ON WEAKLY REFINABLE SPACES

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Abstract. We first prove in this paper that a bounded weak $\kappa\overline{\theta}$-cover of any space has a $B(D,\omega_0)$-refinement for any infinite cardinal number κ. The special case $\kappa = \aleph_0$ had already been proved by R.H.Price and J.C.Smith in [7]. Thus we obtain a characterization of $B(D,\omega_0)$-refinability via bounded weak $\kappa\overline{\theta}$-cover refinements. We also prove that the set of all those points in any space having positive and finite order with respect to a given open family is covered by a σ-discrete closed refinement of that family. Thus a theorem of Bennett and Lutzer on subparacompactness is obtained as a corollary. We finally give a healthy proof of the fact that every weakly $\overline{\theta}$-refinable space is $B(D,\omega_0^2)$-refinable.

0. Introduction The generalized covering properties have been extensively studied in the past. Metacompactness and subparacompactness, the two most widely known weak forms of paracompactness for instance had been defined respectively in the historical papers [1] and [3]. In their well known paper [10], J.M.Worrell and H.H.Wicke, on the other hand, have proved several interesting characterizations of developable spaces via another weak covering property θ-refinability. For instance they proved I) A topological space is paracompact and T_2 iff it is collectionwise normal, θ-refinable and T_1; II) A topological space is developable iff it is essentially T_1 (i.e. the closures of any two singletons are either equal or disjoint), θ-refinable and has a base of countable order.

A sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of a topological space X is called a **θ-cover** iff for each point $x \in X$ there exists an $n_x \in \mathbb{N}$ such that $\mathrm{ord}(x,\mathcal{G}_{n_x}) = \mathrm{card}\{G \in \mathcal{G}_{n_x} : x \in G\} < \omega_0$. The space X is called **θ-refinable** iff each open cover of X has a θ-cover refinement. Spaces that are θ-refinable are also known as **submetacompact** in the literature, since every metacompact and every subparacompact space is evidently θ-refinable. The two generalizations of this concept have been defined as **weakly θ-refinable** and **weakly $\overline{\theta}$-refinable** spaces respectively in [2] and [8]. A cover $\mathcal{G} = \bigcup_{n=1}^{\infty}\mathcal{G}_n$ is called a **weak $\overline{\theta}$-cover** in X iff the following three conditions hold: i) each $\mathcal{G}_n$ is an open family (which is not necessarily a cover), ii) for each $x \in X$ there exists an $n_x \in \mathbb{N}$ such that $0 < \mathrm{ord}(x,\mathcal{G}_{n_x}) < \omega_0$, iii) the countable open cover $\{\bigcup\mathcal{G}_n\}_{n=1}^{\infty}$ is point-finite, see [8]. $\mathcal{G}$ is called a **weak θ-cover** on the other hand if it satisfies solely the two conditions i) and ii). In this note we briefly write $\bigcup\mathcal{A}$ instead of $\bigcup\{A : A \in \mathcal{A}\}$ for any family $\mathcal{A}$ of subsets of X. The space X is called **weakly $\overline{\theta}$-refinable** (resp. **weakly θ-refinable**) iff every open cover of X has a weak $\overline{\theta}$-cover (resp. weak θ-cover). H.R. Bennett and D.J. Lutzer have shown in [2] that quasi-developable spaces are weakly θ-refinable. J.C. Smith has proved in [8] that every θ-refinable space is weakly $\overline{\theta}$-refinable and there exists a weakly $\overline{\theta}$-refinable non θ-refinable T_2 space. Some several related examples and knowledges can also be found in the survey chapter [4] of D.Burke. J.Chaber and H.Junnila on the other hand have observed in [5] that every open cover of any θ-refinable space has a refinement $\mathcal{K} = \bigcup_{n=1}^{\infty}\mathcal{K}_n$ such that

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each $\mathcal{K}_n$ is a locally finite collection of closed subsets in the subspace $X - \bigcup_{1 \leq k < n}(\bigcup \mathcal{K}_k)$. Then it is straightforward to observe by induction that the union set $\bigcup_{k<n}(\bigcup \mathcal{K}_k)$ is closed in X for each $n \in \mathbb{N}$; thus it is understood that every θ-refinable space is $B(LF, \omega_0)$-refinable, (they were actually called the property b_1 for $B(LF, \omega_0)$-refinability), see below for the definition:

Definition 1: Let λ be an ordinal number. A topological space X is called **$B(D, \lambda)$-refinable** (resp. **$B(LF, \lambda)$-refinable**), iff every open cover of X has a refinement $\mathcal{K} = \bigcup_{\alpha<\lambda} \mathcal{K}_\alpha$ such that i) each $\mathcal{K}_\alpha$ is a discrete (resp. locally finite) family of closed subsets in the subspace $X - \bigcup_{\beta<\alpha}(\bigcup \mathcal{K}_\beta)$, ii) the union set $\bigcup_{\beta<\alpha}(\bigcup \mathcal{K}_\beta)$ is closed for each $\alpha < \lambda$, see [9] and [6]. Then $\mathcal{K}$ is called as the ***B(D, λ)*-refinement** of that open cover.

J.Chaber and H.Junnila have proved in the above mentioned paper that a space X is metacompact (resp. submetacompact $\equiv$ θ-refinable) iff X is almost expandable (resp. almost θ-expandable) and $B(LF, \omega_0)$-refinable. Every $B(D, \lambda)$-refinable space is evidently $B(LF, \lambda)$-refinable.

R.H.Price and J.C.Smith, on the other hand have proved in [7] that every bounded weak $\overline{\theta}$-cover of any space has a $B(D, \omega_0)$-refinement. We prove first in this paper that even any bounded weak $\kappa\overline{\theta}$-cover of any space has a $B(D, \omega_0)$-refinement for any infinite cardinal number κ. The following concept has been recently defined by N.Ergun and T.Noiri as a natural generalization of weak $\overline{\theta}$-cover, see [12]:

Definition 2: Let κ be an infinite cardinal number. An open cover $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$ of a topological space X is called a weak $\kappa\overline{\theta}$-cover iff *i)* $\mathcal{G}^* = \{\bigcup \mathcal{G}_\alpha\}_{\alpha<\kappa}$ is a point-finite open cover of X and *ii)* for each $x \in X$ there exists an index $\alpha_x < \kappa$ such that $0 < \mathrm{ord}(x, \mathcal{G}_{\alpha_x}) < \omega_0$.

