THE BEST CONSTANT OF SOBOLEV INEQUALITY IN AN n DIMENSIONAL EUCLIDEAN SPACE

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Abstract. The best constant of Sobolev space of order M defined on $\mathbb{R}^n$: $W^M(\mathbb{R}^n)$ is obtained using the reproducing kernel which is the Green function of some high order elliptic partial differential operator on $\mathbb{R}^n$.

1 Introduction Let $H := W^M(\mathbb{R}^n)$ be the Sobolev space of order M satisfying $2M > n$, then the well-known Sobolev's inequality (Sobolev embedding) [1, Cor. 9.13] asserts

$$(\sup_{y\in\mathbb{R}^n} |u(y)|)^2 \le C\|u\|_H^2, \tag{1}$$

where $\|\cdot\|_H$ is the norm of H which is induced by the certain inner product attached to H. In [2], Hegland and Marti obtained the *best constant* of the inequality (1) when $n = 1$, $M = (1, 2\cdots)$ and $n = 2, M = 2$ assuming the usual inner product

$$(u, v)_H = \sum_{|\alpha|\le M} (\partial^\alpha u, \partial^\alpha v)_{L^2(\mathbb{R}^n)}. \tag{2}$$

Another result for the best constant of the embedding of $W^M(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ is by Morosi and Pizzocchero [3]. It is shown in [3] that the best constant is

$$C(n, M) = (\frac{1}{4\pi})^{\frac{n}{2}} \frac{\Gamma(M - \frac{n}{2})}{\Gamma(M)},$$

under the assumption on the inner product:

$$(u, v)_H = \int_{\mathbb{R}^n} (1 + |\xi|^2)^M \widehat{u}(\xi)\overline{\widehat{v}(\xi)} d\xi. \tag{3}$$

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This paper develops the result of [3], adopting the generalized inner product[1]

(4)

$$(u,v)_H$$

$$:= \int_{\mathbb{R}^n} \left[\sum_{j=0}^{[M/2]} p_{M-2j} \left(\Delta^j u(x)\right) \overline{\left(\Delta^j v(x)\right)} + \sum_{j=0}^{[\frac{M-1}{2}]} p_{M-2j-1} \nabla\left(\Delta^j u(x)\right) \cdot \overline{\nabla\left(\Delta^j v(x)\right)} \right] dx$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \left[\sum_{j=0}^{[M/2]} p_{M-2j} (|\xi|^2)^{2j} \widehat{u}(\xi)\overline{\widehat{v}(\xi)} + \sum_{j=0}^{[\frac{M-1}{2}]} p_{M-2j-1} (|\xi|^2)^{2j+1} \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \right] dx$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} (-1)^M \, p(-|\xi|^2) \, \widehat{u}(\xi) \, \overline{\widehat{v}(\xi)} \, d\xi, \quad (\forall u, v \in H),$$

where

$$p(\lambda) = \prod_{j=0}^{M-1} (\lambda - \alpha_j) = \sum_{j=0}^{M} (-1)^j \, p_j \, \lambda^{M-j}, \tag{5}$$

and $\{\alpha_i\}_{i=0}^{M-1}$ are positive numbers satisfying $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_{M-1}$. Therefore, the result of [3] is understood as the degenerated case ($\alpha_0 = \cdots = \alpha_{M-1} = 1$) of this paper.

2 Reproducing kernel In this section, we compute the reproducing kernel of H which is needed for the proof of Theorem 3. Let $G(\alpha; x)$ be the Green function of the differential operator $(-1)^M \, p(\Delta)$. Then the following theorem holds:

Theorem 1 *Assume $2M > n$ then Green function $K(x, y) = G(\alpha; x-y)$ is the reproducing kernel with respect to the Hilbert space H and an inner product $(u, v)_H$ of Eq. (4). That is to say, for any $y \in \mathbb{R}^n$, $K(x, y)$ belongs to H as a function of x and for almost all $y \in \mathbb{R}^n$ we have*

$$(u(x), K(x, y))_H = u(y). \tag{6}$$

Proof of Theorem 1 : Since

$$\widehat{G}(\alpha; \xi) = (-1)^M p(-|\xi|^2)^{-1}, \tag{7}$$

$\widehat{K}(\xi, y) = e^{-\sqrt{-1}<\xi,y>} \widehat{G}(\alpha; \xi)$ holds. In order to show that $K(x, y) \in H$, it is enough to show that $|\xi|^M |\widehat{G}(\alpha; \xi)| \in L^2(\mathbb{R}^n)$ and Eq. (6), but this is assured by the condition $n < 2M$. By Eq. (4) we have

$$(u(x), K(x, y))_H = (2\pi)^{-n} \int_{\mathbb{R}^n} (-1)^M \, p(-|\xi|^2) \, \widehat{u}(\xi) \, \overline{e^{-\sqrt{-1}<\xi,y>} \, \widehat{G}(\alpha; \xi)} \, d\xi =$$

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{\sqrt{-1}<\xi,y>} \, \widehat{u}(\xi) \, d\xi = u(y)$$

where we have used (7). This completes the proof of Theorem 1. ■

[1]The norm induced by the inner products Eqs. (3) and (4) induce equivalent norms to the one induced by the usual inner product (2).

3 Main Results Using the well-known theory of reproducing kernel we have the main results of this paper.

Theorem 2 *Let $2M > n$, then for every u in H there exists a positive constant C which is independent of u such that the following Sobolev inequality holds.*

$$\left(\sup_{y\in\mathbb{R}^n}\big|u(y)\big|\right)^2 \le C||u||_H^2 = C(u,u)_H \tag{8}$$

Among these constants C the best constant is

$$C(n,M) = \sup_{y\in\mathbb{R}^n} K(y,y) = G(\alpha;0) \tag{9}$$

If we replace C by $C(n,M)$ in the above inequality, the equality holds for $u(x) = K(x,y)$ for every fixed $y \in \mathbb{R}^n$.

The proof of this theorem is easy so we omit it; see [4, p. 812] . The best constants are given by the following theorem.

Theorem 3
(1) For $M = 2,3,4,\cdots,$ we assume that an odd number $n = 2q+2$ satisfies $q - 1/2 = 0,1,2,\cdots,M-2$. Then we have

$$C(2q+2,M) = (-1)^{M+q-1/2}\,\frac{1}{4\Gamma(q+1)}\left(\frac{1}{4\pi}\right)^q \sum_{j=0}^{M-1} e_j\,\alpha_j^q = \tag{10}$$

$$(-1)^{M+q-1/2}\,\frac{1}{4\Gamma(q+1)}\left(\frac{1}{4\pi}\right)^q \left|\begin{array}{c} \alpha_j^i \\ \hline \alpha_j^q \end{array}\right| \Bigg/ \left|\ \alpha_j^i\ \right|$$

where the numerator is the determinant of an $M\times M$ matrix ($0 \le i \le M-2,\ 0 \le j \le M-1$) and the denominator is the determinant of an $M\times M$ Vandermonde matrix and $e_j = 1/p'(\alpha_j)$ $(0 \le j \le M-1)$.
(2) For $M = 2,3,4,\cdots$ we assume that an even number $n = 2q+2$ satisfies $q = 0,1,2,\cdots,M-2$. Then we have

$$C(2q+2,M) = (-1)^{M+q}\,\frac{1}{\Gamma(q+1)}\left(\frac{1}{4\pi}\right)^{q+1} \sum_{j=0}^{M-1} e_j\,\alpha_j^q\,\log\alpha_j = \tag{11}$$

$$(-1)^{M+q}\,\frac{1}{\Gamma(q+1)}\left(\frac{1}{4\pi}\right)^{q+1} \left|\begin{array}{c} \alpha_j^i \\ \hline \alpha_j^q\log\alpha_j \end{array}\right| \Bigg/ \left|\ \alpha_j^i\ \right|$$

As shown in Theorem 2, the best constant is given by

$$G(\alpha;0) = \frac{(-1)^M}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{1}{p(-|\xi|^2)}d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{1}{\prod_{j=0}^{M-1}(|\xi|^2+\alpha_j)}d\xi. \tag{12}$$

Indeed, this integral can be computed by calculus of residues. This approach is seen in Appendix. We remark that the residue computation results in Appendix were deduced from the one obtained by the following more direct computation. For the direct computation approach, we introduce an another expression for $G(\alpha, 0)$.

