SOME RESULTS ON DEDUCTIVE SYSTEMS OF BL-ALGEBRAS

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Abstract. Special sets in a BL-algebra are introduced, and an example in which a special set is not a deductive system is given. Conditions for these special sets to be a deductive system are stated. Using this special set, an equivalent condition of a deductive system is established. We prove that a deductive system can be represented by the union of such special sets. The notion of commutative BL-algebras is also introduced, and its characterizations are given. The notion of sensitive deductive systems is introduced, and some con ditions for a deductive system to be a sensitive deductive system are given.

1. Introduction

Fuzzy logic grows as a new discipline from the necessity to deal with vague data and imprecise information caused by the indistinguishability of objects in certain experimental environments. As mathematical tools fuzzy logic is only using $[0,1]$-valued maps and certain binary operations $*$ on the real unit interval $[0,1]$ known also as left-continuous t-norms. It took sometime to understand partially ordered monoids of the form $([0,1],\leq,*)$ as *algebras* for $[0,1]$-valued interpretations of a certain type of non-classical logic–the so-called monoidal logic. BL-algebras arise naturally in the analysis of the proof theory of propositional fuzzy logics. In [4], Ko and Kim investigated some properties of BL-algebras, and they [5] also studied relationships between closure operators and BL-algebras. Jun and Ko [2] showed that a deductive system of a BL-algebra can be represented as a union of special sets, and gave a characterization of a deductive system of a BL-algebra. They also discussed how to generate a deductive system by a set, and investigated some related properties. In this paper, we introduced special sets in a BL-algebra, and give an example in which a special set is not a deductive system. We state conditions for these special sets to be a deductive system. Using this special set, we provide an equivalent condition of a deductive system. We prove that a deductive system can be represented by the union of such special sets. We also introduce the notion of commutative BL-algebras, and give its characterizations. We discuss the notion of sensitive deductive systems, and we give some condi tions for a deductive system to be a sensitive deductive system.

2. Preliminaries

Definition 2.1. [7] A *BL-algebra* is an algebra $(L,\wedge,\vee,\odot,\rightsquigarrow,0,1)$ of type $(2,2,2,2,0,0)$ that satisfies the following conditions:

(A1) $(L,\wedge,\vee,0,1)$ is a bounded lattice,
(A2) $(L,\odot,1)$ is a commutative monoid,
(A3) $\odot$ and $\rightsquigarrow$ form an adjoint pair, i.e., $z\leq x\rightsquigarrow y$ if and only if $x\odot z\leq y$ for all $x,y,z\in L$, where $\leq$ is the lattice ordering on L,

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(A4) $(\forall x, y \in L)\ (x \wedge y = x \odot (x \rightsquigarrow y))$,
(A5) $(\forall x, y \in L)\ ((x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1)$.

Example 2.2. [4] Let X be a nonempty set and let $\mathcal{P}(X)$ be the family of all subsets of X. Define operations $\odot$ and $\rightsquigarrow$ by

$$A \odot B = A \cap B \ \text{ and } \ A \rightsquigarrow B = A^c \cup B$$

for all $A, B \in \mathcal{P}(X)$, respectively. Then $(\mathcal{P}(X), \subset, \cap, \cup, \odot, \rightsquigarrow, \emptyset, X)$ is a BL-algebra.

We call $\mathcal{P}(X)$ the *power BL-algebra* of X.

Proposition 2.3. [7] *In a BL-algebra* $(L, \leq, \wedge, \vee, \odot, \rightsquigarrow, 0, 1)$*, we have the following properties for all* $x, y, z \in L$*:*

(p1) $1 \rightsquigarrow x = x$, $x \rightsquigarrow x = 1$, $x \leq y \rightsquigarrow x$, $x \rightsquigarrow 1 = 1$,
(p2) $x \leq (x \rightsquigarrow y) \rightsquigarrow y$,
(p3) $x \odot y \leq x, y$,
(p4) $x \odot y \leq x \wedge y$,
(p5) $y \leq x \rightsquigarrow y$,
(p6) $x \odot y \leq x \rightsquigarrow y$,
(p7) $x \leq y \Leftrightarrow 1 = x \rightsquigarrow y$,
(p8) $x = y \Leftrightarrow 1 = x \rightsquigarrow y = y \rightsquigarrow x$,
(p9) $x \odot (x \rightsquigarrow y) \leq y$,
(p10) $x \rightsquigarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightsquigarrow z)$.
(p11) $(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) \leq x \rightsquigarrow (y \rightsquigarrow z)$,
(p12) $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,
(p13) $x \leq y \Rightarrow z \rightsquigarrow x \leq z \rightsquigarrow y,\ y \rightsquigarrow z\ lex \rightsquigarrow z$,
(p14) $(x \odot y) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z)$.

