G^2 PH QUINTIC SPIRAL TRANSITION CURVES AND THEIR APPLICATIONS

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Abstract. A method for family of G^2 Pythagorean hodograph (PH) quintic spiral transition from straight line to circle is discussed in this paper. This method is then extended to a pair of spirals between two straight lines or two circles. We derive a family of quintic transition spiral curves joining them. Due to flexibility and wide range of shape control parameters, our method can be easily applied for practical applications like high way designing, blending in CAD, consumer products such as ping-pong paddles, rounding corners, or designing a smooth path that avoids obstacles.

1 Introduction and Description of Method A method for smooth G^2 planar PH quintic spiral transition from straight line to circle is developed. This method is then extended to a pair of spirals transition between two circles or between two non-parallel straight lines. We also develop a method for drawing a constrained guided planar spiral curve that falls within a closed boundary. The boundary is composed of straight line segments and circular arcs. Our constrained curve can easily be controlled by shape control parameter. Any change in this shape control parameter does not effect the continuity and neighborhood parts of the curve. There are several problems whose solution requires these types of methods. For example

- Transition curves can be used for blending in the plane, i.e., to round corners, or for smooth transition between circles or straight lines.
- Consumer products such as ping-pong paddles or vase cross section can be designed by blending circles.
- User can easily design rounding shapes like satellite antenna by surface of revolution of our smooth transition curve.
- For applications such as the design of highways or railways it is desirable that transitions be fair. In the discussion about geometric design standards in AASHO (American Association of State Highway officials), Hickerson [6] (p. 17) states that "Sudden changes between curves of widely different radii or between long tangents and sharp curves should be avoided by the use of curves of *gradually increasing or decreasing radii* without at the same time introducing an appearance of forced alignment". The importance of this design feature is highlighted in [4] that links vehicle accidents to inconsistency in highway geometric design.
- A user may wish to design a curve that fits inside a given region as, for example, when one is designing a shape to be cut from a flat sheet of material.
- A user may wish to design a smooth path that avoids obstacles as, for example, when one is designing a robot or auto drive car path.

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Parametric cubic curves are popular in CAD applications because they are the lowest degree polynomial curves that allow inflection points (where curvature is zero), so they are suitable for the composition of G^2 blending curves. To be visually pleasing it is desirable that the blend be fair. The Bézier form of a parametric cubic curve is usually used in CAD/CAM and CAGD (Computer Aided Geometric Design) applications because of its geometric and numerical properties. Many authors have advocated their use in different applications like data fitting and font designing. The importance of using fair curves in the design process is well documented in the literature [1, 2, 3, 7, 11]. Cubic curves, although smoother, are not always helpful since they might have unwanted inflection points and singularities (See [9, 10]). A cubic segment has the following undesirable features

- Its arc-length is the integral part of the square root of a polynomial of its parameter.
- Its offset is neither polynomial, nor a rational algebraic function of its parameter.
- It may have more curavture extrema than necessary.

Pythagorean Hodograph (PH) curves do not suffer from the first two of the aforementioned undesirable features. A quintic is the lowest degree PH curve that may have an inflection point, as required for an S-shaped transition curve. Spirals have several advantages of containing neither inflection points, singularities nor curvature extrema (See [5, 13]). Such curves are useful for transition between two circles (See [14]). Walton [12] considered a planar G^2 quintic transition spiral joining a straight line to a circle. Its use enables us to join two circles forming C- and S-shaped curves such that all points of contact are G^2. Recently, Meek [8] presented a method for drawing a guided G^1 continuous planar spline curve that falls within a closed boundary composed of straight line segments and circles. The objectives and shape features of our scheme in this paper are

- To obtain a family of fair G^2 PH quintic transition spiral curve joining a straight line and a circle.
- To obtain a family of transition curves between two circles or between two non parallel straight lines.
- To simplify the analysis of Walton [12].
- To achieve more degrees of freedom and flexible constraints for easy use in practical applications.
- To demonstrate guided G^2 continuous spiral spline that falls within a closed boundary composed of straight line segments and circles. Our guided curve has better smoothness than Meek [8] scheme which has G^1 continuity.
- Any change in shape parameter does not effect continuity and neighborhood parts of our guided spline. So, our scheme is completely local.

