ON THE PROBLEM OF NEARLY DERIVATIVES

DONATELLA BONGIORNO

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Abstract. We provide a minimal constructive integration process of Riemann type which includes the Lebesgue integral and also integrates the derivatives of nearly differentiable functions.

1. Introduction

In [1] a minimal constructive integration process of Riemann-type which includes the Lebesgue integral and also integrates the derivatives of differentiable functions is given. It is called the C-integral and it is obtained from McShane's definition of the Lebesgue integral (see for example [6], [8], [9] and [10]) by imposing a mild regularity condition on McShane's partitions.

Given an interval $[a, b]$ of the real line $\mathbb{R}$ and a positive function δ on $[a, b]$ (in the sequel called gauge), we recall that a δ-fine McShane's partition of $[a, b]$ is (by definition) a collection $P = \{(A_h, x_h)\}_{h=1}^{p}$ of nonoverlapping intervals A_h and points $x_h \in [a, b]$ such that $A_h \subset (x_h - \delta(x_h), x_h + \delta(x_h))$, for each h, and $[a, b] = \bigcup_h A_h$.

Definition 1.1. A function $f : [a, b] \to \mathbb{R}$ is said to be C-integrable on $[a, b]$ if there exists a constant A such that for each $\varepsilon > 0$ there is a gauge δ on $[a, b]$ with

$$\left| \sum_{i=1}^{p} f(x_i)|A_i| - A \right| < \varepsilon \ ,$$

for each δ-fine McShane's partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ of $[a, b]$ satisfying the condition $\sum_{i=1}^{p} \mathrm{dist}(x_i, A_i) < 1/\varepsilon$.

Theorem 1.2. [1, Main Theorem] *A function $f : [a, b] \to \mathbb{R}$ is C-integrable on $[a, b]$ if and only if there exist a Lebesgue integrable function g and a derivative h on $[a, b]$ such that $f(x) = g(x) + h(x)$ for each $x \in [a, b]$.*

The fact that the C-integral is properly included into the Denjoy-Perron integral is shown by A.M. Bruckner, R.J. Fleissner and J. Foran in [4] by the following example:

The function

$$F(x) = \begin{cases} x \sin(1/x^2) & \text{for } x \in (0, 1], \\ 0 & \text{for } x = 0 \end{cases}$$

is ACG^* on $[0, 1]$ and, for each absolutely continuous function G, the function $F - G$ is not differentiable at 0. Therefore the function $f(x) = F'(x)$ for $x \in (0, 1]$ and $f(0) = 0$ is Denjoy-Perron integrable and not C-integrable on $[0, 1]$.

We say that a continuous function F is *nearly differentiable* on $[a, b]$ if there exists a countable set $N \subset [a, b]$ such that F is differentiable on $[a, b] \setminus N$. A function f equivalent to the derivative of a nearly differentiable function is called a *nearly derivative*. If f is

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a nearly derivative, then f is Denjoy-Perron integrable, but it may be not C-integrable (previous example). Now let us consider the following example:

Let $\mathcal{K}$ be the family of all complementary intervals of the Cantor ternary set, $\lambda > \log_3 2$ and $F = \sum_{K\in\mathcal{K}} F_K$, where

$$F_{(\alpha,\beta)}(x) = (x-\alpha)^{\lambda} \cos\left(\frac{\pi}{2}\cdot\frac{|\beta-\alpha|^2}{x-\alpha}\right).$$

In [2] it is proved that F is ACG^* and, for each absolutely continuous function G, the function $F-G$ is not nearly differentiable. This implies that the Denjoy-Perron integral is a solution "too general" for the

Problem of nearly derivatives: Recover, by integration process, the primitive of a nearly derivative.

In this paper we provide a new solution to the above problem, by a constructive integration process of Riemann type, based on a slight modification of the C-integral:

Definition 1.3. We say that a function $f\colon [a,b]\to\mathbb{R}$ is $\tilde{C}$-integrable on $[a,b]$ if there exist a constant A and a countable set N such that for each $\varepsilon > 0$ there is a gauge δ with

$$\left|\sum_{i=1}^{p} f(x_i)|A_i| - A\right| < \varepsilon\ , \tag{1.1}$$

for each δ-fine McShane's partition $\{(A_1,x_1),\ldots,(A_p,x_p)\}$ of $[a,b]$ satisfying the following conditions

$$\begin{cases} \sum_{i=1}^{p}\operatorname{dist}(x_i,A_i) < 1/\varepsilon; \\ \\ \text{if } x_i\in N, \text{ then } x_i\in A_i,\ i=1,\cdots,p. \end{cases} \tag{1.2}$$

The number A is called the $\tilde{C}$-integral of f on $[a,b]$.

In Section 3 we prove that

Theorem 1.4. *A function $f\colon [a,b]\to\mathbb{R}$ is $\tilde{C}$-integrable on $[a,b]$ if and only if there exist a Lebesgue integrable function g and a nearly derivative h on $[a,b]$ such that $f(x) = g(x)+h(x)$ for each $x\in[a,b]$.*

Therefore the $\tilde{C}$-integral provides the minimal extension of the Lebesgue integral which also integrates each nearly derivative.

In Sections 4 and 5 we give two characterizations of the $\tilde{C}$-primitives.

In Section 6 we prove that each BV function is a multiplier for the $\tilde{C}$-integral .

