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DUAL POSITIVE IMPLICATIVE HYPER K-IDEALS OF TYPE 1

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Abstract. In this note first we define the notion of *dual positive implicative hyper K-ideal of type 1*, where for simplicity is written by $DPIHKI - T1$. Then we obtain some basic related results. After that we determine hyper K-algebras of order 3, which have $D_1 = \{1\}$, $D_2 = \{1, 2\}$ and $D_3 = \{0, 1\}$ as a $DPIHKI - T1$. Finally we give some connections between the notions of dual positive implicative hyper K-ideals of types 1, 2, 3 and 4.

1 Introduction

The hyperalgebraic structure theory was introduced by F. Marty [6] in 1934. Imai and Iseki [6] in 1966 introduced the notion of a BCK-algebra. Borzooei, Jun and Zahedi et.al. [2,3,10] applied the hyperstructure to BCK-algebras and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. Recently in [8,9,11] we introduced the notions of dual positive implicative hyper K-ideals of types 2, 3 and 4 and then we characterized them. Now in this note first we define the notion of dual positive implicative hyper K-ideal of type 1, then we obtain some results which have been mentioned in the abstract.

2 Preliminaries

Definition 2.1. [2] Let H be a nonempty set and " $\circ$ " be a *hyperoperation* on H, that is " $\circ$ " is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$. Then H is called a hyper K-algebra if it contains a constant "0" and satisfies the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$
(HK3) $x < x$
(HK4) $x < y,\ y < x \Rightarrow x = y$
(HK5) $0 < x$,
for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{\substack{a \in A \\ b \in B}} a \circ b$ of H.

Theorem 2.2. [2] Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all non-empty subsets A, B and C of H the following hold:

(i) $x \circ y < z \Leftrightarrow x \circ z < y$, (ii) $(x \circ z) \circ (x \circ y) < y \circ z$,
(iii) $x \circ (x \circ y) < y$, (iv) $x \circ y < x$,
(v) $A \subseteq B$ implies $A < B$, (vi) $x \in x \circ 0$,
(vii) $(A \circ C) \circ (A \circ B) < B \circ C$, (viii) $(A \circ C) \circ (B \circ C) < A \circ B$,
(ix) $A \circ B < C \Leftrightarrow A \circ C < B$, (x) $A \circ B < A$.

Definition 2.3. [2] Let $(H, \circ, 0)$ be a hyper K-algebra. If there exists an element $1 \in H$ such that $x < 1$ for all $x \in H$, then H is called a bounded hyper K-algebra and 1 is said to

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be the unit of H.
In a bounded hyper K-algebra, we denote $1 \circ x$ by Nx.

Theorem 2.4. [9] In H we have $1 \circ 0 = \{1\}$.

Definition 2.5. [11] Let H be a bounded hyper K-algebra with unit 1. Then a non-empty subset D of H is called *a dual positive implicative hyper K-ideal of type* 2 *($DPIHKI - T2$)* if it satisfies:
(i)$1 \in D$
(ii)$N((Nx \circ Ny) \circ Nz) < D$ and $N(Ny \circ Nz) \subseteq D$ imply that $N(Nx \circ Nz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.6. [11] Let H be a bounded hyper K-algebra with unit 1 and let D be a subset of H containing 1. Then D is a $DPIHKI - T2$ if and only if $N(Ny \circ Nz) \subseteq D$ implies that $N(Nx \circ Nz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.7. [11] Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3 with unit 1 and let $D_1 = \{1\}$ be a subset of H. Then the following statements hold:
(i) Let $1 \circ 2 = \{1\}$. Then D_1 is a $DPIHKI - T2$ if and only if $1 \in 1 \circ 1$.
(ii) Let $1 \circ 2 = \{2\}$. Then D_1 is a $DPIHKI - T2$ if and only if $2 \circ 2 \neq \{0\}$ and $1 \circ 1 \neq \{0\}$.
(iii) Let $1 \circ 2 = \{1, 2\}$. Then:
  (a) If $1 \circ 1 = \{0\}$, then D_1 is not a $DPIHKI - T2$.
  (b) If $1 \in 1 \circ 1$, then D_1 is a $DPIHKI - T2$.
  (c) If $1 \circ 1 = \{0, 2\}$, then D_1 is a $DPIHKI - T2$ if and only if $2 \circ 1 \neq \{0\}$ or $0 \circ 1 \neq \{0\}$.