The organization of our paper is as follows. Next section gives a brief discussion of the notation and conventions for the PH quintic spiral with some theoretical background and description of method. Its use for the various transitions encountered in general curve, guided curve and practical applications is analysed followed by illustrative examples and concluding remarks.

2 Background and Description of Method Let L be a straight line through origin $\boldsymbol{O}$ and a circle Ω of radius r centered at $\boldsymbol{C}$ (see Fig. 1). First we consider a PH quintic transition $\boldsymbol{z}(t)(=(x(t),y(t)), 0 \leq t \leq 1$ of the form

$$\boldsymbol{z}'(t) = (u(t)^2 - v(t)^2, 2u(t)v(t)) \tag{1}$$

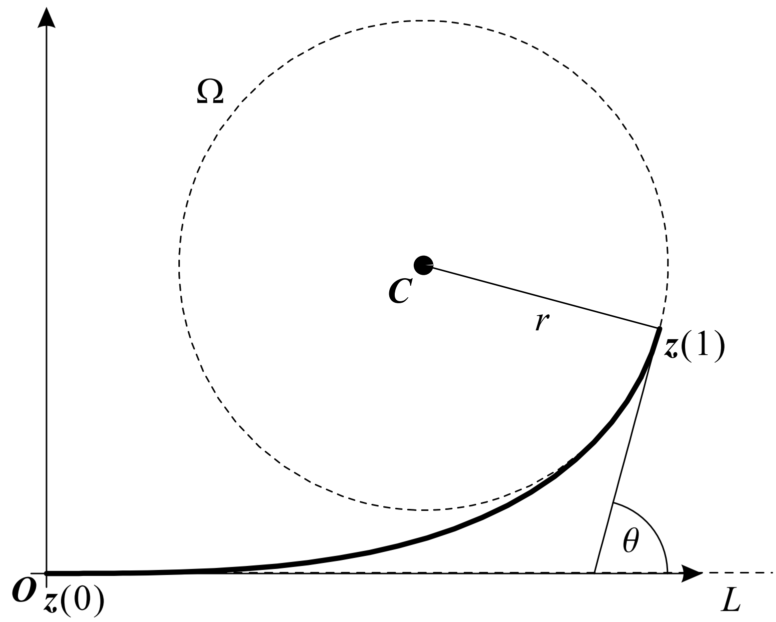

Figure 1: Straight line to circle transition.

where

$$u(t) = u_0(1-t)^2 + 2u_1 t(1-t) + u_2 t^2 \tag{2}$$
$$v(t) = v_0(1-t)^2 + 2v_1 t(1-t) + v_2 t^2$$

Its signed curvature $\kappa(t)$ is given by

$$\kappa(t)\left(= \frac{\boldsymbol{z}'(t)\times\boldsymbol{z}''(t)}{\|\boldsymbol{z}'(t)\|^3}\right) = \frac{2\left\{u(t)v'(t)-u'(t)v(t)\right\}}{\left\{u(t)^2+v^2(t)\right\}^2} \tag{3}$$

where $\times$ stands for the two-dimensional cross product $(x_0, y_0)\times(x_1, y_1) = x_0y_1 - x_1y_0$ and $\|\bullet\|$ means the Euclidean norm. For later use, consider

$$\begin{aligned}\left\{u^2(t)+v^2(t)\right\}^3\kappa'(t) = 2&\left[\left\{u(t)v''(t)-u''(t)v(t)\right\}\left\{u^2(t)+v^2(t)\right\}\right.\\ &\left.-4\left\{u(t)v'(t)-u'(t)v(t)\right\}\left\{u(t)u'(t)+v(t)v'(t)\right\}\right](=w(t))\end{aligned} \tag{4}$$

Then, we require for $0 < \theta \le \pi/2$

$$\boldsymbol{z}(0) = (0,0),\quad \boldsymbol{z}'(0)\parallel(1,0),\quad \kappa(0)=0,\quad \kappa(1)=1/r,\quad \boldsymbol{z}'(1)\parallel(\cos\theta,\sin\theta) \tag{5}$$