2. Preliminaries

The set of all natural numbers, integer numbers, and real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$, respectively. If $E\subset\mathbb{R}$ then $|E|$ denotes the Lebesgue measure of E. Let $A = (\alpha,\beta)\subset[a,b]$ and let F be a real valued function on $[a,b]$. We set $F(A) = F(\beta)-F(\alpha)$. By $L^1[a,b]$ we denote the family of all Lebesgue integrable functions.

In this paper, to distinguish the different integrals, we denote by $\int_a^b f$ the Lebesgue integral, by $(\tilde{C})\int_a^b f$ the $\tilde{C}$-integral, and by $(DP)\int_a^b f$ the Denjoy-Perron integral of f on the interval $[a,b]$.

Given a subset E of $[a,b]$ and a gauge δ, we call δ-fine McShane partial partition anchored on E any collection $\{(A_h,x_h)\}_{h=1}^{p}$ of nonoverlapping intervals $A_h\subset[a,b]$ and points $x_h\in E$ such that $A_h\subset(x_h-\delta(x_h),x_h+\delta(x_h))$, for each h.

Remark 2.1. If $f \in L^1[a,b]$ then f is $\tilde{C}$-integrable on $[a,b]$ (with the same value of the integral).

This follows by the fact that the Lebesgue integral is equivalent to the McShane integral (see [6], [8], [9], and [10]).

Remark 2.2. If f is $\tilde{C}$-integrable on $[a,b]$, then f is Denjoy-Perron integrable on $[a,b]$ (with the same value of the integral).

This follows by the fact that the Denjoy-Perron integral is equivalent to the Henstock-Kurzweil integral (see [6], and [10]) and to the fact that the partitions involved in the definition of the Henstock-Kurzweil integral are McShane's partition $\{(A_1,x_1),\dots,(A_p,x_p)\}$ of $[a,b]$ satisfying the conditions $x_i \in A_i$, for each i.

Remark 2.3. The indefinite integral $F(x) = (\tilde{C})\int_a^x f$ is continuous.

Remark 2.4. If f is $\tilde{C}$-integrable on $[a,b]$, then f is $\tilde{C}$-integrable on each subinterval of $[a,b]$.

Henstock's type lemma. If f is $\tilde{C}$-integrable on $[a,b]$, then there is a countable set N such that for each $\varepsilon > 0$ there exists a gauge δ so that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - (\tilde{C})\int_{A_i} f \right| < \varepsilon \ ,$$

for each δ-fine McShane partition $\{(A_1,x_1),\dots,(A_p,x_p)\}$ of $[a,b]$ satisfying condition (1.2).

Lemma 2.5. *If f is a nearly derivative on $[a,b]$, then f is $\tilde{C}$-integrable on $[a,b]$.*

Proof. Let F be a nearly differentiable function with nearly derivative f. Then there is a sequence $\{a_n\} \subset [a,b]$ such that $F'(x) = f(x)$ for each $x \in [a,b] \setminus \{a_n\}$. Given $0 < \varepsilon < 1/(b-a)$, define $\delta\colon [a,b] \to \mathbb{R}^+$ such that:

$$|f(a_n)| \cdot \delta(a_n) < \frac{\varepsilon}{2^{n+2}}, \tag{2.1}$$

for $n = 1, 2, \cdots$;

$$|F(t) - F(a_n)| < \frac{\varepsilon}{2^{n+2}},$$

for $|t - a_n| < \delta(a_n)$, and $n = 1, 2, \cdots$;

$$\left| \frac{F(y)-F(x)}{y-x} - f(x) \right| < \frac{\varepsilon^2}{8}, \tag{2.2}$$

for $x \in [a,b] \setminus \{a_n\}$, and $y \in [a,b]$ with $|y - x| < \delta(x)$.

Note that if $x \in [a,b] \setminus \{a_n\}$ and if $A = (\alpha, \beta)$ is a subinterval of $[a,b]$ such that $\operatorname{dist}(x, A) < \delta(x)$, then condition (2.2) implies

$$\begin{aligned} &| \, F(A) - f(x)|A|| \\ &\quad \le |F(\beta) - F(x) - f(x)(\beta - x)| \\ &\quad\quad + |F(\alpha) - F(x) - f(x)(\alpha - x)| \\ &\quad < \frac{\varepsilon^2}{8}|\beta - x| + \frac{\varepsilon^2}{8}|\alpha - x| \le \frac{\varepsilon^2}{4}\ (\operatorname{dist}(x, A) + |A|). \end{aligned} \tag{2.3}$$

Thus, given a δ-fine McShane's partition $\{(A_1, x_1), \cdots, (A_p, x_p)\}$ with $\sum_{i=1}^p \operatorname{dist}(x_i, A_i) < 1/\varepsilon$, and $x_i \in A_i$ if $x_i \in \{a_n\}$, by (2.1) and (2.3) we have

$$\begin{aligned}
&\left|\sum_{i=1}^p f(x_i)|A_i| - (F(b) - F(a))\right| \\
&\quad \le \sum_{x_i \in \{a_n\}} |f(x_i)|A_i| - F(A_i)| + \sum_{x_i \notin \{a_n\}} |f(x_i)|A_i| - F(A_i)| \\
&\quad < 2\sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+2}} + \frac{\varepsilon^2}{4} \sum_{i=1}^{p} (\operatorname{dist}(x_i, A_i) + |A_i|) \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon^2}{4} \cdot \left(\frac{1}{\varepsilon} + (b-a)\right) < \varepsilon.
\end{aligned}$$

This complete the proof of $\tilde{C}$-integrability of f. □

An interval $[\alpha, \beta] \subset [a, b]$ is said to be *f-regular* if there exist a Lebesgue integrable function g and a nearly derivative h on $[a, b]$ such that $f(x) = g(x) + h(x)$ for each $x \in [\alpha, \beta]$.