Lemma 1 *The Green function for the differential operator $(-1)^M p(\Delta)$ has the following integral representation:*

$$G(\alpha;x) = \int_0^\infty e(\alpha;t)\, H(x,t)\, dt, \tag{13}$$

$$e(\alpha;t) = (-1)^{M-1} \sum_{j=0}^{M-1} e_j\, e^{-\alpha_j t},$$

$$H(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/(4t)),$$

where $H(x,t)$ is the heat kernel.

Proof of Lemma 1 : Applying well-known formula of expansion by partial fractions

$$p(\lambda)^{-1} = \sum_{j=0}^{M-1} e_j\, (\lambda - \alpha_j)^{-1}$$

to Eq. (7), we have

$$\widehat{G}(\alpha;\xi) = (-1)^M \sum_{j=0}^{M-1} e_j\, (-|\xi|^2 - \alpha_j)^{-1} = (-1)^M \int_0^\infty \sum_{j=0}^{M-1} e_j e^{-(|\xi|^2+\alpha_j)t} dt$$
$$= \int_0^\infty e(\alpha;t)\, e^{-|\xi|^2 t}\, dt.$$

Using well known formula $\widehat{H}(\xi,t) = e^{-|\xi|^2 t}$ we obtain (13). ■

Now we state a fundamental lemma concerning $e(\alpha;t)$.

Lemma 2 *$e(\alpha;t)$ is an entire function of t and can be expressed by a Taylor series*

$$e(\alpha;t) = (-1)^{M-1} \sum_{i=M-1}^{\infty} \left(\sum_{j=0}^{M-1} \alpha_j^i e_j \right) \frac{(-1)^i}{i!}\, t^i. \tag{14}$$

This follows at once from the well-known fact

$$\sum_{j=0}^{M-1} \alpha_j^i\, e_j = \delta_{i,M-1} = \begin{cases} 0 & (0 \le i \le M-2) \\ 1 & (i = M-1). \end{cases} \tag{15}$$

Before, going to the proof of Theorem 3, we prove another important fact that $G(\alpha;\cdot)$ is positive.

Proposition 1 *$G(\alpha;\cdot)$ is positive.*

Proof of Proposition 1: It is enough to show that $e(\alpha; t) > 0$, since by Eq. (13), $H(x, t) > 0$. From (13) and (15), we see that $e(\alpha; t) = e(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}; t)$ is the solution of the initial value problem:

$$(16) \qquad (\frac{d}{dt} + \alpha_{M-1}) \cdots (\frac{d}{dt} + \alpha_1)(\frac{d}{dt} + \alpha_0) e(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}; t) = 0,$$
$$e^{(i)}(\alpha; 0) = \begin{cases} 0 & (0 \le i \le M-2) \\ 1 & (i = M-1). \end{cases}$$

From this fact, we know that $e(\alpha; t) = e(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}; t)$ can be expressed as

$$(17) \qquad \begin{aligned} & e(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}; t) \\ &= \int_0^t e(\alpha_0, \alpha_1, \ldots, \alpha_{M-2}; t-s) \cdot e(\alpha_{M-1}; s) ds \\ &= \int_0^t (e(\alpha_0; \cdot) * \cdots * e(\alpha_{M-2}; \cdot))(t-s) e(\alpha_{M-1}; s) ds \\ &= (e(\alpha_0; \cdot) * \cdots * e(\alpha_{M-1}; \cdot))(t). \end{aligned}$$

Since $e(\alpha_0; t) = e^{-\alpha_0 t} > 0$, by induction, $e(\alpha; t)$ is positive. ■

Now, we prove Theorem 3.