Thus weak $\overline{\theta}$-covers are nothing but weak $\aleph_0\overline{\theta}$-covers. Bounded weak $\kappa\overline{\theta}$-covers, on the other hand, will be defined before Proposition 1 in the sequel. Weakly $\kappa\overline{\theta}$-refinable spaces are investigated in [12]. The purpose of this paper is expressed in the abstract. No separation axiom is assumed in this note unless explicitly stated. $[A]^n$ will denote, as is well known, the set of all special finite subsets of A having cardinality n. The first infinite ordinal number will be denoted by ω_0 as usual. $\mathbb{N}$ denotes the set of all positive integers whereas $\mathbb{N}_n$ denotes the finite set $\{1, 2, \ldots, n\}$ for each $n \in \mathbb{N}$. $\mathcal{A} \prec \mathcal{B}$ simply means throughout the paper that for each $A \in \mathcal{A}$ there exists a $B_A \in \mathcal{B}$ such that $A \subseteq B_A$.

1. Results Let $\mathcal{G}$ be any nonempty family of open subsets in the topological space X. Then it is known that the sets $H_n(\mathcal{G}) = \{x \in X : \mathrm{ord}(x, \mathcal{G}) \leq n\}$ and $U_n(\mathcal{G}) = \{x \in X : n \leq \mathrm{ord}(x, \mathcal{G})\}$ are respectively closed and open in X. These symbols will be utilized throughout the note. Note that $H_n(\mathcal{G}) \subseteq H_{n+1}(\mathcal{G})$.

Let $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$ be a weak $\kappa\overline{\theta}$-cover of X. Then for each $x \in X$ there exists a finite set $\kappa(x) = \{\alpha < \kappa : 0 < \mathrm{ord}(x, \mathcal{G}_\alpha)\}$ and at least one element $\alpha_x \in \kappa(x)$ such that $0 < \mathrm{ord}(x, \mathcal{G}_{\alpha_x}) < \omega_0$. Thus we have

$$1 \leq \min_{\alpha \in \kappa(x)} \mathrm{ord}(x, \mathcal{G}_\alpha) < \omega_0 \qquad (\forall x \in X).$$

We briefly write $\min \mathrm{ord}_{\mathcal{G}}(x)$ (or $\min \mathrm{ord}(x)$ if there is no possibility of confusion) for this minimum. Then $\mathcal{G}$ will be called a **bounded weak $\kappa\overline{\theta}$-cover** of X if there exists a positive integer n_0 such that

$$\max_{x \in X} \min \mathrm{ord}(x) = n_0,$$

i.e. $\min \mathrm{ord}(x) \leq n_0$ for each $x \in X$ and there exists at least one point $x_0 \in X$ satisfying $\min \mathrm{ord}(x_0) = n_0$.

Now we start with the following result. The notations and concepts which are introduce now will be used throughout this proposition.

Proposition 1 *A bounded weak $\kappa\overline{\theta}$-cover of any space has a $B(D,\omega_0)$-refinements for any infinite cardinal number κ.*

Proof: Let $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_\alpha$ be a bounded weak $\kappa\overline{\theta}$-cover of X. Let us write $\mathcal{G}_\alpha = \{G_{\alpha,\beta} : \beta \in I_\alpha\}$ for each $\alpha < \kappa$. We divide the whole proof in two steps.

Step 1: Suppose there exists a fixed positive integer n_0 such that $\min \mathrm{ord}(x) = n_0$ holds for each $x \in X$, i.e. $n_0 \leq \mathrm{ord}(x, \mathcal{G}_\alpha)$ for each $\alpha \in \kappa(x)$ and there exists an $\alpha_x \in \kappa(x)$ such that $n_0 = \mathrm{ord}(x, \mathcal{G}_{\alpha_x})$ for each $x \in X$. Define then

$$\mathcal{U}_\alpha = \{\bigcap_{\beta\in\Lambda} G_{\alpha,\beta} : \Lambda \in [I_\alpha]^{n_0}\} \qquad (\alpha < \kappa)$$

and write $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ and $\mathcal{U}^* = \{\bigcup \mathcal{U}_\alpha\}_{\alpha<\kappa}$. Let $\mathcal{U}_\alpha = \{U(\alpha,\beta) : \beta \in \Lambda_\alpha\}$ as an indexed family for each $\alpha < \kappa$. We evidently suppose in here that each $\mathcal{U}_\alpha$ family is faithfully indexed, i.e. $U(\alpha,\beta_1) \neq U(\alpha,\beta_2)$ whenever $\beta_1 \neq \beta_2$, $\beta_1, \beta_2 \in \Lambda_\alpha$. Notice that $\mathrm{ord}(x,\mathcal{U}_{\alpha_x}) = 1$ for each $x \in X$ and besides $\mathrm{ord}(x,\mathcal{U}_\alpha) = 0$ whenever $\alpha \in \kappa - \kappa(x)$. Thus $\mathcal{U}^*$ is a point-finite open cover of X. Define now

$$A_n = \{x \in X : \mathrm{ord}(x,\mathcal{U}^*) = n\} \qquad (n \in \mathbb{N}).$$

Note first that $A_n \cap A_m = \emptyset$ whenever $n \neq m$ and $\bigcup_{1\leq k\leq n} A_k = H_n(\mathcal{U}^*)$. Besides for any point $x \in A_n$ there exists a unique set $\kappa(x) = \{\alpha_x(1), \alpha_x(2), \ldots, \alpha_x(n)\} \in [\kappa]^n$ such that $\alpha_x(1) < \alpha_x(2) < \cdots < \alpha_x(n) < \kappa$ and $1 \leq \mathrm{ord}(x,\mathcal{U}_{\alpha_x(k)})$ for each $1 \leq k \leq n$ and at least one of these orders is certainly 1 and some of them may possibly greater or equal to ω_0. We are going to define now the families $\mathcal{K}_{n,m}$ for each $n \in \mathbb{N}$ and for each $1 \leq m \leq n$ such that all conditions