Remark 2.6. If an interval is union of two f-regular intervals, then it is f-regular.

Lemma 2.7. *Let f be $\tilde{C}$-integrable on a given interval $[\alpha, \beta]$, and let E be a closed subset of $[\alpha, \beta]$ such that $\alpha, \beta \in E$ and each closed interval disjoint with E is f-regular. If $f \in L^1(E)$ and if*

$$\sum_{p \in \mathbb{N}} \sup_{\substack{J \subset I_p \\ J \text{ interval}}} \left|(\tilde{C}) \int_J f\right| < +\infty ,$$

where $\{I_p\}$ is the sequence of all connected components of $[\alpha, \beta] \setminus E$, then $[\alpha, \beta]$ is f-regular.

Proof. With no loss of generality, we can assume that $E \neq [\alpha, \beta]$ and that $\{I_p\}$ is infinite (indeed, if it is finite we can reduce the proof to the infinite case by adding a suitable sequence of points to E).
By Remark 2.1 the function $f\chi_E$ is $\tilde{C}$-integrable, then $g = f - f\chi_E$ is null on E and $\tilde{C}$-integrable on $[\alpha, \beta]$. So, if we prove that $[\alpha, \beta]$ is g-regular, then $[\alpha, \beta]$ is f-regular. Thus we can assume $f = 0$ on E.
Fix a sequence $\{\varepsilon_n\}$ of positive real numbers such that

$$\sum_{n \in \mathbb{N}} \varepsilon_n < +\infty, \tag{2.4}$$

and

$$\beta - \alpha + 2 < \frac{1}{\varepsilon_n}. \tag{2.5}$$

Moreover, according to the Henstock's type lemma, there is a countable set $N \subset [\alpha, \beta]$ and, for each $n \in \mathbb{N}$, a gauge $\delta_n(x)$ such that

$$\sum_{i=1}^{p} \left|(\tilde{C}) \int_{A_i} f\right| < \varepsilon_n , \tag{2.6}$$

for each δ_n-fine McShane's partial partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored on E and satisfying condition (1.2).

Now, following [1, Section 3], we fix an increasing sequence of compact sets H_k such that $\bigcup_{k=1}^{\infty} H_k = (\alpha, \beta) \setminus E$, $H_k \cap I_p$ is a proper interval for finitely many p and $H_k \cap I_p = \emptyset$ for the remaining indices p, and such that

$$\sum_{p\in\mathbb{N}} \sup_{\substack{J\subset I_p\setminus H_k \\ J \text{ interval}}} \left|(\tilde{C})\int_J f\right| < +\infty\ . \tag{2.7}$$

Then we can repeat the proof of [1, Claim 1] to prove the existence of a family of real numbers $\{\gamma_{p,q}; p\in\mathbb{N}, q\in\mathbb{Z}\}$ such that

(c_1) $\gamma_{p,0} = \frac{\alpha_p+\beta_p}{2}$;

(c_2) $\gamma_{p,q} \le \gamma_{p,q+1}$;

(c_3) $\alpha_p < \gamma_{p,q} < \beta_p$;

(c_4) $\inf_{q\in\mathbb{Z}}\{\gamma_{p,q}\} = \alpha_p$ and $\sup_{q\in\mathbb{Z}}\{\gamma_{p,q}\} = \beta_p$;

and such that the intervals $I_{p,q} = [\gamma_{p,q}, \gamma_{p,q+1}]$ satisfy the following conditions:

(c_5) for each $x\in E$ and each $\varepsilon > 0$ there exists $\delta > 0$ with

$$\left|(\tilde{C})\int_J f\right| \le\ \varepsilon\ \mathrm{dist}(x, I_{p,q})\ ,$$

for any interval J contained in $I_{p,q}$, provided that $I_{p,q}\subset (x-\delta, x+\delta)$;

(c_6) $\sum_{p\in\mathbb{N},q\in\mathbb{Z}} \left|(\tilde{C})\int_{I_{p,q}} f\right| < +\infty$.

To prove condition (c_6) we follows [1] and take $x_{p,q}\in E_{k_{p,q},l_{p,q}}\setminus N$ (to be able to apply (2.6)), instead of $x_{p,q}\in E_{k_{p,q}}$.

Further, for each $p\in\mathbb{N}$ and $q\in\mathbb{Z}$ we put

$$c_{p,q} = \frac{1}{|I_{p,q}|}(\tilde{C})\int_{I_{p,q}} f,$$

and take

$$0 < \eta_{p,q} < \mathrm{dist}^2(I_{p,q}, E) \ \text{ such that } \sum_{p\in\mathbb{N},q\in\mathbb{Z}} \eta_{p,q} < +\infty. \tag{2.8}$$