A BL-algebra L is said to be *implicative* [2] if it satisfies the following inequality:

$$(\forall x, y, z \in L)\, (x \rightsquigarrow (y \rightsquigarrow z) \leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)).$$

Recall that the power BL-algebra $\mathcal{P}(X)$ of a set X is an implicative BL-algebra (see [2]).

Definition 2.4. [1, 6, 7] A subset D of L is called a *deductive system* of L if it satisfies the following conditions:

(d1) $1 \in D$,
(d2) $(\forall x, y \in L)\ (x \in D,\ P(x \backslash y) \in D \Rightarrow y \in D)$.

Proposition 2.5. [4, 7] *Let* D *be a nonempty subset of* L*. Then* D *is a deductive system of* L *if and only if it satisfies:*

(d3) $(\forall a, b \in D)\ (a \odot b \in D)$,
(d4) $(\forall a \in D)(\forall b \in L)\ (a \leq b \Rightarrow b \in D)$.

3. Main results

In what follows, let L denote a BL-algebra unless otherwise specified. For every a_1, $a_2, \cdots, a_n \in L$, we define

$$P(a_1, a_2, \cdots, a_{n-1} \backslash a_n) := \begin{cases} a_n & \text{if } n = 1, \\ a_1 \rightsquigarrow P(a_2, a_3, \cdots, a_{n-1} \backslash a_n) & \text{if } n > 1, \end{cases}$$

and $P(a^n, b \backslash x) = P(a, a, \cdots, a, b \backslash x)$ in which a occurs n-times.

Definition 3.1. For any $a, b \in L$ and $k \in \mathbb{N}$ we define

$$[a^k; b] := \{x \in L \mid P(a^k, b \backslash x) = 1\}.$$

Obviously, $1, a, b \in [a^k; b]$ for all $a, b \in L$ and $k \in \mathbb{N}$. Observe that $[0^m; a] = L = [a^n; 0]$ for all $a \in L$ and $m, n \in \mathbb{N}$. Note that $[a^1; b] \subseteq [a^2; b] \subseteq [a^3; b] \subseteq \cdots$ for all $a, b \in L$.

Proposition 3.2. *Let $a, b \in L$ and $k \in \mathbb{N}$. If $x \in [a^k; b]$, then $P(y\backslash x) \in [a^k; b]$ for all $y \in L$.*

Proof. Assume that $x \in [a^k; b]$. Then

$$P(a^k, b\backslash P(y\backslash x)) = P(a^k, b, y\backslash x) = P(a^k, y, b\backslash x) = P(y\backslash P(a^k, b\backslash x)) = P(y\backslash 1) = 1$$

for all $y \in L$, and so $P(y\backslash x) \in [a^k; b]$. □

Proposition 3.3. *If $a \in L$ satisfies the equality $P(a\backslash x) = 1$ for all $x(\neq 0) \in L$, then $[a^k; b] = L = [b^k; a]$ for all $b \in L$ and $k \in \mathbb{N}$.*

Proof. Since $x \leq 1$ for all $x \in L$, the result follows from (p10). □

Proposition 3.4. *If $a \leq b$ in L, then $[b^k; c] \subseteq [a^k; c]$ for all $c \in L$ and $k \in \mathbb{N}$.*

Proof. Let $a, b, c \in L$ be such that $a \leq b$. If $x \in [b^1; c]$, then $P(b, c\backslash x) = 1$, i.e., $b \leq P(c\backslash x)$. Since $a \leq b$, it follows from (p13) that $1 = P(a\backslash b) \leq P(a, c\backslash x)$ so that $P(a, c\backslash x) = 1$. Hence $x \in [a^1; c]$. If $x \in [b^2; c]$, then $P(b, b, c\backslash x) = P(b^2, c\backslash x) = 1$, i.e., $b \leq P(b, c\backslash x)$. Using (p10) and (p13), we have $1 = P(a\backslash b) \leq P(a, b, c\backslash x) = P(b, a, c\backslash x)$. Thus $P(b, a, c\backslash x) = 1$, and so $b \leq P(a, c\backslash x)$. It follows from (p13) that $1 = P(a\backslash b) \leq P(a, a, c\backslash x) = P(a^2, c\backslash x)$ so that $P(a^2, c\backslash x) = 1$. This shows that $x \in [a^2; c]$. Continuing this process, we obtain that if $x \in [b^k; c]$ then $x \in [a^k; c]$. This completes the proof. □

Proposition 3.5. *If $b \leq c$ in L, then $[a^k; c] \subseteq [a^k; b]$ for all $a \in L$ and $k \in \mathbb{N}$.*