Proof of Theorem 3 : By Theorem 2 we have

$$C(2q+2, M) = G(\alpha; 0) = \int_0^\infty e(\alpha; t)\, H(0; t)\, dt =$$

$$\left(\frac{1}{4\pi}\right)^{q+1} \int_0^\infty e(\alpha; t)\, t^{-q-1}\, dt.$$

We assumed $n = 2q+2 > 2M$ that is $M - q - 1 > 0$ so the above integral is convergent because of the Taylor expansion.

$$e(\alpha; t) = \frac{1}{(M-1)!}\, t^{M-1} + \cdots$$

Differentiating the above expression we also have

$$e^{(j)}(\alpha; t) = \frac{1}{(M-j-1)!}\, t^{M-j-1} + \cdots \quad (0 \le j \le M-1).$$

At first we treat the case (1). Integrating the relation

$$\left[\sum_{j=0}^{q-1/2} e^{(j)}(\alpha; t)\, \Gamma(q-j)\, t^{-(q-j)} \right]' =$$

$$e^{(q+1/2)}(\alpha; t)\, \Gamma(1/2)\, t^{-1/2} - e(\alpha; t)\, \Gamma(q+1)\, t^{-(q+1)}$$

on the interval $0 < t < \infty$ we obtain

$$0 = \sum_{j=0}^{q-1/2} e^{(j)}(\alpha; t)\, \Gamma(q-j)\, t^{-(q-j)} \Big|_{t=0}^{t=\infty} =$$

$$\Gamma(1/2) \int_0^\infty e^{(q+1/2)}(\alpha; t)\, t^{-1/2}\, dt - \Gamma(q+1) \int_0^\infty e(\alpha; t)\, t^{-q-1}\, dt.$$

Hence we have

$$\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt = \frac{\Gamma(1/2)}{\Gamma(q+1)}\int_0^\infty e^{(q+1/2)}(\alpha;t)\,t^{-1/2}\,dt =$$

$$(-1)^{M+q-1/2}\frac{\Gamma(1/2)}{\Gamma(q+1)}\sum_{j=0}^{M-1} e_j\,\alpha_j^{q+1/2}\int_0^\infty e^{-\alpha_j t}\,t^{-1/2}\,dt.$$

Owing to the relation

$$\int_0^\infty e^{-\alpha_j t}\,t^{-1/2}\,dt = \Gamma(1/2)\,\alpha_j^{-1/2}$$

finally we have

$$\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt = (-1)^{M+q-1/2}\,\frac{\Gamma(1/2)^2}{\Gamma(q+1)}\sum_{j=0}^{M-1} e_j\,\alpha_j^q.$$

This proves case (1).
Next we treat the case (2). Integrating the relation

$$\left[\sum_{j=0}^{q-1} e^{(j)}(\alpha;t)\,\Gamma(q-j)\,t^{-(q-j)}\right]' = e^{(q)}(\alpha;t)t^{-1} - e(\alpha;t)\,\Gamma(q+1)\,t^{-q-1}$$

on the interval $0<t<\infty$ we obtain

$$0 = \left[\sum_{j=0}^{q-1} e^{(j)}(\alpha;t)\,\Gamma(q-j)\,t^{-(q-j)} - e^{(q)}(\alpha;t)\,\log t\right]\Bigg|_{t=0}^{t=\infty} =$$

$$-\,\Gamma(q+1)\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt - \int_0^\infty e^{(q+1)}(\alpha;t)\,\log t\,dt.$$

So we have

$$\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt = -\,\frac{1}{\Gamma(q+1)}\int_0^\infty e^{(q+1)}(\alpha;t)\,\log t\,dt =$$

$$\frac{(-1)^{M+q-1}}{\Gamma(q+1)}\sum_{j=0}^{M-1} e_j\,\alpha_j^{q+1}\int_0^\infty e^{-\alpha_j t}\,\log t\,dt.$$