Therefore, by an easy extension of Lemma 1 of [1], for each $p\in\mathbb{N}$ and $q\in\mathbb{Z}$ there exists a nearly derivative $h_{p,q}$ such that $h_{p,q} = 0$ outside $I_{p,q}$, and such that

$$\int_{I_{p,q}} |f - c_{p,q} - h_{p,q}| < \eta_{p,q},$$

and

$$(\tilde{C})\int_{I_{p,q}} h_{p,q} = 0.$$

Now we define

$$h = \sum_{p\in\mathbb{N},q\in\mathbb{Z}} h_{p,q},$$

and we prove the f-regularity of $[\alpha,\beta]$ by showing that $f-h$ is Lebesgue integrable in $[\alpha,\beta]$ and h is a nearly derivative in $[\alpha,\beta]$. Since $f=0$ on E, we have

$$\begin{aligned}\int_\alpha^\beta &|f-h| \\ &\leq \sum_{p\in\mathbb{N},q\in\mathbb{Z}} \int_{I_{p,q}} (|c_{p,q}| + |f-c_{p,q}-h_{p,q}|) \\ &\leq \sum_{p\in\mathbf{N},q\in\mathbf{Z}} \left|(\tilde{C})\int_{I_{p,q}} f\right| + \sum_{p\in\mathbb{N},q\in\mathbb{Z}} \eta_{p,q}.\end{aligned}$$

Then, by condition (c_6) and by (2.8) we conclude that $f-h\in L^1[\alpha,\beta]$. Hence, by Remark 2.1 and Remark 2.2, $h=(h-f)+f$ is Denjoy-Perron integrable. Consequently its primitive

$$H(x)=(DP)\int_\alpha^x h, \quad x\in[\alpha,\beta],$$

is differentiable with derivative $H'(x)=h(x)$ for each $x\notin(E\cup N)$. Thus, to complete the proof, we can follow [1, Page 125-126] to get $H'(x)=0=h(x)$, for each $x\in E$.

□

Corollary 2.8. *If f is $\tilde{C}$-integrable on $[a,b]$, and if each compact sub-interval of (a,b) is f-regular, then $[a,b]$ is f-regular.*

Proof. By Remark 2.6 we have

$$\sup_{\substack{J\subset(a,b)\\ J \text{ interval}}} \left|(\tilde{C})\int_J f\right| = \left|(\tilde{C})\int_a^b f\right| < +\infty .$$

Then we can apply Lemma 2.7 to f and $E=\{a,b\}$. □

3. Proof of Theorem 1.4

Assume that $f=g+h$ with $g\in L^1[a,b]$ and with h nearly derivative. By Remark 2.1, h is $\tilde{C}$-integrable on $[a,b]$, and by Lemma 2.5 also g is $\tilde{C}$-integrable on $[a,b]$. Therefore f is $\tilde{C}$-integrable on $[a,b]$.

Now let f be $\tilde{C}$-integrable on $[a,b]$. By Remark 2.2, f is Denjoy-Perron integrable on $[a,b]$, then there exists an interval $[\alpha,\beta]\subset[a,b]$ such that $f\in L^1[\alpha,\beta]$ (see [12, Chap. VIII, Theorem 1.4]). Consequently, denoted by Ω the union of the interiors of all f-regular intervals contained in $[a,b]$, we have $\Omega\neq\emptyset$. By Corollary 2.8, if $\Omega=(\alpha,\beta)$, then $[\alpha,\beta]$ is f-regular, and the proof is complete.

Assume, by contradiction, that $(a,b)\setminus\Omega\neq\emptyset$. We firstly prove that $(a,b)\setminus\Omega$ does not contain isolated points. In fact, let $\alpha<\gamma<\beta$, with $\alpha,\gamma,\beta\in[a,b]\setminus\Omega$ such that $(\alpha,\gamma)\subset\Omega$ and $(\gamma,\beta)\subset\Omega$. Then, by Corollary 2.8, $[\alpha,\gamma]$ and $[\gamma,\beta]$ are f-regular. So, by Remark 2.6, $[\alpha,\beta]$ is f-regular, which is in contradiction with the assumption $\gamma\notin\Omega$.

Let $E=[a,b]\setminus\Omega$. Since f is Denjoy-Perron integrable on $[a,b]$, then by [12, Chap. VIII, Theorem 1.4] there exists an interval $[\alpha,\beta]\subset[a,b]$ such that $E\cap(\alpha,\beta)\neq\emptyset$, $\int_{E\cap[\alpha,\beta]}|f|<+\infty$ and

$$\sum_{p\in\mathbb{N}} \sup_{\substack{J\subset I_p\\ J \text{ interval}}} \left|(DP)\int_J f\right| < +\infty ,$$

where I_p is the sequence of all connected components of $(\alpha,\beta)\setminus E$. Since $E\cap(\alpha,\beta)$ has no isolated points, we can assume $\alpha,\beta\in E$. Moreover, by Remarks 2.2 and 2.6, $(DP)\int_J f = (\tilde{C})\int_J f$, for each $J\subset[a,b]$. Thus all assumptions of Lemma 2.7 hold with E replaced by

$E \cap [\alpha, \beta]$. So $[\alpha, \beta]$ is f-regular, which, by definition of E means that (α, β) cannot meet E. This is the final contradiction proving that $[a, b]$ is f-regular.

4. $ACG_{\tilde{C}}$–Functions

We start by a characterization of the $\tilde{C}$-primitives in terms of a slight modification of the classical ACG_* notion.

Definition 4.1. A function $F\colon [a, b] \to \mathbb{R}$ is said to be $AC_{\tilde{C}}$ on $E \subset [a, b]$ if for each $\varepsilon > 0$ there exist a constant $\eta > 0$, a countable set $N \subset E$ and a gauge δ such that $\sum_i |F(A_i)| < \varepsilon$ for each δ-fine McShane's partial partition $\{(A_i, x_i)\}_{i=1}^p$ anchored on E, satisfying condition (1.2), and such that $\sum_{i=1}^p |A_i| < \eta$.