Proof. Let $a, b, c \in L$ be such that $b \leq c$. For every $k \in \mathbb{N}$, if $x \in [a^k; c]$ then

$$1 = P(a^k, c\backslash x) = P(c\backslash P(a^k\backslash x)) \leq P(b\backslash P(a^k\backslash x)) = P(a^k, b\backslash x)$$

by using (p10) and (p13), and hence $P(a^k, b\backslash x) = 1$. Therefore $x \in [a^k; b]$, which shows that $[a^k; c] \subseteq [a^k; b]$. □

For every $w \in L$, we define a set

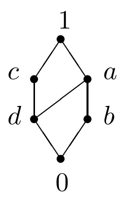
$$[w) := \{x \in L \mid P(w\backslash x) = 1\}.$$

Obviously, $1, w \in [w)$ and $[w)$ is not a deductive system of L in general (see [3]). Note that $[1^k; a] = [a)$ for every $a \in L$ and $k \in \mathbb{N}$. If the following implication

$$(1) \qquad (\forall x, y \in L)(P(a, x\backslash y) = 1,\ P(a\backslash x) = 1 \Rightarrow P(a\backslash y) = 1)$$

is valid, then $[1^k; a]$ is a deductive system of L (see [3]). The following example shows that there exist $a, b \in L$ and $k \in \mathbb{N}$ such that $[a^k; b]$ is not a deductive system of L.

Example 3.6. Let $L = \{0, a, b, c, d, 1\}$ be a set with Hasse diagram and Cayley tables as follows:



$\odot$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

$\rightsquigarrow$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

For every $x, y \in L$, define $x \wedge y = x \odot P(x\backslash y)$ and

$$x \vee y = P(P(x\backslash y)\backslash y) \odot P(P(P(x\backslash y)\backslash y)\backslash P(P(y\backslash x)\backslash x)).$$

Then $(L; \leq, \wedge, \vee, \odot, \rightsquigarrow, 0, 1)$ is a BL-algebra (cf. [3]). We have $[a^2; b] = \{1, a, b\}$ which is a deductive system of L. We know that for every $k \in \mathbb{N}$, $[1^k; a] = \{1, a\}$ is not a deductive system of L.

Definition 3.7. [2] A BL-algebra L is said to be *implicative* if it satisfies the following inequality:

$$(\forall x, y, z \in L)\,(P(x, y\backslash z) \leq P(P(x\backslash y), x\backslash z)).$$

Theorem 3.8. *Let L be a BL-algebra. Then L is implicative if and only if*

$$(\forall x, y \in L)\,(P(x^2\backslash y) = P(x\backslash y)). \tag{2}$$

Proof. Assume that L is implicative. Then

$$(\forall x, y, z \in L)\,(P(x, y\backslash z) = P(P(x\backslash y)\backslash P(x\backslash z)). \tag{3}$$

Substituting y and z by x and y, respectively, in (3) and using (p1), we have $P(x^2\backslash y) = P(x\backslash y)$. Suppose that L satisfies the condition (2). Then

$$\begin{aligned} P(P(x\backslash y), x\backslash z) &= P(P(x\backslash y)\backslash P(x^2\backslash z)) \qquad \text{by (2)} \\ &= P(x, P(x\backslash y)\backslash P(x\backslash z)) \qquad \text{by (p10)} \\ &\geq P(x, y\backslash z) \qquad \text{by (p12) and (p13)} \end{aligned}$$

for all $x, y, z \in L$, and hence L is implicative. □

Corollary 3.9. *If L is a BL-algebra that satisfies the equality $x \odot x = x$ for all $x \in L$, then L is implicative.*

Proof. Using (p14), we obtain $P(x^2\backslash y) = P(x \odot x\backslash y) = P(x\backslash y)$ for all $x, y \in L$. Hence L is implicative. □

We now state a condition for a set $[a^k; b]$ to be a deductive system.

Theorem 3.10. *If L is implicative, then $[a^k; b]$ is a deductive system of L for all $a, b \in L$ and $k \in \mathbb{N}$.*

Proof. Let $x, y \in L$ be such that $P(x\backslash y) \in [a^k; b]$ and $x \in [a^k; b]$. Then

$$\begin{aligned} 1 &= P(a^k, b\backslash P(x\backslash y)) = P(a^{k-1}, a\backslash P(b\backslash P(x\backslash y))) \\ &= P(a^{k-1}, a\backslash P(P(b\backslash x)\backslash P(b\backslash y))) \\ &= P(a^{k-1}\backslash P(a\backslash P(P(b\backslash x)\backslash P(b\backslash y)))) \\ &= P(a^{k-1}\backslash P(P(a, b\backslash x)\backslash P(a, b\backslash y))) \\ &= \cdots \\ &= P(P(a^k, b\backslash x)\backslash P(a^k, b\backslash y)) \\ &= P(1\backslash P(a^k, b\backslash y)) = P(a^k, b\backslash y), \end{aligned}$$

and so $y \in [a^k; b]$. Therefore $[a^k; b]$ is a deductive system of L. □

Using the set $[a^k; b]$ we establish a condition for a subset D of L to be a deductive system of L.