1. $\mathcal{K}_{n,m} \prec \mathcal{U}$,
2. $A_n = \bigcup_{1\leq m\leq n}(\bigcup \mathcal{K}_{n,m})$,
3. $\bigcup \mathcal{K}_{n,m} \cap \big(H_{n-1}(\mathcal{U}^*) \cup \bigcup_{1\leq k<m}(\bigcup \mathcal{K}_{n,k})\big) = \emptyset$,
4. $\mathcal{K}_{n,m}$ is a closed-discrete family in the subspace $X - \big(H_{n-1}(\mathcal{U}^*) \cup \bigcup_{1\leq k<m}(\bigcup \mathcal{K}_{n,k})\big)$,
5. $H_{n-1}(\mathcal{U}^*) \cup \bigcup_{1\leq k<m}(\bigcup \mathcal{K}_{n,k}) = \bigcup_{j<i} \bigcup_{i<n}(\bigcup \mathcal{K}_{i,j}) \cup \bigcup\limits_{1\leq k<m} (\bigcup \mathcal{K}_{n,k})$ is closed in X;

hold for each $n \in \mathbb{N}$ and for each $1 \leq m \leq n$. Define now for this purpose the subset $A_{n,m,\alpha}$ of A_n, which is the set of the whole special points $x \in A_n$ satisfying the following three conditions:

i) $\mathrm{ord}(x,\mathcal{U}_{\alpha_x(m)}) = \min_{1\leq k\leq n} \mathrm{ord}(x,\mathcal{U}_{\alpha_x(k)})$ $(= \min \mathrm{ord}_{\mathcal{U}}(x))$

ii) $1 < \min_{1\leq k<m} \mathrm{ord}(x,\mathcal{U}_{\alpha_x(k)})$ if $1 < m$,

iii) $\alpha_x(m) = \alpha$.

We evidently have $\mathrm{ord}(x,\mathcal{U}_\alpha) = \mathrm{ord}(x,\mathcal{U}_{\alpha_x(m)}) = \min \mathrm{ord}_{\mathcal{U}}(x) = 1$ for any $x \in A_{n,m,\alpha}$ and besides $X = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{n} \bigcup_{\alpha<\kappa} A_{n,m,\alpha}$ holds. Define then the families

$$\begin{aligned}
\mathcal{K}_{1,1} &= \{A_1 \cap U : U \in \mathcal{U}\}, \\
\mathcal{K}_{n,m,\alpha} &= \{A(n,m,\alpha) \cap U(\alpha,\beta) : \beta \in \Lambda_\alpha\}, \\
\mathcal{K}_{n,m} &= \bigcup_{\alpha<\kappa} \mathcal{K}_{n,m,\alpha}
\end{aligned}$$

where $1 < n$ and $1 \leq m \leq n$. Thus we first have $\bigcup \mathcal{K}_{n_1,m_1,\alpha_1} \cap \bigcup \mathcal{K}_{n_2,m_2,\alpha_2} = \emptyset = \bigcup \mathcal{K}_{n_1,m_1} \cap \bigcup \mathcal{K}_{n_2,m_2}$ whenever the subindices of these families are different and $X = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{n} (\bigcup \mathcal{K}_{n,m})$. It is easy to observe that $\mathcal{K}_{1,1}$ is a discrete family of closed subsets in X since $A_1 = H_1(\mathcal{U}^*)$ is closed, $\text{ord}(x,\mathcal{U}_{\alpha_x}) = 1$ and $\text{ord}(x,\mathcal{U}_{\alpha}) = 0$ for each $x \in H_1(\mathcal{U}^*)$ and for each $\alpha \neq \alpha_x$. Similarly the members of each $\mathcal{K}_{n,m,\alpha}$ family are pairwise disjoint since $\text{ord}(x,\mathcal{U}_{\alpha}) = 1$ for any $x \in A(n,m,\alpha)$. Therefore each $U(\alpha,\beta) \in \mathcal{U}_{\alpha}$ intersects only one member of $\mathcal{K}_{n,m,\alpha}$. Notice additionally the basic fact that if the indices α_k satisfy $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ then we necessarily have $\alpha_x(1) = \alpha_1, \ldots, \alpha_x(n) = \alpha_n$ for any point $x \in A_n \cap \bigcap_{1\leq k\leq n}(\bigcup \mathcal{U}_{\alpha_k})$. Now let us first prove that $\mathcal{K}_{n,m}$ is a closed-discrete family in the subspace $X - (H_{n-1}(\mathcal{U}^*) \cup \bigcup_{1\leq k<m}(\bigcup \mathcal{K}_{n,k}))$. Suppose first $1 < m$ and let us take any point x from this complementary set. Then we have $n \leq \text{ord}(x,\mathcal{U}^*) = n'$, $x \in \bigcup \mathcal{K}_{n',m'}$ and $x \in A_{n',m',\beta}$ where $m \leq m' \leq n'$. Now if $n < n'$ then we evidently have $x \in U_{n+1}(\mathcal{U}^*)$ and $U_{n+1}(\mathcal{U}^*) \cap \bigcup \mathcal{K}_{n,m} = \emptyset$. Let us suppose now that $n = n'$. So $x \in \bigcup \mathcal{K}_{n,m'}$ and $x \in A_{n,m',\gamma_x}$ and $m \leq m' \leq n$. Thus we have $2 \leq \text{ord}(x,\mathcal{U}_{\alpha_x(i)})$ for each $1 \leq i < m'$ and $\text{ord}(x,\mathcal{U}_{\alpha_x(m')}) = 1$. Thus by determining the finite subsets $\Lambda_x(i) \subseteq \Lambda_{\alpha_x(i)}$ with $2 \leq \text{card}(\Lambda_x(i))$ for each $1 \leq i < m'$ and the unique $\beta_x \in \Lambda_{\alpha_x(m')}$ with $x \in U(\alpha_x(m'),\beta_x) - \bigcup_{\beta\neq\beta_x} U(\alpha_x(m'),\beta)$ one can define the special open set

$$W_x = \left(\bigcap_{i<m'} \bigcap_{\beta\in\Lambda_x(i)} U(\alpha_x(i),\beta)\right) \cap\ U(\alpha_x(m'),\beta_x)\ \cap \left(\bigcap_{i'=m'+1}^{n} \left(\bigcup \mathcal{U}_{\alpha_x(i')}\right)\right).$$