Definition 4.2. A continuous function $F\colon [a, b] \to \mathbb{R}$ is said to be $ACG_{\tilde{C}}$ on $[a, b]$ if there exists a sequence $\{E_n\}$ of measurable sets such that $[a, b] = \bigcup_n E_n$ and F is $AC_{\tilde{C}}$ on each E_n.

Lemma 4.3. *If F is $ACG_{\tilde{C}}$ on $[a, b]$ and $E \subset [a, b]$ with $|E| = 0$, then given $\varepsilon > 0$ there exist a countable set $N \subset E$ and a gauge δ such that $\sum_{i=1}^p |F(A_i)| < \varepsilon$, for each δ-fine McShane's partial partition $\{(A_i, x_i)\}_{i=1}^p$ anchored on E and satisfying condition (1.2).*

Proof. Since F is $ACG_{\tilde{C}}$, there exists a sequence $\{E_n\}$ of pairwise disjoint measurable sets such that $[a, b] = \bigcup_n E_n$ and F is $AC_{\tilde{C}}$ on each measurable set E_n. Given $\varepsilon > 0$, for a fixed $n \in \mathbb{N}$ there exists a positive number η_n, a countable set $N_n \subset E_n$ and a gauge δ_n such that

$$\sum_{i=1}^{q} |F(J_i)| < \frac{\varepsilon}{2^n}, \tag{4.1}$$

for each δ_n-fine McShane's partial partition $\{(J_i, y_i)\}_{i=1}^q$ anchored on E_n, satisfying condition (1.2), and such that $\sum_i |J_i| < \eta$. Since $|E \cap E_n| = 0$ we can find an open set $O_n \supset E \cap E_n$ such that $|O_n| < \eta_n$. Define $\delta_n^*(x) = \min(\delta_n(x), \mathrm{dist}(x, {}^cO_n))$, where by ${}^cO_n$ we denote the complement of O_n. Therefore (4.1) holds for each δ_n^*-fine McShane's partial partition $\{(J_i, y_i)\}_{i=1}^q$ anchored on $E \cap E_n$ and satisfying condition (1.2).

Set $N = \bigcup_n (E \cap N_n)$ and define $\delta(x) = \delta_n^*(x)$, for $x \in E \cap E_n$, $n \in \mathbb{N}$. Then, given a δ-fine McShane's partial partition $\{(A_i, x_i)\}_{i=1}^p$ anchored on E and satisfying condition (1.2), we have

$$\sum_{i=1}^{p} |F(A_i)| = \sum_{n=1}^{\infty} \sum_{x_i \in E_n} |F(A_i)| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

$\square$

Lemma 4.4. *If F is a function differentiable at x, then given $\varepsilon > 0$ there exists $\gamma(x) > 0$ such that*

$$|F(I) - F'(x)|I|| < \varepsilon\ (dist(x, I) + |I|),$$

for each interval $I \subset (x - \gamma(x), x + \gamma(x))$.

Proof. By the definition of derivative, there exists $\gamma(x) > 0$ such that

$$|F(y) - F(x) - F'(x)(y - x)| < \frac{\varepsilon}{2}|y - x|,$$

for each $y \in (x - \gamma(x), x + \gamma(x))$.
Therefore, given $I = (\alpha, \beta) \subset (x - \gamma(x), x + \gamma(x))$ we have

$$\begin{aligned}
&|F(\beta) - F(\alpha) - F'(x)(\beta - \alpha)| \\
&\quad \le |F(\beta) - F(x) - F'(x)(\beta - x)| + |F(\alpha) - F(x) - F'(x)(\alpha - x)| \\
&\quad < \frac{\varepsilon}{2}|\beta - x| + \frac{\varepsilon}{2}|\alpha - x| \\
&\quad < \frac{\varepsilon}{2}\ \mathrm{dist}(x, I) + \frac{\varepsilon}{2}\ (\mathrm{dist}(x, I) + |I|) \\
&\quad < \varepsilon\ (\mathrm{dist}(x, I) + |I|).
\end{aligned}$$

□

Theorem 4.5. *F is $ACG_{\tilde{C}}$ on $[a, b]$ if and only if there exists a $\tilde{C}$-integrable function $f\colon [a, b] \to \mathbb{R}$ such that*

$$F(x) - F(a) = (\tilde{C})\int_a^x f(t)\, dt, \quad \textit{for each} \quad x \in [a, b]. \tag{4.2}$$

Proof. Note that if F is $ACG_{\tilde{C}}$ on $[a, b]$ then it is ACG_δ, according to Gordon's definition (see [5]). Therefore it is differentiable a.e. in $[a, b]$ (see [5, Theorem 6] and [12, Chapter VII, Theorem 7.2]). Let $E = \{x \in [a, b] : F \text{ is not differentiable at } x\}$. Then $|E| = 0$ and, by Lemma 4.3, given $0 < \varepsilon < 2/(b - a)$ there exist a countable set $N \subset E$ and a gauge τ such that

$$\sum_{i=1}^{p} |F(A_i)| < \frac{\varepsilon}{4},$$

for each τ-fine McShane's partial partition $\{(A_i, x_i)\}_{i=1}^p$ anchored on E and satisfying condition (1.2).

If $x \notin E$, by Lemma 4.4 there exists $\gamma(x) > 0$ such that

$$|\,F(I) - F'(x)|I|\,| < \frac{\varepsilon^2}{4}(\mathrm{dist}(x, I) + |I|),$$

for each interval $I \subset [a, b] \cap (x - \gamma(x), x + \gamma(x))$.