Considering that

$$\int_0^\infty e^{-\alpha_j t}\,\log t\,dt = -\,\alpha_j^{-1}\,(\log\alpha_j+\gamma)$$

where $\gamma = 0.577\cdots$ is the constant of Euler, finally we obtain

$$\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt = \frac{(-1)^{M+q}}{\Gamma(q+1)}\left[\sum_{j=0}^{M-1} e_j\,\alpha_j^q\,\log\alpha_j + \gamma\sum_{j=0}^{M-1} e_j\,\alpha_j^q\right] =$$

$$\frac{(-1)^{M+q}}{\Gamma(q+1)}\sum_{j=0}^{M-1} e_j\,\alpha_j^q\,\log\alpha_j.$$

Here we used the fact

$$\sum_{j=0}^{M-1} e_j\, \alpha_i^q = 0 \quad (0 \le q \le M-2).$$

This completes the proof of case (2). ■

4 Special case We here treat the simplest case $M = 2$, where the condition $n < 2M$ means that $n = 2, 3$. In this case, the Sobolev space $H = W^2(\mathbb{R}^n)$ consists of all the functions $u(x) \in L^2(\mathbb{R}^n)$ satisfying $\Delta u(x) \in L^2(\mathbb{R}^n)$. We assume that the inner product for any u, v in H is given by

$$(u,v)_H = \int_{\mathbb{R}^n} \left[(\Delta u(x))\, \overline{(\Delta v(x))} + p_1\, (\nabla u(x)) \cdot \overline{(\nabla v(x))} + p_2\, u(x)\, \overline{v(x)} \right] dx,$$

where $p_1 = \alpha_0 + \alpha_1,\ p_2 = \alpha_0 \alpha_1$ for some pair of positive numbers α_0 and α_1 $(0 < \alpha_0 < \alpha_1)$.

As a special case of Theorem 2 and 3 we have

Theorem 4 *We assume that $n = 2, 3$. For every $u(x)$ in $H = W^2(\mathbb{R}^n)$ there exists a positive constant C which is independent of $u(x)$ such that the following Sobolev inequality holds.*

$$\left(\sup_{y \in \mathbb{R}^n} |u(y)| \right)^2 \le C \int_{\mathbb{R}^n} (|\Delta u(x)|^2 + p_1\, |\nabla u(x)|^2 + p_2\, |u(x)|^2)\, dx. \tag{18}$$

Among these constants C the best constant is

$$C(n,2) = \begin{cases} \dfrac{1}{4\pi}\, \dfrac{\log \alpha_1 - \log \alpha_0}{\alpha_1 - \alpha_0} & (n = 2) \\ \dfrac{1}{4\pi}\, \dfrac{1}{\alpha_0^{1/2} + \alpha_1^{1/2}} & (n = 3). \end{cases}$$

If we substitute $C(n,2)$ into C in the above inequality (18), the equality holds for $u(x) = K(x,y)$ for every fixed $y \in \mathbb{R}^n$. The reproducing kernel $K(x,y)$ is given by the following formula.

$$\begin{aligned} K(x,y) = G(\alpha_0, \alpha_1; x-y) = \int_0^\infty e(\alpha_0, \alpha_1; t)\, H(x-y,t)\, dt = \\ \int_0^\infty \frac{1}{\alpha_1 - \alpha_0}\, (e^{-\alpha_0 t} - e^{-\alpha_1 t})\, H(x-y,t)\, dt \end{aligned} \tag{19}$$

where

$$H(x,t) = \begin{cases} (4\pi t)^{-1} \exp\bigl(-(x_1^2 + x_2^2)/(4t) \bigr) & (n = 2) \\ (4\pi t)^{-3/2} \exp\bigl(-(x_1^2 + x_2^2 + x_3^2)/(4t) \bigr) & (n = 3) \end{cases}$$

is the heat kernel.