Then W_x contains x and furthermore

$$W_x \cap \left(H_{n-1}(\mathcal{U}^*) \cup \bigcup_{i<m'} \left(\bigcup \mathcal{K}_{n,i}\right) \cup \bigcup_{\beta\neq\gamma_x} A(n,m',\beta)\right) = \emptyset.$$

In fact notice that if there exists a point $y \in A_{n,m'',\alpha} \cap W_x$ where $m'' < m'$, then we have in particular $\alpha_y(m'') = \alpha_x(m'')$, $y \in \bigcap_{\beta\in\Lambda_x(m')} U(\alpha_x(m''),\beta)$ and thus we would have $2 \leq \text{card}\Lambda_x(m'') \leq \text{ord}(y,\mathcal{U}_{N_x(m'')}) = \text{ord}(y,\mathcal{U}_{N_y(m'')}) = 1$. Additionally if a point $y \in A_{n,m',\beta} \cap W_x$ does exist then we easily have $y \in A_n \cap W_x$ and $\beta = \alpha_y(m') = \alpha_x(m') = \gamma_x$ which yields $W_x \cap \bigcup_{\beta\neq\gamma_x} A_{n,m',\beta} = \emptyset$. Therefore it is understood that the condition 4) in above is satisfied whenever $1 < m \leq n$. Now let us examine the case $m = 1$. The complementary set written at the right side of 4) would simply be $X - H_{n-1}(\mathcal{U}^*)$ in this case and for any point x of this set we either have i) $n < \text{ord}(x,\mathcal{U}^*)$, i.e. $x \in U_{n+1}(\mathcal{U}^*)$ or ii) $x \in A_{n,1,\alpha_x(1)}$ or iii) $x \in A_{n,m',\gamma}$ where $1 < m' \leq n$ and $k \in \mathbb{N}$. In each of these cases an open nbhd W_x of x missing all but (possibly) one member of $\mathcal{K}_{n,1}$ can easily be defined. We have for instance $U_{n+1}(\mathcal{U}^*) \cap \bigcup_{1\leq m\leq n} (\bigcup \mathcal{K}_{n,m}) = \emptyset$ for the case i) and the case iii) has already been proved a little while ago; if finally $x \in A_{n,1,\alpha_x(1)}$ then the unique member of $\mathcal{U}_{\alpha_x(1)}$ containing x can be taken as W_x. All these intermediate results easily verify the conditions 3), 4) and 5). Observing the conditions 1) and 2) is just straightforward. We also have $X = \bigcup_{n=1}^{\infty} A_n = \bigcup_{1\leq m\leq n} \bigcup_{n=1}^{\infty} (\bigcup \mathcal{K}_{n,m})$. By using lexicographic ordering now, one can define the families $\mathcal{K}_1^* = \mathcal{K}_{1,1}, \mathcal{K}_2^* = \mathcal{K}_{2,1}, \mathcal{K}_3^* = \mathcal{K}_{2,2}, \mathcal{K}_4^* = \mathcal{K}_{3,1}, \ldots$ and in general $\mathcal{K}_k^* = \mathcal{K}_{n_k,m_k}$ where (n_k, m_k) is the k^{th} couple (n, m) with respect to this ordering in which $n \in \mathbb{N}$ and $1 \leq m \leq n$. It is easy to see that now $\mathcal{K}^* = \bigcup_{k=1}^{\infty} \mathcal{K}_k^*$ is the required refinement of $\mathcal{U}$ satisfying all the $B(D,\omega_0)$-refinability conditions.

Step 2: Suppose now there exists a positive integer n_0 such that the boundedness conditions written before this proposition holds for the weak $\kappa\overline{\theta}$-cover $\mathcal{G} = \bigcup_{\alpha<\kappa} \mathcal{G}_{\alpha}$. Then the family $\mathcal{G}^* = \bigcup_{n\leq n_0} \bigcup_{\alpha<\kappa} \mathcal{G}_{\alpha,n}$ would be a bounded weak $\kappa\overline{\theta}$-cover refinement of $\mathcal{G}$

whereas

$$\mathcal{G}_{\alpha,n} = \{\bigcap_{\beta\in\Lambda} G_{\alpha,\beta} : \Lambda \in [I_\alpha]^n\} \qquad (\alpha, n) \in \kappa \times \mathbb{N}_{n_0},$$

since if $\min \mathrm{ord}_{\mathcal{G}}(x) = n_x \leq n_0$ and $n_x = \mathrm{ord}(x, \mathcal{G}_{\alpha_x})$ then we have $\mathrm{ord}(x, \mathcal{G}_{\alpha_x,n_x}) = 1 = \min \mathrm{ord}_{\mathcal{G}^*}(x)$ for each $x \in X$ and therefore the procedure of the previous step works. Thus the proof is over now.

Corollary 1 (R.H.Price&J.C.Smith [7]) *A bounded weak $\overline{\theta}$-cover of any space has a $B(D, \omega_0)$-refinement.*

Corollary 2 *A space X is $B(D, \omega_0)$-refinable iff there exists an infinite cardinal number $\kappa = \kappa(\mathcal{G})$ for every open cover $\mathcal{G}$ of X such that $\mathcal{G}$ has a bounded weak $\kappa\overline{\theta}$-cover refinement.*