Theorem 2.8. [11] Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3 with unit 1 and let $D_2 = \{1, 2\}$ be a subset of H. Then D_2 is a $DPIHKI - T2$ if and only if $1 \in (1 \circ 1) \bigcap (1 \circ 2)$.

Theorem 2.9. [11] Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3 with unit 1 and let $D_3 = \{0, 1\}$ be a subset of H. Then the following statements hold:
(i) Let $1 \circ 2 = \{1\}$. Then D_3 is a $DPIHKI - T2$ if and only if $1 \circ 1 \neq \{0, 2\}$.
(ii) Let $1 \circ 2 = \{2\}$. Then D_3 is a $DPIHKI - T2$ if and only if $2 \circ 2 \neq \{0\}$ and $2 \in 1 \circ 1$.
(iii) Let $1 \circ 2 = \{1, 2\}$. Then:
  (a) If $1 \circ 1 \subseteq \{0, 1\}$, then D_3 is not a $DPIHKI - T2$.
  (b) If $1 \circ 1 = \{0, 1, 2\}$, then D_3 is a $DPIHKI - T2$.
  (c) If $1 \circ 1 = \{0, 2\}$, then D_3 is a $DPIHKI - T2$ if and only if $2 \circ 1 \neq \{0\}$ or $0 \circ 1 \neq \{0\}$.

Definition 2.10. [8] Let H be a bounded hyper K-algebra. Then a non-empty subset D of H is called *a dual positive implicative hyper K-ideal of type 3 (DPIHKI-T3)* if it satisfies:
(i)$1 \in D$
(ii)$N((Nx \circ Ny) \circ Nz) < D$ and $N(Ny \circ Nz) < D$ imply $N(Nx \circ Nz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.11. [8] Let H be a bounded hyper K-algebra and let be a subset of H containing 1. Then D is a $DPIHKI - T3$ if and only if $N(Nx \circ Nz) \subseteq D$, for all $x, z \in H$.

Theorem 2.12. [8] Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3 with unit 1 and let $D = \{0, 1\}$ in H. Then D is a $DPIHKI - T3$ if and only if $2 \notin 1 \circ 2$ and $2 \notin 1 \circ 1$.

Definition 2.13. [9] Let H be a bounded hyper K-algebra. Then a non-empty subset D of H is called *a dual positive implicative hyper K-ideal of type* 4 *($DPIHKI - T4$)* if it

satisfies:
(i)$1 \in D$
(ii) $N((Nx \circ Ny) \circ Nz) \subseteq D$ and $N(Ny \circ Nz) < D$ imply that $N(Nx \circ Nz) \subseteq D, \forall x, y, z \in H$.

Theorem 2.14. [9] Let H be a bounded hyper K-algebra and let D be a subset of H containing 1. Then D is a $DPIHKI-T4$ if and only if $N((Nx \circ Ny) \circ Nz) \subseteq D$ implies that $N(Nx \circ Nz) \subseteq D$, $\forall x, y, z \in H$.

Theorem 2.15. (See [9]) Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3 with unit 1 and let $D_1 = \{1\}$ be a subset of H. Then the following statements hold:
(i) Let $1 \circ 2 = \{1\}$. Then D_1 is a $DPIHKI-T4$ if and only if $1 \in 1 \circ 1$.
(ii) Let $1 \circ 2 = \{2\}$. Then D_1 is a $DPIHKI-T4$ if and only if $2 \circ 2 \neq \{0\}$ and $1 \circ 1 \neq \{0\}$.
(iii) Let $1 \circ 2 = \{1, 2\}$. Then:
(a) If $1 \circ 1 = \{0\}$, then D_1 is not a $DPIHKI-T4$.
(b) If $1 \in 1 \circ 1$, then D_1 is a $DPIHKI-T4$.
(c) If $1 \circ 1 = \{0, 2\}$, then D_1 is a $DPIHKI-T4$ if and only if $0 \circ 1 \neq \{0\}$ or $2 \circ 1 \neq \{0\}$.