Theorem 3.11. *Let D be a nonempty subset of L. Then D is a deductive system of L if and only if $[a^k; b] \subseteq D$ for every $a, b \in D$ and $k \in \mathbb{N}$.*

Proof. Assume that D is a deductive system of L and let $a, b \in D$ and $k \in \mathbb{N}$. If $x \in [a^k; b]$ then

$$P(a\backslash P(a^{k-1}, b\backslash x)) = P(a^k, b\backslash x) = 1 \in D.$$

Since $a \in D$, it follows from (d2) that $P(a^{k-1}, b\backslash x) \in D$. Continuing this process, we have $P(b\backslash x) \in D$ and thus $x \in D$ by (d2). Hence $[a^k; b] \subseteq D$. Conversely suppose that $[a^k; b] \subseteq D$

for every $a, b \in D$ and $k \in \mathbb{N}$. Note that $1 \in [a^k; b] \subseteq D$. Let $x, y \in L$ be such that $x \in D$ and $P(x\backslash y) \in D$. Then

$$\begin{aligned} &P(x^k, P(x\backslash y)\backslash y) = P(x^{k-1}, x, P(x\backslash y)\backslash y) \\ &\quad = P(x^{k-1}\backslash P(P(x\backslash y)\backslash P(x\backslash y))) = P(x^{k-1}\backslash 1) = 1, \end{aligned}$$

and so $y \in [x^k; P(x\backslash y)] \subseteq D$. Therefore D is a deductive system of L. □

Theorem 3.12. *If D is a deductive system of L, then $D = \bigcup_{a,b\in D} [a^k; b]$ for every $k \in \mathbb{N}$.*

Proof. Let D be a deductive system of L. The inclusion $\bigcup_{a,b\in D} [a^k; b] \subseteq D$ follows from Theorem 3.11. Let $x \in D$. Since $x \in [1^k; x]$, it follows that

$$D \subseteq \bigcup_{x\in D} [1^k; x] \subseteq \bigcup_{a,b\in D} [a^k; b].$$

This completes the proof. □

Definition 3.13. A subset D of L is called a *slow deductive system* of L if it satisfies (d1) and

(d5) for any nonzero elements x and y of L with $P(x\backslash y) \neq 1$, $P(x\backslash y) \in D$ implies $x, y \in D$.

The notion of a slow deductive system is different from the notion of a deductive system in a BL-algebra. A set $D := \{1, a, b\}$ in Example 3.6 is a deductive system of L, but not a slow deductive system of L, since $P(d\backslash b) = a \in D$, but $d \notin D$. On the other hand, a set $E := \{1, a, b, c, d\}$ in Example 3.6 is a slow deductive system of L, but not a deductive system of L.

Theorem 3.14. *Let D be a slow deductive system of L. Then D is a deductive system of L if whenever $P(x\backslash y) = 1$ and $x \in D$ then $y \in D$.*

Proof. Let $x, y \in L$ be such that $x \in D$ and $P(x\backslash y) \in D$. If $P(x\backslash y) = 1$ (resp. $P(x\backslash y) \neq 1$), then $y \in D$ by assumption (resp. by (d5)). Hence D is a deductive system of L. □

Let us denote $x \Cup y = P(P(y\backslash x)\backslash x)$ and consider the following equality:

$$(\forall x, y \in L)(x \Cup y = y \Cup x). \tag{4}$$

We note that in a BL-algebra L, the equality (4) is not true in general. In fact, consider the Gödel structure, i.e., if we define binary operations $\odot$ and $\rightsquigarrow$ on the real unit interval I as follows:

$$x \odot y = \min\{x, y\},$$

$$x \rightsquigarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise,} \end{cases}$$

then $(I, \leq, \min, \max, \odot, \rightsquigarrow, 0, 1)$ is a BL-algebra. It is easy to verify that this BL-algebra does not satisfy the equality (4).

Definition 3.15. A BL-algebra L satisfying the equality (4) is called a *commutative BL-algebra.*

Example 3.16. The BL-algebra L in Example 3.6 is commutative.