Proof: Sufficiency follows from the above proposition. Let X be any $B(D, \omega_0)$-refinable space now and let an open cover $\mathcal{G}$ of X be given. It is actually known that $\mathcal{G}$ has a bounded weak $\overline{\theta}$-cover refinement and we are going to give this proof for the sake of self containment, see [7]. Let $\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$ be a $B(D, \omega_0)$-refinement of $\mathcal{G}$. Therefore $\bigcup_{1\leq k<n}(\bigcup \mathcal{K}_k)$ is closed in X for each $n \in \mathbb{N}$ and $\mathcal{K}_n$ is a closed-discrete family in the open subspace $O_n = X - \bigcup_{1\leq k<n}(\bigcup \mathcal{K}_k)$ for each $n \in \mathbb{N}$. Thus we have $\bigcup \mathcal{K}_n \subseteq O_n$ $(n \in \mathbb{N})$. Now let $\mathcal{K}_n = \{K_{n,\alpha} : \alpha \in \Lambda_n\}$ for each $n \in \mathbb{N}$ and determine a unique member $G(K_{n,\alpha})$ from $\mathcal{G}$ for each couple $(n, \alpha) \in \mathbb{N} \times \Lambda_n$ such that $K_{n,\alpha} \subseteq G(K_{n,\alpha})$ holds. Then, there exists a uniquely determined couple (n_x, α_x) for any $x \in X$ such that $x \in \bigcup \mathcal{K}_{n_x} - \bigcup_{1\leq k<n_x}(\bigcup \mathcal{K}_k) = K_{n_x} \cap O_{n_x}$ and $x \in K_{n_x,\alpha_x} - \bigcup\{K_{n_x,\alpha} : \alpha \in \Lambda_{n_x}, \alpha \neq \alpha_x\}$. Let us briefly write Λ'_{n_x} for $\Lambda_{n_x} - \{\alpha_x\}$. Since $\mathcal{K}_{n_x}$ is a discrete family of closed sets in O_{n_x} we have $cl_{O_{n_x}}(\bigcup_{\alpha\in\Lambda'_{n_x}} K_{n_x,\alpha}) = \bigcup_{\alpha\in\Lambda'_{n_x}} K_{n_x,\alpha}$ and thus

$$x \in (K_{n_x,\alpha_x} \cap O_{n_x}) - cl_{O_{n_x}}\left(\bigcup_{\alpha\in\Lambda'_{n_x}} K_{n_x,\alpha}\right) = K_{n_x,\alpha_x} - \overline{\bigcup_{\alpha\in\Lambda'_{n_x}} K_{n_x,\alpha}}.$$

Thus we have $x \in G^*_{n_x,\alpha_x}$ where the open $G^*_{n,\alpha}$ sets are defined as

$$G^*_{n,\alpha} = G(K_{n,\alpha}) - \left[\bigcup_{1\leq k<n}\left(\bigcup \mathcal{K}_k\right) \cup \overline{\bigcup_{\beta\in\Lambda_n-\{\alpha\}} K_{n,\beta}}\right], \quad (n, \alpha) \in \mathbb{N} \times \Lambda_n$$

and thus an open refinement $\mathcal{G}^*$ of $\mathcal{G}$ could be defined as $\mathcal{G}^* = \bigcup_{n=1}^{\infty} \mathcal{G}^*_n$, whereas $\mathcal{G}^*_n = \{G^*_{n,\alpha} : \alpha \in \Lambda_n\}$ $(n \in \mathbb{N})$. Notice that $x \notin G^*_{n_x,\alpha}$ for each $\alpha \in \Lambda'_{n_x}$, i.e. $\mathrm{ord}(x, \mathcal{G}^*_{n_x}) = 1 = \min \mathrm{ord}_{\mathcal{G}^*}(x)$ for each $x \in X$. Besides we evidently have $x \in \bigcup_{1\leq k<n}(\bigcup \mathcal{K}_k)$ for each $n > n_x$ and therefore $\mathrm{ord}(x, \bigcup_{n_x<n} \mathcal{G}^*_n) = 0$. Thus $\mathcal{G}^*$ is a bounded weak $\overline{\theta}$-cover of $\mathcal{G}$.

Proposition 2 *Let $\omega_0 \leq \lambda < \omega_1$. Then every $B(LF, \lambda)$-refinable space is weak θ-refinable.*

Proof: Let X be a $B(LF, \lambda)$-refinable space and let $\mathcal{U}$ be any open cover of X. Suppose that the refinement $\mathcal{K} = \bigcup_{\alpha<\lambda} \mathcal{K}_\alpha$ of $\mathcal{U}$ satisfy the conditions i) (with locally finite property) and ii) of the Definition. Write $\mathcal{K}_\alpha = \{K_{\alpha,\beta} : \beta \in \Lambda_\alpha\}$ for each $\alpha < \lambda$ and define

$$G_{\alpha,n}(\Lambda) = U_\alpha(\Lambda) - (\bigcup_{\beta\notin\Lambda} K_{\alpha,\beta} \cup \bigcup_{\gamma<\alpha}(\bigcup \mathcal{K}_\gamma))$$

for each $\Lambda \in [\Lambda_\alpha]^n$. Note that first, this set is open in X since $X_\alpha = X - \bigcup_{\beta<\alpha}(\bigcup \mathcal{K}_\beta)$ is open in X by the condition ii), $\bigcup_{\beta\notin\Lambda} K_{\alpha,\beta}$ is closed in X_α and therefore $G_{\alpha,n}(\Lambda) = (U_\alpha(\Lambda) \cap X_\alpha) - \bigcup_{\beta\notin\Lambda} K_{\alpha,\beta}$ is open in X_α. We have supposed in above that a well determined unique member $U_{\alpha,\beta} \in \mathcal{U}$ have already been chosen for each $\alpha < \lambda$ and $\beta \in \Lambda_\alpha$ such that $K_{\alpha,\beta} \subseteq U_{\alpha,\beta}$; we write above then

$$U_\alpha(\Lambda) = \bigcap\{U_{\alpha,\beta} : \beta \in \Lambda\} \qquad (\Lambda \in [\Lambda_\alpha]^n).$$

Define now the open families $\mathcal{G}_{\alpha,n} = \{G_{\alpha,n}(\Lambda) : \Lambda \in [\Lambda_\alpha]^n\}$ for each $(\alpha, n) \in [0, \lambda) \times \mathbb{N}$. Since $[0, \lambda)$ is well ordered, $X = \bigcup_{\alpha<\lambda}(\bigcup \mathcal{K}_\alpha)$ and $\mathcal{K}_\alpha$ is a locally finite family in X_α, it is not difficult to observe that there exists an $\alpha_x < \lambda$ and an $n_x \in \mathbb{N}$ for each $x \in X$ such that $\mathrm{ord}(x, \mathcal{G}_{\alpha_x,n_x}) = 1$. But since $[0, \lambda) \times \mathbb{N}$ is countable, we can write the open family $\bigcup_{n=1}^\infty \bigcup_{\alpha<\lambda} \mathcal{G}_{\alpha,n}$ as $\bigcup_{k=1}^\infty \mathcal{G}_{\alpha_k,n_k}$ and this family evidently constitutes a weak θ-cover refinement of $\mathcal{U}$. Thus proposition follows easily.