Theorem 2.16. (See [9]) Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3 with unit 1 and let $D_2 = \{1, 2\}$ be a subset of H. Then the following statements hold:
(i) Let $1 \circ 2 = \{1\}$. Then D_2 is a $DPIHKI-T4$ if and only if $1 \in 1 \circ 1$.
(ii) Let $1 \circ 2 = \{2\}$. Then:
(a) If $1 \circ 1 = \{0\}$, then D_2 is not a $DPIHKI-T4$.
(b) If $1 \circ 1 = \{0, 1\}$, then D_2 is a $DPIHKI-T4$ if and only if $1 \in 2 \circ 1$.
(c) If $1 \circ 1 = \{0, 2\}$, then:
(c_1) If $2 \circ 2 \subseteq \{0, 2\}$, then D_2 is not a $DPIHKI-T4$.
(c_2) If $2 \circ 2 = \{0, 1, 2\}$, then D_2 is a $DPIHKI-T4$.
(c_3) If $2 \circ 2 = \{0, 1\}$, then D_2 is a $DPIHKI-T4$ if and only if $1 \in 0 \circ 2$.
(d) If $1 \circ 1 = \{0, 1, 2\}$, then D_2 is a $DPIHKI-T4$ if and only if $1 \in (0 \circ 2)$ or $(2 \circ 2) = \{0, 1, 2\}$.
(iii) Let $1 \circ 2 = \{1, 2\}$. Then:
(a) If $1 \in 1 \circ 1$, then D_2 is a $DPIHKI-T4$.
(b) If $1 \circ 1 = \{0\}$, then D_2 is not a $DPIHKI-T4$.
(c) If $1 \circ 1 = \{0, 2\}$, then:
(c_1) If $2 \circ 2 = \{0, 1\}$, then D_2 is a $DPIHKI-T4$.
(c_2) If $2 \circ 2 = \{0, 1, 2\}$, then D_2 is a $DPIHKI-T4$ if and only if $2 \circ 1 \neq \{0, 1\}$ or $0 \circ 1 \neq \{0\}$.
(c_3) If $2 \circ 2 \subseteq \{0, 2\}$, then D_2 is a $DPIHKI-T4$ if and only if $1 \in 0 \circ 2$.

Theorem 2.17. (See [9]) Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3 with unit 1 and let $D_3 = \{0, 1\}$ be a subset of H. Then the following statements hold:
(i) Let $1 \circ 2 = \{1\}$. Then D_3 is a $DPIHKI-T4$ if and only if $1 \circ 1 \neq \{0, 2\}$.
(ii) Let $1 \circ 2 = \{2\}$. Then:
(a) If $2 \in 1 \circ 1$, then D_3 is a $DPIHKI-T4$ if and only if $2 \circ 2 \neq \{0\}$.
(b) If $1 \circ 1 = \{0, 1\}$, then D_3 is a $DPIHKI-T4$ if and only if $2 \in 2 \circ 2$.
(c) If $1 \circ 1 = \{0\}$, then D_3 is a $DPIHKI-T4$ if and only if $2 \in (2 \circ 2) \bigcap (2 \circ 1)$.
(iii) Let $1 \circ 2 = \{1, 2\}$. Then:
(a) If $1 \in 1 \circ 1$, then D_3 is a $DPIHKI-T4$.
(b) If $1 \circ 1 = \{0\}$, then D_3 is a $DPIHKI-T4$ if and only if $2 \in (2 \circ 2) \bigcap (2 \circ 1)$.
(c) If $1 \circ 1 = \{0, 2\}$, then D_3 is a $DPIHKI-T4$ if and only if $0 \circ 1 \neq \{0\}$ or $2 \circ 1 \neq \{0\}$.

Theorem 2.18. [11] Let $1 \in 1 \circ x$; $\forall x \in H$. If $0 \notin D$, then D is a $DPIHKI - T2$.