Now we have $y \le P(P(y\backslash x)\backslash x) = x \Cup y$, and $P(y\backslash x) \le 1 = P(x\backslash x)$ implies $x \le P(P(y\backslash x)\backslash x) = x \Cup y$. Hence we know that $x \Cup y$ is a common upper bound of x and y. As is easily seen, we have

$$x \Cup x = x \ \text{ and } \ x \Cup 1 = 1 \Cup x = 1. \tag{5}$$

In general, $x \Cup y$ is not the least upper bound of x and y. But if the equality (4) holds, then L is a semilattice with respect to $\Cup$ as follows:

Theorem 3.17. *A BL-algebra L is commutative if and only if it is a semilattice with respect to $\Cup$.*

Proof. If L is a semilattice with respect to $\Cup$, then $x \Cup y$ is the least upper bound of x and y, and so L is commutative. Conversely assume that L is commutative. Let $z \in L$ be a common upper bound of x and y. Then $P(x\backslash z) = 1 = P(y\backslash z)$, and so

$$z = P(1\backslash z) = P(P(x\backslash z)\backslash z) = z \Cup x = x \Cup z$$

$$z = P(1\backslash z) = P(P(y\backslash z)\backslash z) = z \Cup y.$$

It follows from (p2) and (p13) that

$$\begin{aligned} z &= x \Cup z = P(P(z\backslash x)\backslash x) = P(P((z \Cup y)\backslash x)\backslash x) \\ &= P(P((P(y\backslash z)\backslash z)\backslash x)\backslash x) \ge P(P(y\backslash x)\backslash x) = x \Cup y. \end{aligned}$$

This shows that $x \Cup y$ is the least upper bound of x and y. Therefore we have the associative law on $\Cup$. Hence L is a semilattice with respect to $\Cup$. □

Theorem 3.18. *A BL-algebra L is commutative if and only if it satisfies*

$$(\forall a, b \in L)\,([a) \cap [b) = [a \Cup b)). \tag{6}$$

Proof. Assume that L is commutative. If $x \in [a) \cap [b)$, then $P(a\backslash x) = 1$ and $P(b\backslash x) = 1$. Hence x is a common upper bound of a and b. Since L is commutative, $a \Cup b$ is the least upper bound of a and b, and thus $a \Cup b \le x$, i.e., $P((a \Cup b)\backslash x) = 1$. Therefore $x \in [a \Cup b)$. If $y \in [a \Cup b)$, then $P((a \Cup b)\backslash y) = 1$, and so y is a common upper bound of a and b since $a \Cup b$ is the least upper bound of a and b. Hence $P(a\backslash y) = 1$ and $P(b\backslash y) = 1$, that is, $y \in [a)$ and $y \in [b)$ and therefore $y \in [a) \cap [b)$. Conversely assume that L satisfies the condition (6). Then $[a \Cup b) = [a) \cap [b) = [b) \cap [a) = [b \Cup a)$, and thus $a \Cup b \in [b \Cup a)$ and $b \Cup a \in [a \Cup b)$. Therefore $P((b \Cup a)\backslash(a \Cup b)) = 1 = P((a \Cup b)\backslash(b \Cup a))$, which implies $a \Cup b = b \Cup a$. Consequently L is commutative. □

Proposition 3.19. *In a commutative BL-algbera L, we have*

$$(\forall x, y, z \in L)\,(P(z\backslash y) \Cup x \le P((z \Cup x)\backslash(y \Cup x))). \tag{7}$$

Proof. Using (p2), (p10), (p12) and (p13), we get

$$P((z \Cup x)\backslash(y \Cup x)) = P((x \Cup z)\backslash(x \Cup y)) = P(P(y\backslash x), z\backslash x) \ge P(z\backslash y).$$

Also, $P((z \Cup x)\backslash(y \Cup x)) \ge y \Cup x \ge x$. Hence $P((z \Cup x)\backslash(y \Cup x))$ is a common upper bound of $P(z\backslash y)$ and x, and so $P(z\backslash y) \Cup x \le P((z \Cup x)\backslash(y \Cup x))$. □

Let D be a deductive system of a commutative BL-algebra L and $a \in L$. we define

$$a^{-1}D := \{x \in L \mid a \Cup x \in D\}.$$

Obviously, $1 \in a^{-1}D$ for all $a \in L$.