Then, the above conditions require

Lemma 1 *With a positive parameter d,*

$$u_1 = \frac{d^3}{4r\sin\frac{\theta}{2}},\ u_2 = d\cos\frac{\theta}{2},\ v_0 = 0,\ v_1 = 0,\ v_2 = d\sin\frac{\theta}{2} \tag{6}$$

where $\boldsymbol{z}'(0) = (u_0^2, 0)$ and $\boldsymbol{z}'(1) = d^2(\cos\theta, \sin\theta)$.

We introduce a pair of parameters (m, q) for (u_0, d) as

$$u_0 = mu_1, \quad d = \sqrt{qr} \tag{7}$$

Then, note

$$\begin{aligned} u(t) &= \frac{\sqrt{qr}}{4\sin\frac{\theta}{2}}\left[q\left\{m(1-t)+2t\right\}(1-t)+2t^2\sin\theta\right] \\ v(t) &= \sqrt{qr}t^2\sin\frac{\theta}{2} \end{aligned} \tag{8}$$

Here Walton require with additional condition $\kappa'(1) = 0$

$$u_2 = \frac{1}{2}\sqrt{7r\sin\theta},\ u_0 = u_1 = \frac{7u_2}{4(1+\cos\theta)},\ v_0 = v_1 = 0,\ v_2 = u_2\tan\frac{\theta}{2}$$

from which follow

$$\begin{aligned} u(t) &= \frac{\sqrt{7r\sin\theta}}{16\cos^2\frac{\theta}{2}}\left(7-3t^2+4t^2\cos\theta\right) \\ v(t) &= \frac{t^2\sqrt{7r\sin\theta}}{2}\tan\frac{\theta}{2} \end{aligned} \tag{9}$$

i.e., their case is derived from ours with $m = 1$ and $q = \frac{7}{2}\tan\frac{\theta}{2}$ ([12]). With help of a symbolic manipulator, we obtain

$$w\left(\frac{1}{1+s}\right) = \frac{q^3r^2}{16(1+s)^5\sin^2\frac{\theta}{2}}\sum_{i=0}^{5}a_is^i \tag{10}$$

where

$$\begin{aligned} &a_0 = 16\left\{q\sin\theta-(6+m)\sin^2\frac{\theta}{2}\right\}, \quad a_1 = 8\left\{2q^2-q(4-3m)\sin\theta-14m\sin^2\frac{\theta}{2}\right\} \\ &a_2 = 4q\left\{(-2+9m)q-3m(4-m)\sin\theta\right\}, \quad a_3 = 4mq\left\{(-3+7m)q-3m\ sin\theta\right\} \\ &a_4 = m^2q^2(-2+7m), a_5 = m^3q^2 \end{aligned}$$

Hence, we have a sufficient spiral condition for a transition curve $\boldsymbol{z}(t), 0 \le t \le 1$, i.e., $a_i \ge 0, 0 \le i \le 5$

Lemma 2 *The quintic segment $\boldsymbol{z}(t), 0 \le t \le 1$ of the form (2) is a spiral satisfying (5) if $m > 3/7$ and*

$$\begin{aligned} &q \ge q(m,\theta)\left(= Max\left[\frac{6+m}{2}\tan\frac{\theta}{2},\ \frac{3m(4-m)\sin\theta}{9m-2},\ \frac{3m\sin\theta}{7m-3},\right.\right. \\ &\left.\left.\frac{1}{4}\left\{(4-3m)\sin\theta+\sqrt{56m(1-\cos\theta)+(4-3m)^2\sin^2\theta}\right\}\right]\right) \end{aligned} \tag{11}$$

As in [12], we can require an additional condition $\kappa'(1) = 0$. Then $b_0 = 0$, i.e., $q = \frac{6+m}{2}\tan\frac{\theta}{2}$. Then Lemma 2 gives a spiral condition

Lemma 3 *The quintic segment* $\boldsymbol{z}(t), 0 \leq t \leq 1$ *of the form (2) is a spiral satisfying (5) and* $\kappa'(1) = 0$ *for* $\theta \in (0, \pi/2]$ *if* $m \geq \frac{2}{7}(-3 + \sqrt{30})(\approx 0.707)$.