As is well known a topological space X is called **perfect** iff each open set in X is an $\mathcal{F}_\sigma$-set. We work now in the class of perfect spaces.

Proposition 3 *Let $\mathcal{G}$ be any non empty open family in any perfect space X. Then the set $A(\mathcal{G}) = \{x \in X : 0 < ord(x, \mathcal{G}) < \omega_0\}$ is covered by a σ-discrete closed refinement of $\mathcal{G}$ in X.*

Proof: Let an open family $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ in a perfect space X be given. We are going to prove that $A(\mathcal{G})$ can be covered by a σ-discrete family of closed sets in X. Notice that $A(\mathcal{G}) = \bigcup_{n=1}^\infty (U_1(\mathcal{G}) \cap H_n(\mathcal{G}))$. We may naturally suppose that the index set Λ is well ordered by $<$. Let $H_n(\mathcal{G}) = \bigcap_{k=1}^\infty U_{n,k}$ for each $n \in \mathbb{N}$ and $G_\alpha = \bigcup_{i=1}^\infty K_{\alpha,i}$ for each $\alpha \in \Lambda$, where each $U_{n,k}$ set is open and each $K_{\alpha,i}$ is closed in X. Define now

$$\begin{aligned} F_{n,m,i}(\alpha) &= (K_{\alpha,i} \cap H_n(\mathcal{G})) - \left(\bigcup_{\beta<\alpha} G_\beta \cup U_{n-1,m}\right), \\ \mathcal{K}_{n,m,i} &= \{F_{n,m,i}(\alpha) : \alpha \in \Lambda\}, \\ \mathcal{K} &= \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{\infty} \mathcal{K}_{n,m,i}. \end{aligned}$$

All $U_{0,m}$ sets are taken as the empty set. Note that $F_{n,m,i}(\alpha) \subseteq K_{\alpha,i} \subseteq G_\alpha \subseteq U_1(\mathcal{G})$ and $\mathcal{K} \prec \mathcal{G}$. Let $(n, m, i) \in \mathbb{N}^3$ be fixed. Notice that if $x \notin H_n(\mathcal{G}) - U_{n-1,m}$ then x has evidently an open nbhd missing all members of $\mathcal{K}_{n,m,i}$ since $F_{n,m,i}(\alpha) \subseteq H_n(\mathcal{G}) - U_{n-1,m}$. If on the other hand $x \in H_n(\mathcal{G}) - U_{n-1,m}$ then we have first $\mathrm{ord}(x, \mathcal{G}) = n$ and thus there exists a set $\Lambda_x \in [\Lambda]^n$ such that $x \in W_x = \bigcap_{\alpha\in\Lambda_x} G_\alpha$ and so $W_x \cap F_{n,m,i}(\alpha) \subseteq \bigcap_{\gamma\in\Lambda_x} G_\gamma \cap G_\alpha \cap H_n(\mathcal{G}) = \emptyset$ for any $\alpha < \gamma_x = \min \Lambda_x$ and $W_x \cap F_{n,m,i}(\alpha) \subseteq G_{\gamma_x} - \bigcup_{\beta<\alpha} G_\beta = \emptyset$ for any $\gamma_x < \alpha$. Thus each family $\mathcal{K}_{n,m,i}$ is a closed-discrete family in X. Besides for any point $x \in A(\mathcal{G})$, the positive integers $n_x = \mathrm{ord}(x, \mathcal{G})$ and m_x in which $x \notin U_{n_x-1,m_x}$ as well as the index $\alpha_x \in \Lambda$ in which $x \in G_{\alpha_x} - \bigcup_{\alpha<\alpha_x} G_\alpha$ and the positive integer i_x in which $x \in F_{n_x,m_x,i_x}(\alpha_x)$ holds, are all well defined. Thus we have $A(\mathcal{G}) = \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty \bigcup_{i=1}^\infty (\bigcup \mathcal{K}_{n,m,i})$.

Corollary 3 (H.R.Bennett and D.J.Lutzer, [2]) *A weakly θ-refinable perfect space is subparacompact.*

Proof: This statement follows easily after the preceding result since if $\mathcal{G} = \bigcup_{n=1}^\infty \mathcal{G}_n$ is a weak θ-cover of X then $X = \bigcup_{n=1}^\infty A(\mathcal{G}_n)$ holds.

Corollary 4 (R.Hodel, [11]) *Every perfect metacompact space is subparacompact.*

Remark: The following Corollary 5 has been given in [9], see Theorem 2.2 of [9]. But the proof of it, given in that paper is unfortunately not correct. The family $\mathcal{F}_k$ defined in that proof should be indexed with (i,j) and therefore should be written as $\mathcal{F}_k(i,j)$. This family is closed and discrete in the open subspace $X-P(i,j)$ but since $\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty}(\bigcup\mathcal{F}_k(i,j))\subseteq P(i,j+1)$ holds, one should also define all the families $\mathcal{F}_k(i,j+1)$ similarly for $X-P(i,j+1)$; that makes totaly ω_0^3 families! We given here in this context a correct and healthy proof of this interesting result.