Theorem 2.19. [11] Let $1 \circ y = \{1\}$; $\forall y \in H - \{1\}$, $1 \circ 1 = \{0\}$. Then D is a $DPIHKI - T2$ if and only if $0 \in D$

Theorem 2.20. [11] Let $1 \in 1 \circ x$; $\forall x \in H$ and $x^{'} \in 1 \circ 1$ for some $x^{'} \in H - \{0, 1\}$. If $x^{'} \notin D$, then D is a $DPIHKI - T2$.

3 Dual positive Implicative Hyper K-Ideals of Type 1

From now on H is a bounded hyper K-algebra with unit 1.

Definition 3.1. A non-empty subset D of H is called *a dual positive implicative hyper K-ideal of type* 1 *($DPIHKI - T1$)* if it satisfies:
(i)$1 \in D$
(ii)$N((Nx \circ Ny) \circ Nz) \subseteq D$ and $N(Ny \circ Nz) \subseteq D$ imply that $N(Nx \circ Nz) \subseteq D$, $\forall x, y, z \in H$.

Example 3.2. The following tables show some hyper K-algebra structures on $\{0, 1, 2\}$.

H_1	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0, 1\}$

H_2	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0, 1, 2\}$	$\{0, 2\}$

Then 1 is the unit of H_1 and H_2, also $D_1 = \{1\}$ and $D_3 = \{0, 1\}$ are $DPIHKI - T1$ in H_1 and H_2, while $D_2 = \{1, 2\}$ is a $DPIHKI - T2$ in H_1 and it is not of type 2 in H_2.

In the sequel we let D be a non-empty subset of H containing 1.

Theorem 3.3. If D is a $DPIHKI - T2$, $T3$ or $T4$, then D is a $DPIHKI - T1$.

Proof. The proof follows from Theorems 2.6, 2.11 and 2.14, respectively.

The following example shows that the converse of Theorem 3.3 is not true in general.
Example 3.4. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0, 2\}$
1	$\{1\}$	$\{0, 2\}$	$\{1\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, 2\}$

Then we will see that $D_1 = \{1\}$, $D_2 = \{1, 2\}$ and $D_3 = \{0, 1\}$ are $DPIHKI - T1$, but they are not $DPIHKI - T2$, $T3$ and $T4$.

Theorem 3.5. Let $1 \in 1 \circ x$; $\forall x \in H$. Then:
(i) If $0 \notin D$, then D is a $DPIHKI - T1$.
(ii) If $x \in 1 \circ 1$ for some $x \in H - \{0, 1\}$ and $x \notin D$, then D is a $DPIHKI - T1$.

Proof. The proof follows from Theorems 2.18 , 2.20 and 3.3.

Example 3.6. Let $H = \{0, 1, 2, 3\}$. Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$
2	$\{2\}$	$\{0,3\}$	$\{0,3\}$	$\{3\}$
3	$\{3\}$	$\{0\}$	$\{0\}$	$\{0\}$

Also $D_1 = \{1\}$, $D_2 = \{1, 2\}$, $D_3 = \{1, 3\}$, $D_4 = \{1, 2, 3\}$, $D_5 = \{0, 1\}$ and $D_6 = \{0, 1, 3\}$ are $DPIHKI - T1$, by Theorem 3.5.

Theorem 3.7. Let $x \in H$ and $1 \circ x = \{x\}$. If $x \circ x = \{0\}$ and $x \notin D$, then D is not a $DPIHKI - T1$.

Proof. By hypothesis and Theorem 2.4 we get that $1 \circ (((1 \circ 0) \circ (1 \circ x)) \circ (1 \circ x)) = 1 \circ ((1 \circ x) \circ (1 \circ x)) = 1 \circ (x \circ x) = 1 \circ 0 = \{1\} \subseteq D$ and $1 \circ ((1 \circ x) \circ (1 \circ x)) = \{1\} \subseteq D$, while $1 \circ ((1 \circ 0) \circ (1 \circ x)) = 1 \circ (1 \circ x) = 1 \circ x = \{x\} \not\subseteq D$. Thus D is not a $DPIHKI - T1$.