Let

$$\delta(x) = \begin{cases} \tau(x) & \text{if} \quad x \in E \\ \gamma(x) & \text{if} \quad x \notin E, \end{cases}$$

and let

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in E \\ F'(x) & \text{if} \quad x \notin E. \end{cases}$$

So, for each $x \in (a, b]$ and for each δ-fine McShane's partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ of $[a, x]$ satisfying conditions (1.2), we have

$$\begin{aligned}
\left| \sum_{i=1}^{p} f(x_i)|A_i| - (F(x) - F(a)) \right| &\le \sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| \\
&\le \sum_{x_i \in E} |F(A_i)| + \sum_{x_i \notin E} |F'(x_i)|A_i| - F(A_i)| \\
&< \frac{\varepsilon}{4} + \sum_{x_i \notin E} \frac{\varepsilon^2}{4}(\mathrm{dist}(x_i, A_i) + |A_i|) \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon^2}{4}(b - a) < \varepsilon.
\end{aligned}$$

This completes the proof of the "if" part of the theorem.

Now assume that F is the $\tilde{C}$-primitive of a $\tilde{C}$-integrable function f. For each natural n we set $E_n = \{x \in [a,b] : |f(x)| \le n\}$. Then $[a,b] = \bigcup_n E_n$. To complete the proof of the "only if" part, it is enough to prove that F is $AC_{\tilde{C}}$ on each E_n.

Let $n \in \mathbb{N}$ and $\varepsilon > 0$. By Henstock's type lemma there exist a countable set $N \subset [a,b]$ and a gauge δ such that

$$\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \frac{\varepsilon}{2},$$

for each δ-fine McShane's partial partition $P = \{(A_h, x_h)\}_{h=1}^{p}$ of $[a,b]$ satisfying condition (1.2).
So, if P is anchored on E_n, and $\sum_i |A_i| < \varepsilon/2n$, then

$$\begin{aligned} &\sum_{i=1}^{p} |F(A_i)| \\ &\quad \le \sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| + \sum_{i=1}^{p} |f(x_i)| \cdot |A_i| \\ &\quad < \frac{\varepsilon}{2} + n \sum_i |A_i| < \varepsilon. \end{aligned}$$

Therefore F is $AC_{\tilde{C}}$ on E_n, and the proof is complete. □

5. The Variational Measure $V_{\tilde{C}}$

Let F be an additive interval function defined on the family of all sub-intervals of $[a,b]$. Given a gauge δ, a set $E \subset [a,b]$, a subset N of E, and $\varepsilon > 0$, we set

$$\mathrm{V}(F, \delta, E, N, \varepsilon) = \sup \sum_i |F(A_i)|,$$

where the "sup" is taken over all δ-fine McShane's partial partitions anchored on E and satisfying condition (1.2).

The variational measure $V_{\tilde{C}}$ is defined as follows:

$$\mathrm{V}_{\tilde{C}}F(E) = \lim_{\varepsilon \to 0} \inf\nolimits_{\delta} \inf_{N} \{\mathrm{V}(F, \delta, E, N, \varepsilon) : N \text{ countable}\}.$$

$\mathrm{V}_{\tilde{C}}F$ is a regular Borel measure in $[a,b]$ (we can easily follows the proofs of Theorems 3.7 and 3.15 of [11]).

Theorem 5.1. *An additive function F is a $\tilde{C}$-primitive if and only if the variational measure $\mathrm{V}_{\tilde{C}}F$ is absolutely continuous with respect to the Lebesgue measure.*

Proof. Let $F(x) = F(a) + (\tilde{C}) \int_a^x f(t)\,dt$ be the indefinite $\tilde{C}$-integral of a function f, and let $E \subset [a,b]$ be a Lebesgue null set. Without loss of generality we can assume $f(x) = 0$ for each $x \in E$. Then, by the Henstock's type lemma, there exists a countable set $N \subset [a,b]$ such that for each $\varepsilon > 0$ there is a gauge δ with

$$\sum_{i=1}^{p} |F(I_i)| < \varepsilon \ ,$$

for each δ-fine McShane partial partition $\{(A_1, x_1), \ldots , (A_p, x_p)\}$ anchored on E and satisfying condition (1.2).
Consequently $\mathrm{V}(F, \delta, E, N, \varepsilon) \le \varepsilon$, hence $\mathrm{V}_{\tilde{C}}F(E) = 0$. This completes the proof of the "if" part of the Theorem.

Now assume that $\mathrm{V}_{\tilde{C}}F$ is absolutely continuous with respect to the Lebesgue measure, and remark that

$$\mathrm{V}(F,\delta,E,E,+\infty) \leq \mathrm{V}(F,\delta,E,N,\varepsilon),$$

for each δ, E, N and ε. Then

$$\inf_{\delta} \mathrm{V}(F,\delta,E,E,+\infty) \leq \mathrm{V}_{\tilde{C}}F(E),$$

for each set $E \subset [a,b]$. This implies that the Henstock-Kurzweil variational measure $\mathrm{V}_{HK}F$, defined by $\mathrm{V}_{HK}F(E) = \inf_{\delta} \mathrm{V}(F,\delta,E,E,+\infty)$, is absolutely continuous with respect to the Lebesgue measure. Thus, by [3, Theorem 2], the function F is differentiable outside a Lebesgue null set E. Therefore, since $\mathrm{V}_{\tilde{C}}F(E) = 0$, for each $\varepsilon > 0$ there exist a positive η, a gauge δ_1, and a countable set $N \subset E$ such that

$$\sum_{i=1}^{q} |F(B_i)| < \frac{\varepsilon}{2},$$

for each δ_1-fine McShane partial partition $\{(B_1,x_1),\ldots,(B_p,x_q)\}$ anchored on E and satisfying condition (1.2), with B_i instead of A_i.