Proposition 4 *A weak $\overline{\theta}$-cover of any topological space has a $B(D,\omega_0^2)$-refinement.*

Proof: Let $\mathcal{G}=\bigcup_{n=1}^{\infty}\mathcal{G}_n$ be a weak $\overline{\theta}$-cover of X and let $\mathcal{G}_n=\{G(n,\alpha):\alpha\in\Lambda_n\}$ for each $n\in\mathbb{N}$. We suppose in here that each $\mathcal{G}_n$ family is faithfully indexed, i.e. $G(n,\alpha)\neq G(n,\beta)$ whenever $\alpha\neq\beta$ and $\alpha,\beta\in\Lambda_n$. Notice that the open cover $\mathcal{G}^*=\{\bigcup\mathcal{G}_n\}_{n=1}^{\infty}$ is point-finite in X. Thus we have $X=\bigcup_{n=1}^{\infty}A_n$ where $A_n=\{x\in X:\operatorname{ord}(x,\mathcal{G}^*)=n\}$ for each $n\in\mathbb{N}$. Then $A_n\cap A_m=\emptyset$ holds whenever $n\neq m$ and there exists a unique set $\{N_x(k):k\in\mathbb{N}_n\}\in[\mathbb{N}]^n$ for each $x\in A_n$ such that $N_x(1)<N_x(2)<\cdots<N_x(n)$ and $1\leq\operatorname{ord}(x,\mathcal{G}_{N_x(k)})$ for each $k\in\mathbb{N}_n$ and furthermore at least one of these orders is certainly finite, i.e. $\exists k_x\in\mathbb{N}_n$, $\operatorname{ord}(x,\mathcal{G}_{N_x(k_x)})<\omega_0$ and some of them may possibly be $\geq\omega_0$. Let us define now the following sets for each four-tuples $(n,m,N,k)\in\mathbb{N}\times\mathbb{N}_n\times\mathbb{N}\times\mathbb{N}$:

$$A_{n,m,N,k}=\{x\in A_n:\textbf{Con 1 , Con 2, Con 3}\text{ holds for }x\}$$

whereas our basic conditions for any $x\in A_n$ are respectively

Con 1: $\operatorname{ord}(x,\mathcal{G}_{N_x(m)})=\min\operatorname{ord}_{\mathcal{G}}(x)=N$,

Con 2: $\operatorname{ord}(x,\mathcal{G}_{N_x(m)})<\min_{1\leq i<m}\operatorname{ord}(x,\mathcal{G}_{N_x(i)})$ if $1<m$,

Con 3: $N_x(m)=k$.

We will write from now on $N=\min\operatorname{ord}(x)$ for each $x\in A_{n,m,N,k}$. Notice furthermore that $\operatorname{ord}(x,\mathcal{G}_k)=N$ holds for each $x\in A_{n,m,N,k}$ and besides if the positive integers $N_1<N_2<\cdots<N_n$ are given, then we necessarily have $N_x(k)=N_k$ for each $k\in\mathbb{N}_n$ and for any point $x\in A_n\cap\bigcap_{1\leq k\leq n}(\bigcup\mathcal{G}_{N_k})$.

Now let

$$\begin{aligned}\mathcal{K}_{n,m,N,k}&=\{A_{n,m,N,k}\cap\bigcap_{\alpha\in\Lambda}G(n,\alpha):\Lambda\in[\Lambda_n]^N\},\\ \mathcal{K}_{n,m,N}&=\bigcup_{k=1}^{\infty}\mathcal{K}_{n,m,N,k}.\end{aligned}$$

It is not difficult to see that, all the orders $\operatorname{ord}(x,\mathcal{G}_{N_x(k)})$ $(k\in\mathbb{N}_n)$ and the finite subsets $\mathbb{N}(x)=\{N_x(k):k\in\mathbb{N}_n\}$ are uniquely determined for each $x\in A_n$ and thus $A_{n_1,m_1,N_1,k_1}\cap A_{n_2,m_2,N_2,k_2}=\emptyset$ iff $(n_1,m_1,N_1,k_1)\neq(n_2,m_2,N_2,k_2)$ and therefore

$$\bigcup\mathcal{K}_{n_1,m_1,N_1,k_1}\cap\bigcup\mathcal{K}_{n_2,m_2,N_2,k_2}=\emptyset=\bigcup\mathcal{K}_{n_1,m_1,N_1}\cap\bigcup\mathcal{K}_{n_2,m_2,N_2}$$

whenever the subindexes are different. Let us define finally an ordering among $\mathcal{K}_{n,m,N}$ families as in the following: We will write $\mathcal{K}_{n,m_1,N_1}<\mathcal{K}_{n,m_2,N_2}$ for any $n\in\mathbb{N}$ iff we either have $N_1<N_2$ or $N_1=N_2$ and $m_1<m_2$. Besides we define for all positive integers $m(\leq n_1),m'(\leq n_2),N,N'$: $\mathcal{K}_{n_1,m,N}<\mathcal{K}_{n_2,m',N'}$ whenever $n_1<n_2$. Thus we have $\mathcal{K}_{n,1,1}<\mathcal{K}_{n,2,1}<\cdots<\mathcal{K}_{n,n,1}<\mathcal{K}_{n,1,2}<\mathcal{K}_{n,2,2}<\cdots$ and $\mathcal{K}_{n,m,N}<\mathcal{K}_{n+1,m',N'}$ for all $n\in\mathbb{N}$

and for all positive integers $m \leq n$, $m' \leq n+1$ and N, N'. By using the refinement notation $\prec$ we have now the following: $\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{N=1}^{\infty}\mathcal{K}_{n,m,N} \prec \mathcal{G}$. Besides we have $A_n = \bigcup_{m=1}^{\infty}\bigcup_{N=1}^{\infty}(\bigcup \mathcal{K}_{n,m,N})$ $(n \in \mathbb{N})$ and $X = \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{N=1}^{\infty}(\bigcup \mathcal{K}_{n,m,N})$. We furthermore have

$$H_n(\mathcal{G}^*) = \bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n}\bigcup_{m=1}^{\infty}\bigcup_{N=1}^{\infty}(\bigcup \mathcal{K}_{k,m,N}) \qquad (n \in \mathbb{N}), \tag{1}$$

$$(A_{n,m,N,k} \cap \bigcap_{\alpha\in\Lambda_1} G(k,\alpha)) \cap \bigcap_{\alpha\in\Lambda_2} G(k,\alpha) \text{ if } \Lambda_1, \Lambda_2 \in [\Lambda_\alpha]^N,\ \Lambda_1 \neq \Lambda_2, \tag{2}$$

$$\bigcup \mathcal{K}_{n,m,N,k_0} \cap \overline{\bigcup_{k\neq k_0}(\bigcup \mathcal{K}_{n,m,N,k})} = \emptyset \qquad (k_0 \in \mathbb{N}). \tag{3}$$