Example 3.8. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H with unit 1.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

Then $D_1 = \{1\}$ and $D_2 = \{0, 1\}$ are not $DPIHKI - T1$.

Theorem 3.9. Let $1 \circ x = \{1\}$; $\forall x \in H - \{1\}$ and $1 \circ 1 = \{0\}$. Then D is a $DPIHKI - T1$.

Proof. We consider two cases: (i) $0 \in D$ (ii) $0 \notin D$.
(i) If $0 \in D$, then by Theorems 2.19 and 3.3 we conclude that D is a $DPIHKI - T1$.
(ii) Let $0 \notin D$ and on the contrary let D does not be a $DPIHKI - T1$. Then there are $x, y, z \in H$ such that

$$1 \circ (((1 \circ x) \circ (1 \circ y)) \circ (1 \circ z)) \subseteq D, \tag{1}$$

and

$$1 \circ ((1 \circ y) \circ (1 \circ z)) \subseteq D, \tag{2}$$

while

$$1 \circ ((1 \circ x) \circ (1 \circ z)) \not\subseteq D. \tag{3}$$

If x and $z \in H - \{1\}$ or $x = z = 1$, then by some manipulations we conclude that (3) does not hold, which is a contradiction.
If $x \in H - \{1\}$ and $z = 1$, then for $y = 1$, the inclusion (1) does not hold and for $y \in H - \{1\}$, (2) does not hold. So this case is impossible.
If $x = 1$ and $z \in H - \{1\}$. Then we consider two cases: (a) $1 \in 0 \circ 1$, (b) $1 \notin 0 \circ 1$.
(a) If $1 \in 0 \circ 1$, then (1) does not hold, which is a contradiction.
(b) If $1 \notin 0 \circ 1$, then (3) does not hold, which is not true.

Therefore in this case also D is a $DPIHKI-T1$.

Example 3.10. Let $H=\{0,1,2,3\}$. Then the following table shows a hyper K-algebra structure on H with unit 1 such that D is a $DPIHKI-T1$, where $D=\{1\}$, $\{0,1\}$, $\{1,2\}$, $\{1,3\}$, $\{0,1,2\}$, $\{0,1,3\}$ or $\{1,2,3\}$.

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{0,1\}$	$\{3\}$	$\{0,1,3\}$

4 $DPIHKI-T1$ of Hyper K-algebras of Order 3

Henceforth we let $H=\{0,1,2\}$ be a bounded hyper K-algebra of order 3 with unit 1 and $D_1=\{1\}$, $D_2=\{1,2\}$ and $D_3=\{0,1\}$ be subsets of H.

Theorem 4.1. Let $1\circ 1=\{0\}$ and $1\circ 2=\{1\}$. Then D_1, D_2 and D_3 are $DPIHKI-T1$.

Proof. The proof follows from Theorem 3.9.

Example 4.2. Let $H=\{0,1,2\}$. Then the following table shows a hyper K-algebra structure on H such that D_1, D_2 and D_3 are $DPIHKI-T1$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

Theorem 4.3. Let $1\circ 1=\{0\}$ and $1\circ 2=\{2\}$. Then the following statements hold:
(i) D_1 is a $DPIHKI-T1$ if and only if $2\circ 2\neq\{0\}$.
(ii) D_2 is not a $DPIHKI-T1$.
(iii) D_3 is a $DPIHKI-T1$ if and only if $2\in(2\circ 2)\bigcap(2\circ 1)$.

Proof. (i) Let D_1 be a $DPIHKI-T1$. We prove that $2\circ 2\neq\{0\}$. On the contrary let $2\circ 2=\{0\}$. Then $1\circ(((1\circ 0)\circ(1\circ 2))\circ(1\circ 2))=1\circ((1\circ 2)\circ(1\circ 2))=1\circ(2\circ 2)=1\circ 0=\{1\}=D_1$ and $1\circ((1\circ 2)\circ(1\circ 2))=D_1$, while $1\circ((1\circ 0)\circ(1\circ 2))=1\circ(1\circ 2)=1\circ 2=\{2\}\nsubseteq D_1$. Thus D_1 is not a $DPIHKI-T1$, which is a contradiction. Therefore $2\circ 2\neq\{0\}$. Conversely, let $2\circ 2\neq\{0\}$. On the contrary let D_1 do not be $DPIHKI-T1$. Then there are $x,y,z\in H$ such that