Proof. Letting $z = \tan(\theta/2)$, we only have to notice that the terms in brackets of (11) reduce

$$\frac{(6+m)z}{2}(= A_1), \quad \frac{6m(4-m)z}{(9m-2)(1+z^2)}(= A_2), \quad \frac{6mz}{(7m-3)(1+z^2)}(= A_3),$$
$$\frac{z\left\{4 - 3m + \sqrt{16 + 9m^2 + 4m(1+7z^2)}\right\}}{2(1+z^2)}(= A_4)$$

where

$$A_1 \geq A_2 \quad (m > \frac{2}{3}), \quad A_1 \geq A_3 \quad (m > \frac{3}{14}(-9 + \sqrt{137}) \approx 0.5795)$$
$$A_1 \geq A_4 \quad (m > \frac{2}{7}(-3 + \sqrt{30}) \approx 0.7077)$$

2.1 Circle to Circle Spiral Transition A method for straight line to circle transition is being extended to C- or S-shaped transition between two circles Ω_0, Ω_1 with centers C_0, C_1 and radii r_0, r_1 respectively. Now, $z(t, r)(= (x(t, r), y(t, r)), 0 \leq t \leq 1$ denotes the quintic spline satisfying (5). Then, a pair of quintic curves $(x(t, r_0), y(t, r_0))$ and $(-x(t, r_1), y(t, r_1))$ is considered to be a pair of C-shaped spiral transition curves from $\varOmega_0$ to $\varOmega_1$. Then, the distance $\rho(= \|\mathrm{C}_0 - \mathrm{C}_1\|)$ of the centers of the two circles is given with $q \geq q(m, \theta)$

$$\rho = \sqrt{g_1^2(r_0 + r_1)^2 + g_2^2(r_0 - r_1)^2} \tag{12}$$

where

$$g_1\,(= g_1(m, \theta, q)) = \left(\frac{1}{240 \sin^2 \frac{\theta}{2}}\right)\left\{q^3(2 + 3m + 3m^2) + 2q^2(3+m)\sin\theta\right.$$
$$\left. + 48q\sin^2(\theta/2)\cos\theta - 240\sin^2(\theta/2)\sin\theta\right\} \tag{13}$$
$$g_2\,(= g_2(m, \theta, q)) = \frac{1}{60}\left\{q^2(3+m) + 12q\sin\theta + 60\cos\theta\right\}$$

A simple calculation gives

$$(i) \qquad (g_1(m, \theta, q(m, \theta)), g_2(m, \theta, q(m, \theta))) \to (0, 1) \quad (\theta \to 0)$$
$$(ii) \qquad g_1(m, \theta, q(m, \theta)) = \frac{1}{120}\left\{(2 + 3m + 3m^2)q^3(m, \theta) + 2(3+m)q^2(m, \theta)\right.$$
$$\left. -120\right\}(= \phi(m)) \quad (\theta = \frac{\pi}{2})$$

Here, for $m \geq \frac{2}{7}(-3 + \sqrt{30})$, note $q(m, \frac{\pi}{2}) = \frac{6+m}{2}$ to obtain

$$\phi(m) = \frac{1}{320}(-32 + 384m + 356m^2 + 128m^3 + 19m^4 + m^5) \tag{14}$$

Hence we have

Theorem 1 *If* $|r_0 - r_1| < \|C_0 - C_1\| < \phi(m)(r_0 + r_1)$, *a* C*-shaped pair of quintic curves* $(x(t, r_0), y(t, r_0))$ *and* $(-x(t, r_1), y(t, r_1))$ *is a* G^2*-continuity transition curve from* $\varOmega_0$ *to* $\varOmega_1$ *for a selection of* $\theta \in (0, \pi/2]$ *where* $\phi(1) \approx 2.675$ *and* $\phi(m) \to \infty$ $(m \to \infty)$ *([12]).*