Now, by Lemma 4.4, to each $t \in [a,b] \setminus N$ there exists $\delta_2(t) > 0$ such that

$$|f(t)|B| - F(B)| < \frac{\varepsilon^2}{4}\,(\mathrm{dist}(t,B) + |B|),$$

for each interval $B \subset (t-\delta_1(t), t+\delta_1(t))$.

Then, to complete the proof of the "only if" part, it is enough to prove that F is the indefinite $\tilde{C}$-integral of the following function:

$$f(t) = \begin{cases} F'(t) & \text{if } t \in [a,b] \setminus N, \\ 0 & \text{if } t \in N. \end{cases}$$

Let

$$\delta(t) = \begin{cases} \delta_1(t) & \text{if } t \in N, \\ \delta_2(t) & \text{if } t \in [a,b] \setminus N, \end{cases}$$

and, for $x \in (a,b]$, let $\{(A_1,x_1),\ldots,(A_p,x_p)\}$ be a δ-fine McShane partition of $[a,x]$, satisfying condition (1.2).
Then

$$\begin{aligned} \left|\sum_{i=1}^{p} f(x_i)|I_i| - (F(x) - F(a))\right| &\leq \sum_{i=1}^{p} |f(x_i)|I_i| - F(I_i)| \\ &< \sum_{x_i \in N} |F(I_i)| + \frac{\varepsilon^2}{4} \sum_{x_i \notin N} (\mathrm{dist}(x_i,I_i) + |I_i|) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon^2}{4}(b-a) < \varepsilon. \end{aligned}$$

By the arbitrariness of ε, the function f is $\tilde{C}$-integrable on $[a,x]$, and

$$(\tilde{C}) \int_a^x f(t)\,dt = F(x) - F(a).$$

So, by the arbitrariness of $x \in [a,b]$, the function F is the indefinite $\tilde{C}$-integral of f on $[a,b]$. $\square$

6. The multipliers

In this section we prove that each BV function is a multiplier for the $\tilde{C}$-integral; in the sense that if f is a $\tilde{C}$-integrable function and g is a BV function, then fg is $\tilde{C}$-integrable. Recall that a function $g\colon [a,b] \to \mathbb{R}$ is said to be a BV function whenever there exists a function of bounded variation $\tilde{g}\colon [a,b] \to \mathbb{R}$ such that $g = \tilde{g}$ a.e. in $[a,b]$.

Theorem 6.1. *Each BV function is a multiplier for the $\tilde{C}$-integral.*

Proof. Let f be a $\tilde{C}$-integrable function and let F be its primitive. If g is a BV function, then by [12, Chap. 8, Theorem 2.5], fg is Denjoy-Perron integrable, and for each $x \in [a,b]$,

$$(DP)\int_a^x fg\,dt = [Fg]_a^x - \int_a^x F\,dg, \tag{6.1}$$

for each $x \in [a,b]$. Let $\Phi(x) = (DP)\int_a^x fg\,dt$, and let

$$E = \{x \in [a,b] : \Phi^{'}(x) = f(x)g(x)\}.$$

Then $N = [a,b] \setminus E$ is a Lebesgue null set. Without loss of generality we can assume that $f(x) = 0$ for each $x \in N$, and $g(x)$ is increasing and positive on $[a,b]$.

Now fix an $\varepsilon > 0$. By Lemma 4.4, for each $x \in E$ there exists $\delta_1(x) > 0$ such that

$$|f(x)g(x)|I| - \Phi(I)| < \frac{\varepsilon^2(\mathrm{dist}(x,I) + |I|)}{6}, \tag{6.2}$$

for each interval $I \in (x - \delta_1(x), x + \delta_1(x))$.
As the variational measure $V_{\tilde{C}}F$ is absolutely continuous with respect to the Lebesgue measure, by Theorem 5.1, there exists a gauge δ_2 such that

$$\sum_{i=1}^p |F(I_i)| < \frac{\varepsilon}{3(\|g\|_\infty + 1)}, \tag{6.3}$$

for each δ_2-fine McShane partial partition $\{(A_1,x_1),\cdots,(A_p,x_p)\}$ anchored on N and satisfying condition (1.2).
Choose $\sigma > 0$ so that

$$|F(x) - F(y)| < \frac{\varepsilon}{6(\|\,g\,\|_\infty + 1)}, \tag{6.4}$$

for each $x, y \in I$ with $|x - y| < \sigma$, and define the function δ by the formula

$$\delta(x) = \begin{cases} \delta_1(x) & \text{if} \quad x \in E \\ \min(\delta_2(x),\sigma) & \text{if} \quad x \in N. \end{cases} \tag{6.5}$$

Let $\{(A_1,x_1),\cdots,(A_p,x_p)\}$ be a δ-fine McShane partition of $[a,b]$ satisfying condition (1.2). Then

$$\begin{aligned} &\left|\sum_i f(x_i)g(x_i)|I_i| - \Phi([a,b])\right| \\ &\quad \le \sum_i |f(x_i)g(x_i)|I_i| - \Phi(I_i)| \le \sum_{x_i\in E} + \sum_{x_i \in N}. \end{aligned} \tag{6.6}$$