$$1\circ(((1\circ x)\circ(1\circ y))\circ(1\circ z))\subseteq D_1, \tag{1}$$

and

$$1\circ((1\circ y)\circ(1\circ z))\subseteq D_1, \tag{2}$$

while

$$1\circ((1\circ x)\circ(1\circ z))\nsubseteq D_1. \tag{3}$$

If $x=z=1$ or $x=z=0$, then (3) does not hold, which is not true.
If $x=0$ and $z=1$, $x=0$ and $z=2$, $x=2$ and $z=2$ or $x=2$ and $z=1$, then by some

manipulations we can see that (1) or (2) does not hold, which is a contradiction.
If $x = 2$ and $z = 0$ we consider two cases: (a) $2 \circ 1 = \{0\}$, (b) $2 \circ 1 \neq \{0\}$.
In (a) we can see that (3) does not hold. In (b) we can check that one of (1)or (2) does not hold. So this case is impossible.
If $x = 1$ and $z = 0$, then by considering two cases $0 \circ 1 = \{0\}$ or $0 \circ 1 \neq \{0\}$, and by some arguments similar as above , we get a contradiction.
If $x = 1$ and $z = 2$, then by considering two cases $0 \circ 2 = \{0\}$ or $0 \circ 2 \neq \{0\}$ we will obtain a contradiction. Therefore D_1 is a $DPIHKI - T1$.
(ii) By hypothesis, $(HK2)$ and Theorem 2.4 we have $2\circ 0 = (1\circ 2)\circ 0 = (1\circ 0)\circ 2 = 1\circ 2 = \{2\}$. Thus $1 \circ (((1 \circ 0) \circ (1 \circ 2)) \circ (1 \circ 1)) = 1 \circ ((1 \circ 2) \circ 0) = 1 \circ (2 \circ 0) = 1 \circ 2 = \{2\} \subseteq D_2$ and $1 \circ ((1 \circ 2) \circ (1 \circ 1)) = 1 \circ (2 \circ 0) = 1 \circ 2 = \{2\} \subseteq D_2$, while $1 \circ ((1 \circ 0) \circ (1 \circ 1)) = 1 \circ (1 \circ 0) = 1 \circ 1 = \{0\} \not\subseteq D_2$. Therefore D_2 is not a $DPIHKI - T1$.
(iii) Let $2 \in (2 \circ 2) \bigcap (2 \circ 1)$. Then by Theorems 2.17 (ii-c) and 3.3 we get that D_3 is a $DPIHKI - T1$. Conversely, let D_3 be a $DPIHKI - T1$. On the contrary let $2 \notin (2 \circ 2)$ or $2 \notin 2\circ 1$. If $2 \notin 2\circ 2$, then $1\circ(((1\circ 0)\circ(1\circ 2))\circ(1\circ 2)) = 1\circ((1\circ 2)\circ 2) = 1\circ(2\circ 2) \subseteq 1\circ(\{0, 1\}) = \{0, 1\} = D_3$ and $1\circ((1\circ 2)\circ(1\circ 2)) \subseteq D_3$, while $1\circ((1\circ 0)\circ(1\circ 2)) = 1\circ(1\circ 2) = 1\circ 2 = \{2\} \not\subseteq D_3$. Thus D_3 is not a $DPIHKI - T1$, which is a contradiction.
If $2 \notin 2 \circ 1$, then $1 \circ (((1 \circ 2) \circ (1 \circ 0)) \circ (1 \circ 1)) = 1 \circ ((2 \circ 1) \circ 0) \subseteq 1 \circ (\{0, 1\} \circ 0) = \{0, 1\} = D_3$ and $1 \circ ((1 \circ 0) \circ (1 \circ 1)) = \{0\} \subseteq D_3$, but $1 \circ ((1 \circ 2) \circ (1 \circ 1)) = 1 \circ (2 \circ 0) = \{2\} \not\subseteq D_3$. Thus D_3 is not a $DPIHKI - T1$, which is a contradiction. Therefore $2 \in (2 \circ 2) \bigcap (2 \circ 1)$.