Next, a pair of an S-shaped quintic curves $(x(t,r_0), y(t,r_0))$ and $(-x(t,r_1), -y(t,r_1))$ are considered a pair of an S-shaped transition curves from Ω_0 to Ω_1. Then, the distance $\rho(=\|C_0 - C_1\|)$ of the centers of the two circles is given by

$$\rho = \sqrt{g_1^2 + g_2^2}(r_0 + r_1) \tag{15}$$

from which follows

$$g_2(m, \theta, q(m,\theta)) = \frac{1}{60}\left\{(3+m)q^2(m,\theta) + 12q(m,\theta)\right\} (= \psi(m)) \quad (\theta = \frac{\pi}{2}) \tag{16}$$

Here, for $m \geq \frac{2}{7}(-3+\sqrt{30})$,

$$\psi(m) = \frac{1}{240}(6+m)(42+9m+m^2) \tag{17}$$

Letting $\gamma(m) = \sqrt{\phi(m)^2 + \psi(m)^2}$, then

Theorem 2 *If $r_0 + r_1 < \|C_0 - C_1\| < \gamma(m)(r_0 + r_1)$, a pair of an S-shaped quintic curves $(x(t,r_0), y(t,r_0))$ and $(-x(t,r_1), -y(t,r_1))$ is a G^2-continuity transition curve from Ω_0 to Ω_1 for a selection of $\theta \in (0, \pi/2]$ where $\gamma(1) \approx 3.075$ and $\gamma(m) \to \infty$ $(m \to \infty)$ ([12]).*

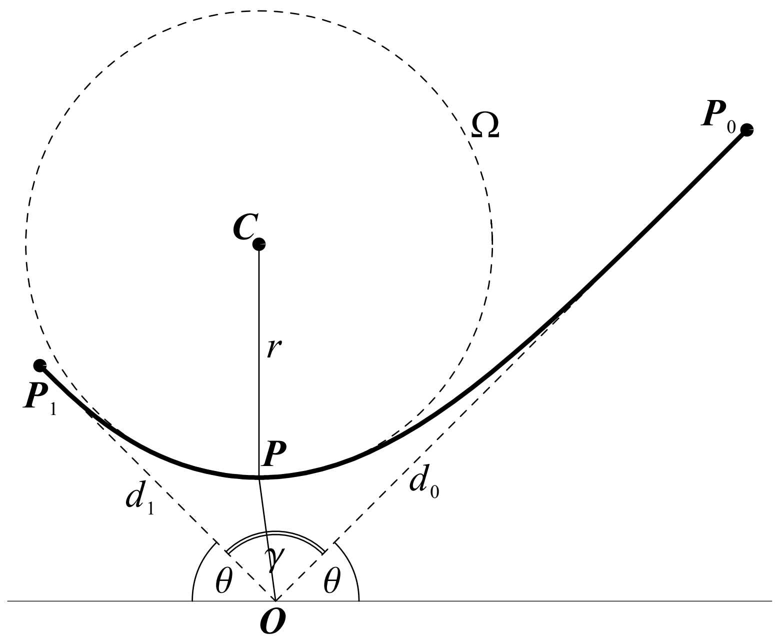

Figure 2: Transition between two straight lines.

2.2 Straight Line to Straight Line Spiral Transition Here, we have extended the idea of straight line to circle transition and derived a method for Bézier spiral transition between two nonparallel straight lines (see Fig. 2). Finally we note the following result that is of use for joining two non-parallel lines. Require for $0 < \theta \leq \pi/2$

$$\boldsymbol{z}(0) = (0,0), \quad \boldsymbol{z}'(0) \parallel (1,0), \quad \kappa(0) = 1/r, \quad \boldsymbol{z}'(1) \parallel (\cos\theta, \sin\theta), \quad \kappa(1) = 0 \tag{18}$$

then, transformation, i.e., rotation, translation, reflection with respect to y-axis and change of variable t with $1-t$ to $\boldsymbol{z}(t)$ by (2) and (9) gives $(u(t), v(t))$ of the form

$$\begin{aligned} u(t) &= \frac{\sqrt{qr}}{4\sin\frac{\theta}{2}}\left[qt\left\{2-(2-m)t\right\}\cos\frac{\theta}{2} + 4(1-t)^2\sin\frac{\theta}{2}\right] \\ v(t) &= \frac{qt\sqrt{qr}}{4}\left\{2-(2-m)t\right\} \end{aligned} \tag{19}$$