Theorem 3.20. *For every deductive system D of a commutative BL-algebra L an d $a \in L$, the set $a^{-1}D$ is a deductive system of L containing D.*

Proof. Let $x, y \in L$ be such that $x \in a^{-1}D$ and $P(x\backslash y) \in a^{-1}D$. Then $x \Cup a = a \Cup x \in D$ and $P(x\backslash y) \Cup a = a \Cup P(x\backslash y) \in D$. Since $P(x\backslash y) \Cup a \leq P((x \Cup a)\backslash(y \Cup a))$ by Proposition 3.19, it follows from (d4) that $P((x \Cup a)\backslash(y \Cup a)) \in D$ so from (d2) that $a \Cup y = y \Cup a \in D$. This means $y \in a^{-1}D$ which proves that $a^{-1}D$ is a deductive system of L. Now let $y \in D$. Then $y \leq a \Cup y$ implies that $a \Cup y \in D$ by (d4). Hence $y \in a^{-1}D$, that is, $D \subseteq a^{-1}D$. This completes the proof. □

Proposition 3.21. *Let D be a deductive system of a commutative BL-algebra L. Then*

(i) $(\forall a \in L)\,(a^{-1}D = L \Leftrightarrow a \in D)$.
(ii) $(\forall a, b \in L)\,(a \leq b \Rightarrow a^{-1}D \subseteq b^{-1}D)$.
(iii) $(\forall a, b \in L)\,((a \Cup b)^{-1}D = b^{-1}(a^{-1}D))$.
(iv) *If C is a deductive system of L such that $C \subseteq D$, then $a^{-1}C \subseteq a^{-1}D$ for all $a \in L$.*
(v) *For every deductive system C of L and $a \in L$, we have*

$$a^{-1}(C \cap D) = a^{-1}C \cap a^{-1}D.$$

Proof. (i) Sufficiency is obvious. Assume that $a \in D$ and let $x \in L$. Since $a \leq x \Cup a$, it follows from (d4) that $a \Cup x = x \Cup a \in D$ so that $x \in a^{-1}D$. Hence $a^{-1}D = L$.

(ii) Let $a, b \in L$ be such that $a \leq b$. If $x \in a^{-1}D$, then $a \Cup x \in D$. Now $a \leq b$ implies $a \Cup x \leq b \Cup x$ by (p13). It follows from (d4) that $b \Cup x \in D$, i.e., $x \in b^{-1}D$. Therefore $a^{-1}D \subseteq b^{-1}D$.

(iii) We have

$$\begin{aligned} x \in (a \Cup b)^{-1}D &\Leftrightarrow (a \Cup b) \Cup x \in D \Leftrightarrow a \Cup (b \Cup x) \in D \\ &\Leftrightarrow b \Cup x \in a^{-1}D \Leftrightarrow x \in b^{-1}(a^{-1}D). \end{aligned}$$

(iv) Let $x \in a^{-1}C$. Then $a \Cup x \in C \subseteq D$, and so $x \in a^{-1}D$.

(v) We have

$$\begin{aligned} x \in a^{-1}(C \cap D) &\Leftrightarrow a \Cup x \in C \cap D \Leftrightarrow a \Cup x \in C,\, a \Cup x \in D \\ &\Leftrightarrow x \in a^{-1}C,\, x \in a^{-1}D \Leftrightarrow x \in a^{-1}C \cap a^{-1}D. \end{aligned}$$

This completes the proof. □

Let D be a deductive system of a commutative BL-algebra L. For any subset K of L we define

$$[K : D] = \{a \in L \mid a \Cup K \subseteq D\}$$

where $a \Cup K = \{a \Cup x \mid x \in K\}$. Note that $1 \in [K : D]$. If $a \in [K : D]$, then $a \Cup K \subseteq D$, and so $K \subseteq a^{-1}D$. This implies that $[K : D] = \{a \in L \mid K \subseteq a^{-1}D\}$.

Proposition 3.22. *If D is a deductive system of a commutative BL-algebra L and G is a subset of L such that $G \subseteq D$, then $x \Cup G \subseteq D$ for all $x \in L$.*

Proof. The proof is straightforward. □

Theorem 3.23. *If D is a deductive system of a commutative BL-algebra L and K is a subset of L, then $[K : D]$ is a deductive system of L containing D.*

Proof. Let $x, y \in L$ be such that $x \in [K : D]$ and $P(x\backslash y) \in [K : D]$. Then $x \Cup K \subseteq D$ and $P(x\backslash y) \Cup K \subseteq D$, which imply that $x \Cup a \in D$ and $P(x\backslash y) \Cup a \in D$ for all $a \in K$. Since $P(x\backslash y) \Cup a \leq P((x \Cup a)\backslash(y \Cup a))$ by (7), it follows from (d4) that $P((x \Cup a)\backslash(y \Cup a)) \in D$ so from (d2) that $y \Cup a \in D$ for all $a \in K$. Thus $y \Cup K \subseteq D$, and consequently $y \in [K : D]$. This proves that $[K : D]$ is a deductive system of L. Now let $x \in D$. Since $x \leq a \Cup x$ for all $a \in K$, we have $x \Cup a = a \Cup x \in D$ for all $a \in K$ by (d4). This shows that $x \in [K : D]$ and thus $D \subseteq [K : D]$. □