Now we give some examples about the above theorem.
Example 4.4. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0, 1\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0, 1, 2\}$	$\{0, 2\}$

H_2	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, 1\}$

H_3	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 1\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0, 1\}$	$\{0\}$

H_4	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{0, 1\}$	$\{0, 2\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0, 1, 2\}$. Moreover:
(a) In H_1, H_2, H_3 and H_4, D_2 is not a $DPIHKI - T1$, by Theorem 4.3 (ii)
(b) In H_1, D_1 and D_3 are $DPIHKI - T1$.
(c) In H_3, D_1 and D_3 are not $DPIHKI - T1$.
(d) In H_2 and H_4, D_1 is a $DPIHKI - T1$, while D_3 is not.

Theorem 4.5. Let $1 \circ 1 = \{0\}$ and $1 \circ 2 = \{1, 2\}$. Then:
(i) D_1 and D_2 are $DPIHKI - T1$.
(ii) D_3 is a $DPIHKI - T1$ if and only if $2 \in 2 \circ 1$.

Proof. By $(HK2)$ and hypothesis we have $0 \circ 2 = (1 \circ 1) \circ 2 = (1 \circ 2) \circ 1 = \{1, 2\} \circ 1 = (1 \circ 1) \bigcup (2 \circ 1) = \{0\} \bigcup (2 \circ 1)$. Since $0 \in (2 \circ 1) \bigcap (0 \circ 2)$, then we conclude that $2 \circ 1 = 0 \circ 2$. Now we prove (i) for D_1, the proof of D_2 is similar to D_1. On the contrary let D_1 does not be a $DPIHKI - T1$. Then there are $x, y, z \in H$ such that

$$1 \circ (((1 \circ x) \circ (1 \circ y)) \circ (1 \circ z)) \subseteq D_1, \qquad (1)$$

and

$$1 \circ ((1 \circ y) \circ (1 \circ z)) \subseteq D_1, \tag{2}$$

while

$$1 \circ ((1 \circ x) \circ (1 \circ z)) \not\subseteq D_1. \tag{3}$$

If $x = z = 0$ or $x = z = 1$, then (3) does not hold, which is a contradiction.
If $x \in \{0, 2\}$ and $z \in \{1, 2\}$ or $x = 1$ and $z = 2$, then by some calculations we conclude that (1) or (2) does not hold. So this case is impossible.
If $x = 1$ and $z = 0$, then by considering two cases $0 \circ 1 \neq \{0\}$ or $0 \circ 1 = \{0\}$, we see that (1) or (3) does not hold, respectively, which is a contradiction.
If $x = 2$ and $z = 0$, then by considering two cases $2 \circ 1 = \{0\}$ or $2 \circ 1 \neq \{0\}$, and by some calculations we obtain a contradiction, by (3) or (1), respectively. Note that for the case $2 \circ 1 \neq \{0\}$, we need some calculations.
(ii) The proof is similar to Theorem 4.3 (i).

Now we give some examples about the above theorem.
Example 4.6. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0\}$	$\{1,2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0, 1, 2\}$. Moreover:
(a) In H_1 and H_2, D_1 and D_2 are $DPIHKI - T1$.
(b) In H_1, D_3 is $DPIHKI - T1$, while it is not a $DPIHKI - T1$ in H_2.

Theorem 4.7. Let $1 \in (1 \circ 1) \bigcap (1 \circ 2)$. Then D_1, D_2 and D_3 are $DPIHKI - T1$.

Proof. The proof follows from Theorems 3.5 (i), 2.17(i), (iii-a) and 3.3.

Now we give some examples about the above theorem.
Example 4.8. Let $H = \{0, 1, 2\}$. Then the following tables show some hyper K-algebra structures on H such that D_1, D_2 and D_3 are $DPIHKI - T1$.