Now, $z(t,m,\theta)(=(x(t,m,\theta), y(t,m,\theta)), 0 \le t \le 1$ denotes the quintic spline satisfying (18). Assume that the angle between two lines is γ $(< \pi)$. Then, $(x(t,m,\theta_0), y(t,m,\theta_0))$ and $(-x(t,m,\theta_1), y(t,m,\theta_1))$ with $\theta_0+\theta_1 = \pi-\gamma$ is a pair of spiral transition curves if q satisfies the inequality (11) in Lemma 2. In addition, when $\theta_0 = \theta_1 = \theta$ and $q = q(m,\theta)(= \frac{6+m}{2}\tan\frac{\theta}{2}$ for $m \ge \frac{2}{7}(-3+\sqrt{30}))$, the distance between the intersection of the two lines and the end points of the transition curve are given by $d(m,n,r,\theta)(=d_0)$ and $d(n,m,r,\theta)(=d_1)$ where

$$d(m,n,r,\theta) = \frac{r\left\{2a(m,n)+3b(m,n)\cos\theta\right\}\tan\frac{\theta}{2}}{960(1+\cos\theta)\cos\theta} \tag{20}$$

where

$$\begin{aligned} a(m,n) &= 504 + 96(m+n) + 15(m^2+n^2) + m^3 + n^3 \\ b(m,n) &= 480 + 400m + 356m^2 + 128m^3 + 19m^4 + m^5 + 16n \end{aligned}$$

This result enables the pair of the spirals to pass through the given points of contact on the non-parallel two straight lines and note the following quantity in ([12])

$$d(1,1,r,\theta) = \frac{7r(26+75\cos\theta)\tan\theta}{120(1+\cos\theta)^2} \tag{21}$$

To keep the transition curve within a closed boundary, value of shape control parameter r can be derived from (20) when control points and boundary information are given. A constrained guided curve is shown in Figure 12.

3 Numerical Examples and Conclusion This section gives numerical examples for $q = q(m,\theta)$ to assure our theoretical analysis. Figures 3-4 give family of transition curves between two circles with $m = 1.5$ (dotted), 1 (bold) and 0.8 (normal). Figures 5-6 give family of transition curves with fixed distance between two circles for (m_0, m_1) =(3, 3) (dotted), (1, 1) (bold), (0.8, 0.8) (normal). Figure 7 shows the graphs of the family of transition curves between two straight lines for $(m_0, m_1) = (3, 0.8)$ (dotted) and (1, 1) (bold). Figure 8 shows curves for $r = 0.1$ (dotted) and 0.2 (bold). In Figure 9, the distances between the intersection of the two curves and the points of contact are 3 and 1 with $r = 0.1$ (dotted) and 0.2 (bold). By (20), for $r = 0.1$, $(m_0, m_1) \approx (4.06, 2.44)$ and for $r = 0.2$, $(m_0, m_1) \approx (3.09, 1.55)$. Figure 10 illustrates a vase cross-section incorporating one C transition and two S transitions. Ends of curves are shown with small circles. Figure 11 is a satellite antenna obtained by surface of revolution of PH quintic spiral. In Figure 12, a constrained G^2 continuous spiral spline uses straight line to straight line transition and straight line to circle transition. The boundary is composed of straight line segments and circular arcs. This figure also shows that all segments have completely local control. The data for boundary and control points has been taken from Figure 8 in [8].

We introduced a family of G^2 spiral transition curves between two circles and straight

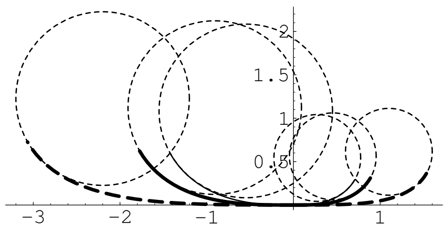

Figure 3: Graphs of $\boldsymbol{z}(t)$ with $(r_0, r_1, \theta) = (0.5, 1, \pi/3)$.

