamical system level of a well known adjunction between the categories **Set** and **Dgr**.

Define, first, the functor Vrt : **Dgr** $\to$ **Set** from the category of digraphs to the category of sets as the underlying vertex set functor. That is, Vrt maps a digraph $G = \langle I, E\rangle$ to its vertex set I and a digraph morphism $h : \langle I_1, E_1\rangle \to \langle I_2, E_2\rangle$ to the set mapping $h : I_1 \to I_2$, which is the restriction of h to vertices.

Next, define the functor Cmp : **Set** $\to$ **Dgr** from the category of sets to the category of digraphs as the complete digraph functor on a given set of vertices. That is, Cmp maps a set I to the digraph $G = \langle I, E\rangle$, where $E = I \times I$, and a mapping $h : I_1 \to I_2$ to the diagraph morphism $h : \langle I_1, E_1\rangle \to \langle I_2, E_2\rangle$, whose restriction on vertices is h.

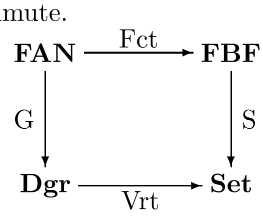
Finally, define the natural transformations $\eta : I_{\mathbf{Dgr}} \to \mathrm{Cmp} \circ \mathrm{Vrt}$, where $I_{\mathbf{Dgr}} : \mathbf{Dgr} \to \mathbf{Dgr}$ is the identity functor on **Dgr**, by $\eta_G : G \to \mathrm{Cmp}(\mathrm{Vrt}(G))$, with η_G being the identity on vertices and $\epsilon : \mathrm{Vrt} \circ \mathrm{Cmp} \to I_{\mathbf{Set}}$, where $I_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ is the identity functor on **Set**, by $\epsilon_I : \mathrm{Vrt}(\mathrm{Cmp}(I)) \to I$, with ϵ_I being the identity map. It is not difficult to check that with these definitions the quadruple $\langle \mathrm{Vrt}, \mathrm{Cmp}, \eta, \epsilon\rangle : \mathbf{Dgr} \to \mathbf{Set}$ forms an adjunction from the category of digraphs to the category of sets, i.e., that the following triangles commute (the first in **Set** and the second in **Dgr**) for all $G \in |\mathbf{Dgr}|$ and all $I \in |\mathbf{Set}|$,

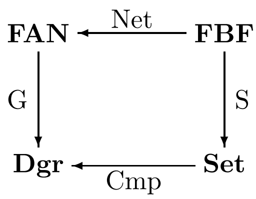




The adjunction described above lifts to an adjunction $\langle \mathrm{Fct}, \mathrm{Net}, \zeta, \xi\rangle : \mathbf{FAN} \to \mathbf{FBF}$ from the category of FANs to the category of FBFs. Fct : **FAN** $\to$ **FBF** is the functor that maps a given FAN to the FBF that has as its first component the set of vertices of the graph of the FAN and as its function the global dynamics of the FAN. Net : **FBF** $\to$ **FAN** is the functor that maps a given FBF $\langle X, f\rangle$ to the FAN that has as its underlying graph the complete graph on the set X and as its local update functions the components of the function f.

Finally, it is not difficult to see that "lifting" may be formally expressed by saying that the two constructions above are such that the following diagrams of categories and functors between them commute.





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