Proposition 3.24. *Let D be a deductive system of a commutative BL-algebra L and let G and H be subsets of L. Then the following hold:*

(i) $G \subseteq H \Rightarrow [H : D] \subseteq [G : D]$.
(ii) $G \subseteq [[G : D] : D]$.
(iii) $[L : D] = D$.
(iv) $[G : D] = [[[G : D] : D] : D]$.
(v) $D \subseteq G \Rightarrow [G : D] \cap G = D$.
(vi) $[G : D] \cap [[G : D] : D] = D$.
(vii) $[G \cup H : D] = [G : D] \cap [H : D]$.
(viii) *If G is a deductive system of L, then* $[H : D] \cap [H : G] = [H : D \cap G]$.
(ix) $[G : D] = L \Leftrightarrow G \subseteq D$.

Proof. (i) If $x \in [H : D]$, then $x \Cup H \subseteq D$. Since $G \subseteq H$, we have $x \Cup G \subseteq x \Cup H \subseteq D$ and so $x \in [G : D]$. Consequently $[H : D] \subseteq [G : D]$.

(ii) Let $x \in G$ and $y \in [G : D]$. Then $x \in G \subseteq y^{-1}D$, which implies that $x \Cup y = y \Cup x \in D$ for all $y \in [G : D]$. Hence $x \Cup [G : D] \subseteq D$. This proves that $x \in [[G : D] : D]$ and consequently $G \subseteq [[G : D] : D]$.

(iii) If $x \in [L : D]$, then $L \subseteq x^{-1}D$ and so $L = x^{-1}D$. It follows from Proposition 3.21(i) that $x \in D$ so that $[L : D] \subseteq D$. The reverse inclusion is by Theorem 3.23.

(iv) We first get $[G : D] \subseteq [[[G : D] : D] : D]$ by (ii). Since $G \subseteq [[G : D] : D]$ by (ii), it follows from (i) that $[[[G : D] : D] : D] \subseteq [G : D]$.

(v) Let $x \in [G : D] \cap G$. Then $x \in G \subseteq x^{-1}D$, and so $x = x \Cup x \in D$. Hence $[G : D] \cap G \subseteq D$. Note that $D \subseteq [G : D]$ by Theorem 3.23, and hence $D \subseteq [G : D] \cap G$.

(vi) It is by (v) and Theorem 3.23.

(vii) We have

$$\begin{aligned} x \in [G \cup H : D] &\Leftrightarrow G \cup H \subseteq x^{-1}D \\ &\Leftrightarrow G \subseteq x^{-1}D,\ H \subseteq x^{-1}D \\ &\Leftrightarrow x \in [G : D],\ x \in [H : D] \\ &\Leftrightarrow x \in [G : D] \cap [H : D]. \end{aligned}$$

Thus $[G \cup H : D] = [G : D] \cap [H : D]$.

(viii) Note that

$$\begin{aligned} x \in [H : D \cap G] &\Leftrightarrow x \Cup H \subseteq D \cap G \\ &\Leftrightarrow x \Cup H \subseteq D,\ x \Cup H \subseteq G \\ &\Leftrightarrow x \in [H : D],\ x \in [H : G] \\ &\Leftrightarrow x \in [H : D] \cap [H : G], \end{aligned}$$

and hence $[H : D] \cap [H : G] = [H : D \cap G]$.

(ix) Assume that $G \subseteq D$. Then $x \Cup G \subseteq D$ for all $x \in L$ (see Proposition 3.22). Thus $[G : D] = L$. Conversely suppose that $[G : D] = L$. If $G \nsubseteq D$, then there exists $a \in G$ such that $a \notin D$. Since $[G : D] = L$, therefore $x \Cup G \subseteq D$ for all $x \in L$. In particular, $a \Cup G \subseteq D$, which implies that $a = a \Cup a \in a \Cup G \subseteq D$. This is a contradiction. Hence $G \subseteq D$. This completes the proof. □

Definition 3.25. Let w be a fixed element of L. A subset D of L is called a *sensitive deductive system* of L with respect to w (*w-sensitive deductive system,* for short) if it satisfies (d1) and

(d6) $(\forall x, y \in L)\,(P(w, x\backslash y) \in D,\ P(w\backslash x) \in D \Rightarrow y \in D)$.

A sensitive deductive system of L with respect to all $w(\neq 0, 1)$ is called a *sensitive deductive system* of L.

Clearly, a sensitive deductive system with respect to 0 is the whole algebra L, and a sensitive deductive system with respect to 1 is coincident with a deductive system.