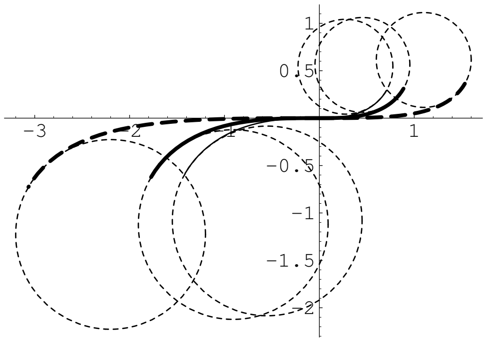

Figure 4: Graphs of $\boldsymbol{z}(t)$ with $(r_0, r_1, \theta) = (0.5, 1, \pi/3)$.

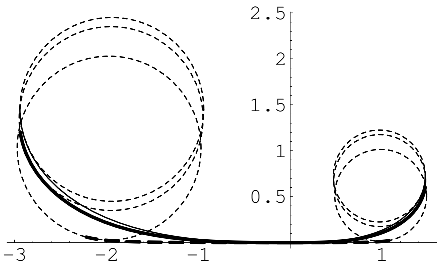

Figure 5: Graphs of $\boldsymbol{z}(t)$ with $(r_0, r_1) = (0.5, 1)$ and $\|\mathrm{C}_0 - \mathrm{C}_1\| = 3$.

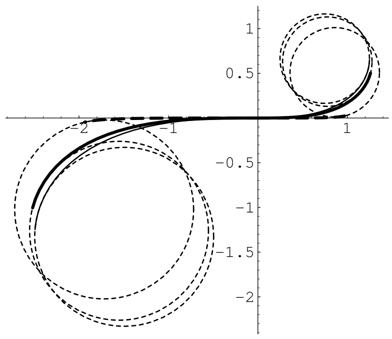

Figure 6: Graphs of $\boldsymbol{z}(t)$ with $(r_0, r_1) = (0.5, 1)$ and $\|\mathrm{C}_0 - \mathrm{C}_1\| = 3$.

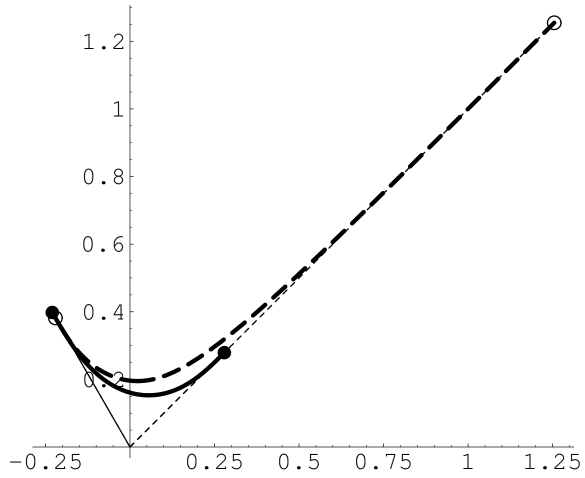

Figure 7: Graphs of $\boldsymbol{z}(t)$ with $(r, \gamma, \theta_0, \theta_1) = (0.2, 5\pi/12, \pi/4, \pi/3)$.

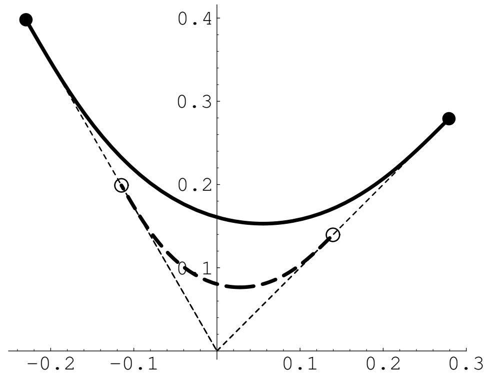

Figure 8: Graphs of $\boldsymbol{z}(t)$ with $(m, n, \gamma, \theta_0, \theta_1) = (1, 1, 5\pi/12, \pi/4, \pi/3)$.