An estimate of $\sum_{x_i\in E}$ follows from (6.2):

$$\begin{aligned} &\sum_{x_i\in E} |f(x_i)g(x_i)|I_i| - \Phi(I_i)| \\ &\quad < \frac{\varepsilon^2}{6}\sum_{x_i\in E}(\mathrm{dist}(x_i,I_i) + |I_i|) < \frac{\varepsilon^2}{6}\cdot\frac{1}{\varepsilon} + \frac{\varepsilon^2(b-a)}{6} < \frac{\varepsilon}{3}. \end{aligned} \tag{6.7}$$

Next we estimate $\sum_{x_i \in N}$. Note that $f(x_i) = 0$ for $x_i \in N$. Then, using (6.1) and letting $I_i = [\alpha_i, \beta_i]$ we obtain

$$\begin{aligned}
\textstyle\sum_{x_i \in N} &= \textstyle\sum_{x_i \in N} |\Phi(I_i)| \\
&= \textstyle\sum_{x_i \in N} \left| \left(F(\beta_i)g(\beta_i) - F(\alpha_i)g(\alpha_i) - \int_{\alpha_i}^{\beta_i} F\,dg \right) \right| \\
&= \textstyle\sum_{x_i \in N} |(F(\beta_i) - F(\alpha_i))\, g(\beta_i) + \\
&\quad + F(\alpha_i)\,(g(\beta_i) - g(\alpha_i)) - F(\xi_i)\,(g(\beta_i) - g(\alpha_i))| \\
&\leq \textstyle\sum_{x_i \in N} |(F(\beta_i) - F(\alpha_i))\, g(\beta_i)| + \\
&\quad + \textstyle\sum_{x_i \in N} |F(\alpha_i) - F(\xi_i)|\,(g(\beta_i) - g(\alpha_i)),
\end{aligned}$$

where $\xi_i \in [\alpha_i, \beta_i]$. Moreover, by (6.3), we have

$$\sum_{x_i \in N} |(F(\beta_i) - F(\alpha_i))\, g(\beta_i)| \leq \frac{\varepsilon}{3(\|g\|_\infty + 1)} \cdot \|g\|_\infty < \frac{\varepsilon}{3}, \tag{6.8}$$

and, by (6.4) and (6.5), we infer

$$\begin{aligned}
&\sum_{x_i \in N} |F(\alpha_i) - F(\xi_i)|\,(g(\beta_i) - g(\alpha_i)) \\
&\qquad \leq \frac{\varepsilon}{6(\|\, g\,\|_\infty + 1)} \cdot 2\,\|\, g\,\|_\infty \leq \frac{\varepsilon}{3}.
\end{aligned} \tag{6.9}$$

Finally, summing up the inequalities (6.7), (6.8) and (6.9) and taking into account (6.6) and (6.7), we obtain

$$\left| \sum_i f(x_i)g(x_i)|I_i| - \Phi([a,b]) \right| < \varepsilon,$$

which completes the proof. ☐

Corollary 6.2. *The product of a nearly derivative and a BV function is a nearly derivative modulo a Lebesgue integrable function having arbitrarily small L^1-norm.*

Proof. Let f be a nearly derivative and let g be a BV function. By Lemma 2.5, f is $\tilde{C}$-integrable, and, by Theorem 6.1, fg is $\tilde{C}$-integrable. Thus, by Theorem 1.4 there exists a nearly derivative f_1 such that $fg - f_1$ is a Lebesgue integrable function. Choose an $\varepsilon > 0$. The absolute continuity of the Lebesgue integral and Lusin's theorem imply that there exists a continuous function h_1 such that

$$\int_a^b |fg - f_1 - h_1| < \frac{\varepsilon}{4}.$$

Let h_2 be a continuous function such that $h_2(a) = f_1(a) + h_1(a)$, $h_2(b) = f_1(b) + h_1(b)$, and $\int_a^b |h_2| < \varepsilon/4$. Moreover, let h_3 be a continuous function for which $h_3(a) = h_3(b) = 0$ and

$$\begin{aligned}
\textstyle\int_a^b h_3 &= \textstyle\int_a^b (f_1 + h_1 - h_2) - \int_a^b fg \\
&= \textstyle\int_a^b (f_1 + h_1 - fg) - \int_a^b h_2.
\end{aligned}$$

Clearly, we may assume $h_3 \geq 0$ or $h_3 \leq 0$. Thus

$$\begin{aligned} \int_a^b |h_3| = \left|\int_a^b h_3\right| &\leq \int_a^b |fg - f_1 - h_1| + \int_a^b |h_2| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

Observe that the function $h_\varepsilon = f_1 + h_1 - h_2 - h_3$ is a nearly derivative, since a continuous function is a derivative and the sum of nearly derivatives is a nearly derivative. Furthemore,

$$\begin{aligned} \int_a^b |fg - h_\varepsilon| &\leq \int_a^b |fg - f_1 - h_1| + \int_a^b |h_2| + \int_a^b |h_3| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Consequently, the claim follows by the obvious identity $fg = h_\varepsilon + (fg - h_\varepsilon)$. □

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Department of Mathematics, Università di Palermo, Facoltà di Ingegneria, Viale delle Scienze, 90100 Palermo (Italy)

E-mail address: `donatell@math.unipa.it`