Only the last assertion requires a proof. Let $k_0 \in \mathbb{N}$ and the point $x \in \bigcup \mathcal{K}_{n,m,N,k_0}$ be given. Then x belongs to open $O_x = \bigcap_{1\leq i\leq n}(\bigcup \mathcal{G}_{N_x(i)})$ and $O_x \cap \bigcup_{k\neq k_0}(\bigcup \mathcal{K}_{n,m,N,k}) \subseteq O_x \cap \bigcup_{k\neq k_0} A_{n,m,N,k} = \emptyset$ since, if there is a point $y \in O_x \cap A_{n,m,N,k}$ then we necessarily have $y \in O_x \cap A_n$ and so $\alpha_0 = \alpha_x(m) = \alpha_y(m) = \alpha$ as we have observed in above. Besides the following union

$$\bigcup_{\mathcal{K}_{n,m,N}\leq\mathcal{K}_{n_0,m_0,N_0}}(\bigcup \mathcal{K}_{n,m,N})$$

is closed in X for any fixed triple (n_0, m_0, N_0). After (1) in above this union is nothing but

$$\bigcup_{1\leq n<n_0} A_n \cup \bigcup \mathcal{K}_{n_0,1,1} \cup \bigcup \mathcal{K}_{n_0,2,1} \cup \cdots \cup \bigcup \mathcal{K}_{n_0,m_0,N_0}$$

and if we briefly write E_0 for this union, we either have $\operatorname{ord}(x,\mathcal{G}^*) < n_0$ or $\operatorname{ord}(x,\mathcal{G}^*) = n_0$ and $\min\operatorname{ord}(x) \leq N_0$ for each $x \in E_0$. Take now any point $x_0 \in X - E_0$. If $n_0 < \operatorname{ord}(x,\mathcal{G}^*)$ then $x_0 \in U_{n_0+1}(\mathcal{G}^*)$ and we evidently have $U_{n_0+1}(\mathcal{G}^*) \cap E_0 = \emptyset$. If $n_0 = \operatorname{ord}(x_0,\mathcal{G}^*)$ and $N_0 < \min\operatorname{ord}(x)$, then, by determining the finite set $\{N_x(k) : k \in \mathbb{N}_{n_0}\} \in [\mathbb{N}]^{n_0}$ where $1 \leq \operatorname{ord}(x_0, \mathcal{G}_{N_{x_0}(k)})$ for each $k \in \mathbb{N}_{n_0}$ and the subsets $\Lambda_k(x_0) \in [\Lambda_{N_{x_0}(k)}]^{N_0+1}$ such that $x_0 \in \bigcap_{\alpha\in\Lambda_k(x_0)} G(N_{x_0}(k),\alpha)$, we define the open set $U_{x_0} = \bigcap_{1\leq k\leq n_0}(\bigcap_{\alpha\in\Lambda_k(x_0)} G(N_{x_0}(k),\alpha))$ which apparently satisfy $x_0 \in U_{x_0}$ and $U_{x_0} \cap E_0 = \emptyset$ since we would have $n_0 = \operatorname{ord}(x,\mathcal{G}^*)$ and $N_0 + 1 \leq \min\operatorname{ord}(x) \leq N_0$ for any $x \in U_{x_0} \cap E_0$. Finally if $n_0 = \operatorname{ord}(x,\mathcal{G}^*)$ and $N_0 = \min\operatorname{ord}(x)$, then, there exists a positive integer $m > m_0$ such that $x_0 \in \bigcup \mathcal{K}_{n_0,m,N_0}$ and therefore there exists an $k_0 \in \mathbb{N}$ with $x_0 \in A_{n_0,m,N_0,k_0}$ and $N_{x_0}(i) \leq N_x(m_0) < N_x(m)$, $N_0 = \operatorname{ord}(x_0, \mathcal{G}_{N_{x_0}(m)}) < \operatorname{ord}(x_0, \mathcal{G}_{N_{x_0}(i)})$ for each $1 \leq i < m$. Now by taking $\Lambda_i(x_0) \in [\Lambda_{N_{x_0}(i)}]^{N_0+1}$ for each $1 \leq i < m$ and $\Lambda_j(x_0) \in [\Lambda_{N_j(x_0)}]^{N_0}$ for each $m \leq j \leq n_0$, one can define the open set

$$V_{x_0} = \bigcap_{1\leq i<m}\left(\bigcap_{\beta\in\Lambda_i(x_0)} G(N_{x_0}(i),\beta)\right) \cap \bigcap_{m\leq j\leq n_0}\left(\bigcap_{\mu\in\Lambda_j(x_0)} G(N_{x_0}(j),\mu)\right)$$

which evidently satisfy $x_0 \in V_{x_0}$ and $V_{x_0} \cap E_0 = \emptyset$, since if a point $x \in V_{x_0} \cap E_0$ does exist, then we first have $n_0 = \operatorname{ord}(x,\mathcal{G}^*)$ and $N_0 + 1 \leq \operatorname{ord}(x, \mathcal{G}_{N_x(i)})$ and $N_x(i) = N_{x_0}(i)$ for each $1 \leq i \leq m$ and since $x \in \bigcup \mathcal{K}_{n_0,m_x,M_x}$ we would finally have $N_0 + 1 \leq \operatorname{ord}(x, \mathcal{G}_{N_x(m_x)})) = M_x \leq N_0$ by the aid of positive integers m_x and M_x whereas $m_x \leq m_0 < m$ and $M_x \leq N_0$. By repeating the same arguments, one can easily prove after (2) and (3) that, each $\mathcal{K}_{m_0,n_0,N_0}$ is a closed and discrete family in the open subspace $X - \bigcup_{\mathcal{K}_{n,m,N}<\mathcal{K}_{n_0,m_0,N_0}}(\bigcup \mathcal{K}_{n,m,N})$. So it is clearly understood now that, the family $\mathcal{K} = \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{n}\bigcup_{N=1}^{\infty}\mathcal{K}_{n,m,N}$ with this ordering is the required refinement of $\mathcal{G}$ satisfying all the $B(D,\omega_0^2)$-refinability conditions.

Corollary 5 (J.C.Smith [9]) *Every weakly $\overline{\theta}$-refinable space is $B(D, \omega_0^2)$-refinable.*

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