5 Appendix First we show that the integral of Eq. (12) when n is odd can be computed by calculus of residues. Representing the integral by the polar coordinate $(r, \theta_1, \dots, \theta_{n-1}) \in [0,\infty) \times [0,\pi] \times \cdots [0,\pi] \times [0, 2\pi]$, we have

$$I := \frac{|S_{n-1}|}{(2\pi)^n} \int_0^\infty \frac{r^{n-1}}{\prod_{k=0}^{M-1} (r^2 + \alpha_k)} dr. \tag{20}$$

If n is odd, by calculus of residues, we obtain

$$\begin{aligned} I &= \frac{1}{(2\pi)^n} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{1}{2} \int_{-\infty}^\infty \frac{r^{n-1}}{\prod_{k=0}^{M-1} (r^2 + \alpha_k)} dr \\ &= \frac{2\pi\sqrt{-1}}{(4\pi)^{\frac{n}{2}} \Gamma(\frac{n}{2})} \sum_{j=0}^{M-1} \frac{(-\alpha_j)^{\frac{n-1}{2}}}{2\sqrt{-\alpha_j} \prod_{k=0, k\neq j}^{M-1} (-\alpha_j + \alpha_k)} \\ &= (-1)^{M-1+\frac{n-1}{2}} \frac{1}{4\Gamma(\frac{n}{2})} \left(\frac{1}{4\pi}\right)^{\frac{n-2}{2}} \sum_{j=0}^{M-1} e_j \alpha_j^{\frac{n-2}{2}}. \end{aligned} \tag{21}$$

This coincides with the result of Theorem 3 (i). Next, we compute the case, that n is even. By the change of the variable, we have

$$I = \frac{|S_{n-1}|}{2(2\pi)^n} \int_0^\infty \frac{z^{\frac{n-2}{2}}}{\prod_{k=0}^{M-1} (z + \alpha_k)} dz \tag{22}$$

Let $f(z) = z^{\frac{n-2}{2}} / \Pi_{k=0}^{M-1} (z + \alpha_k)$ and integral path of f be as in Figure 1. Then we have

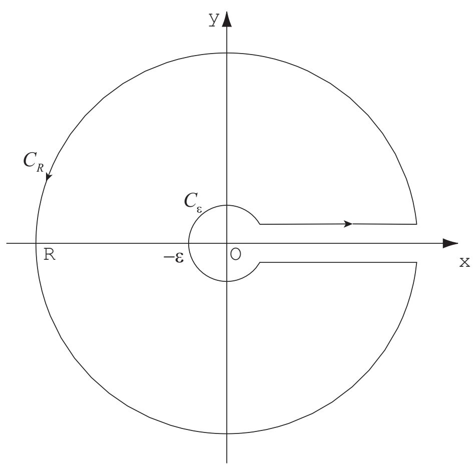

Figure 1: Integral path of f.

$$\begin{aligned} &\int_\epsilon^R (\log x) f(x)dx - \int_\epsilon^R (\log x + 2\pi\sqrt{-1}) f(x)dx \\ + \quad &\{\text{integral on } C_R \text{ and } C_\epsilon\} = 2\pi\sqrt{-1}\mathrm{Res}((\log z)f(z)) \end{aligned}$$

By taking the limit $R \to \infty$ and $\epsilon \to 0$, we have

$$\int_0^\infty f(x)dx = -\mathrm{Res}((\log z)f(z)) = -\sum_{j=0}^{M-1} (\log(-\alpha_j)) \frac{(-\alpha_j)^{\frac{n-2}{2}}}{\prod_{k=0,k\neq j} (-\alpha_j + \alpha_k)}$$

$$= \quad (-1)^{M-q} \sum_{j=0}^{M-1} (\log(\alpha_j) + \pi\sqrt{-1})\alpha_j^q e_j = (-1)^{M-q} \sum_{j=0}^{M-1} e_j \alpha_j^q \log \alpha_j$$

Substituting this to Eq. (22), we obtain the result.

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