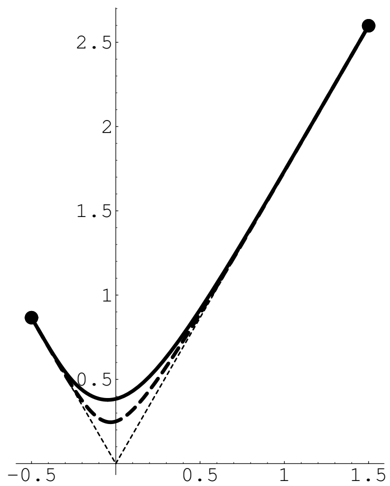

Figure 9: Graphs of $\boldsymbol{z}(t)$ with $\alpha = \theta_0 = \theta_1 = \pi/3$.

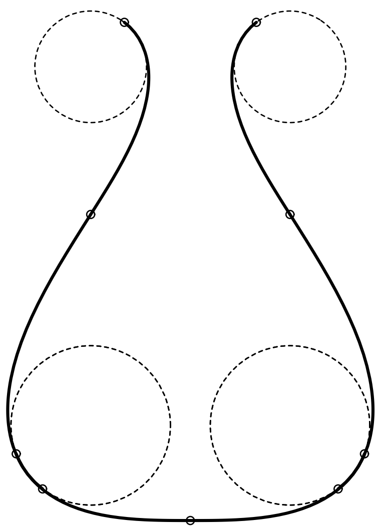

Figure 10: Vase cross-section.

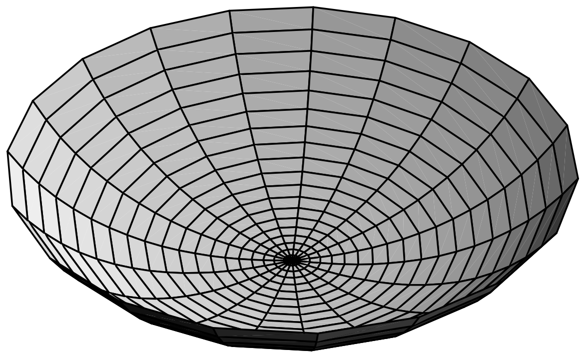

Figure 11: Satellite antenna.

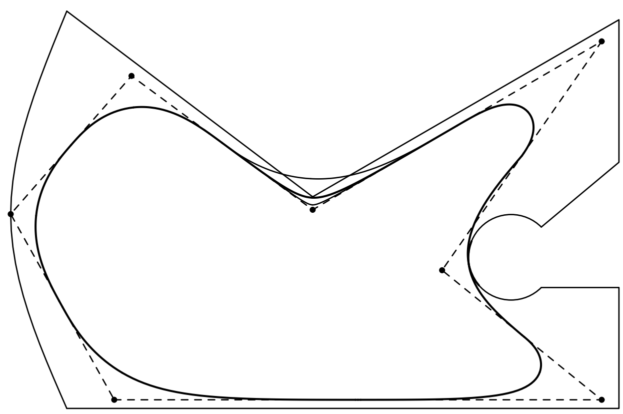

Figure 12: A guided curve constrained by a closed boundary.

lines. Further, we presented a very simple and flexible scheme offering more degree of freedom. To guarantee the absence of interior curvature extremum (i.e., spiral segment) in transition curve, user can select any value of parameter m greater than 0.707. On the other hand it must be 1 in Walton [12] scheme. Similarly, straight line to straight line transition has wide selection of shape control parameters and their values as demonstrated in Figures 3-12. We provided four degrees of freedom for the curve designer to use as shape control parameters by choosing their values whereas Walton offered only one degree of freedom in [12]. We also presented constrained guided G^2 continuous spiral spline within a closed boundary. Our guided curve scheme is more smoother and offer better local control than recent research work of Meek in [8] which has G^1 continuity. So our scheme is not only very simple but also reasonable and comfortable for lot of different kind of practical applications.

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