Example 3.26. Let L be a BL-algebra in Example 3.6. It is routine to verify that $D := \{1, a, b\}$ is a sensitive deductive system of L with respect to a and b, but not with respect to c and d, because

$$P(c, b\backslash d) = 1 \in D,\ P(c\backslash b) = b \in D, \text{ but } d \notin D,$$

$$P(d, b\backslash c) = 1 \in D,\ P(d\backslash b) = a \in D, \text{ but } c \notin D.$$

Proposition 3.27. *Every w-sensitive deductive system contains w itself.*

Proof. If $w = 0, 1$, it is trivial. Assume that $w \neq 0$. Let D be a w-sensitive deductive system of L. Note that $P(w, 1\backslash w) = P(w\backslash w) = 1 \in D$ and $P(w\backslash 1) = 1 \in D$. It follows from (d6) that $w \in D$. □

Theorem 3.28. *Every sensitive deductive system is a deductive system.*

Proof. Let D be a sensitive deductive system of L and let $x, y \in L$ be such that $x \in D$ and $P(x\backslash y) \in D$. Then $P(1\backslash x) = x \in D$ and $P(1, x\backslash y) = P(x\backslash y) \in D$. It follows from (d6) that $y \in D$. Hence D is a deductive system of L. □

The converse of Theorem 3.28 is not true in general. In fact, we know that, in Example 3.6, $D := \{1, a, b\}$ is a deductive system of L, but not a c-sensitive deductive system of L and hence not a sensitive deductive system of L.

We give conditions for a deductive system to be a sensitive deductive system.

Theorem 3.29. *Let D be a deductive system of L. Then D is a sensitive deductive system of L if and only if it satisfies*

(d7) $(\forall x, y, z \in L)\,(P(x, y\backslash z) \in D \Rightarrow P(P(x\backslash y)\backslash z) \in D)$.

Proof. Let D be a deductive system of L satisfying the condition (d7). Let $x, y, z \in L$ be such that $P(x, y\backslash z) \in D$ and $P(x\backslash y) \in D$. Then $P(P(x\backslash y)\backslash z) \in D$ by (d7), and so $z \in D$ by (d2). Therefore D is sensitive. Conversely, suppose that D is a sensitive deductive system of L. Let $x, y, z \in L$ be such that $P(x, y\backslash z) \in D$. Using (p10) and (p12), we have

$$\begin{aligned} &P(x, P(y\backslash z)\backslash P(P(x\backslash y)\backslash z)) = P(P(y\backslash z), x\backslash P(P(x\backslash y)\backslash z)) \\ &\quad = P(P(y\backslash z), P(x\backslash y)\backslash P(x\backslash z)) = 1 \in D. \end{aligned}$$

Since $P(x\backslash P(y\backslash z)) = P(x, y\backslash z) \in D$, it follows from (d6) that $P(P(x\backslash y)\backslash z) \in D$. This completes the proof. □

Theorem 3.30. *Let D be a deductive system of L. Then D is a sensitive deductive system of L if and only if it satisfies*

(d8) $(\forall x, y \in L)\,(P(x^2\backslash y) \in D \Rightarrow y \in D)$.

Proof. It is sufficient to show that (d7) and (d8) are equivalent. Putting $x = y$ in (d7) and using (p1) induces the condition (d8). Assume that (d8) holds and let $x, y, z \in L$ be such that $P(x, y\backslash z) \in D$. Using (p1), (p10), (p12) and (p13), we have

$$\begin{aligned} &P(P(x, y\backslash z), x, x\backslash P(P(x\backslash y)\backslash z)) = P(1, P(x, y\backslash z), x, x\backslash P(P(x\backslash y)\backslash z)) \\ &\quad = P(P(P(y\backslash z), x\backslash P(P(x\backslash y)\backslash z)), P(x, y\backslash z), x, x\backslash P(P(x\backslash y)\backslash z)) \\ &\quad = P(P(x, y\backslash z), P(P(y\backslash z), x\backslash P(P(x\backslash y)\backslash z)), x, x\backslash P(P(x\backslash y)\backslash z)) \\ &\quad \geq P(P(x, y\backslash z)\backslash P(x, y\backslash z)) = 1, \end{aligned}$$

which implies that

$$P(P(x, y\backslash z)\backslash P(x^2\backslash P(P(x\backslash y)\backslash z))) = P(P(x, y\backslash z), x, x\backslash P(P(x\backslash y)\backslash z)) = 1 \in D.$$

Since D is a deductive system of L, it follows from (d2) that $P(x^2\backslash P(P(x\backslash y)\backslash z)) \in D$ so from (d8) that $P(P(x\backslash y)\backslash z) \in D$. This completes the proof. □

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