H_1	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

H_2	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

H_3	0	1	2
0	$\{0,2\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_4	0	1	2
0	$\{0,1\}$	$\{0,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

Theorem 4.9. Let $1 \circ 1 = \{0, 1\}$ and $1 \circ 2 = \{2\}$. Then:
(i) D_1 is a $DPIHKI - T1$ if and only if $2 \circ 2 \neq \{0\}$.
(ii) D_2 is a $DPIHKI - T1$ if and only if $1 \in 2 \circ 1$.
(iii) D_3 is a $DPIHKI - T1$ if and only if $2 \in 2 \circ 2$.

Proof. (i) Let $2 \circ 2 \neq \{0\}$. Then by Theorems 2.15 (ii) and 3.3 we conclude that D_1 is a $DPIHKI - T1$. Conversely, let D_1 be a $DPIHKI - T1$. We prove that $2 \circ 2 \neq \{0\}$. On the contrary let $2 \circ 2 = \{0\}$. Then we have $1 \circ (((1 \circ 0) \circ (1 \circ 2)) \circ (1 \circ 2)) = 1 \circ ((1 \circ 2) \circ (1 \circ 2)) = 1 \circ (2 \circ 2) = 1 \circ 0 = \{1\} = D_1$ and $1 \circ ((1 \circ 2) \circ (1 \circ 2)) = D_1$, while $1 \circ ((1 \circ 0) \circ (1 \circ 2)) = 1 \circ (1 \circ 2) = \{2\} \nsubseteq D_1$. Thus D_1 is not a $DPIHKI - T1$, which is a contradiction.
(ii) Let $1 \in 2 \circ 1$. Then by Theorems 2.16 (ii-b) and 3.3 we conclude that D_2 is a $DPIHKI - T1$. Conversely, let D_2 be a $DPIHKI - T1$. We prove that $1 \in 2 \circ 1$. On the contrary let $1 \notin 2 \circ 1$. Then $1 \circ (((1 \circ 0) \circ (1 \circ 2)) \circ (1 \circ 0)) = 1 \circ ((1 \circ 2) \circ 1)) = 1 \circ (2 \circ 1) \subseteq 1 \circ \{0, 2\} = \{1, 2\} = D_2$ and $1 \circ ((1 \circ 2) \circ (1 \circ 0)) = 1 \circ (2 \circ 1) \subseteq 1 \circ \{0, 2\} = \{1, 2\} = D_2$ while $1 \circ ((1 \circ 0) \circ (1 \circ 0)) = 1 \circ \{0, 1\} = \{0, 1\} \nsubseteq D_2$. Thus D_2 is not a $DPIHKI - T1$, which is a contradiction. Therefore $1 \in 2 \circ 1$.
(iii) The proof is similar to (i).

Now we give some examples about the above theorem.
Example 4.10. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0, 2\}$	$\{0, 1\}$
1	$\{1\}$	$\{0, 1\}$	$\{2\}$
2	$\{2\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$

H_2	0	1	2
0	$\{0\}$	$\{0, 2\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0, 1\}$	$\{2\}$
2	$\{2\}$	$\{0, 1, 2\}$	$\{0, 1\}$

H_3	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 1\}$
1	$\{1\}$	$\{0, 1\}$	$\{2\}$
2	$\{2\}$	$\{0, 1, 2\}$	$\{0\}$

H_4	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 2\}$
1	$\{1\}$	$\{0, 1\}$	$\{2\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, 1\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0, 1, 2\}$. Moreover:
(a) In H_1, D_1, D_2 and D_3 are $DPIHKI - T1$.
(b) In H_2, D_1 and D_2 are $DPIHKI - T1$, while D_3 is not.
(c) In H_3, D_2 is a $DPIHKI - T1$, while D_1 and D_3 are not.
(d) In H_4, D_1 is a $DPIHKI - T1$, while D_2 and D_3 are not.

Theorem 4.11. Let $1 \circ 1 = \{0, 1, 2\}$ and $1 \circ 2 = \{2\}$. Then:
(i) $D_1(D_3)$ is a $DPIHKI - T1$ if and only if $2 \circ 2 \neq \{0\}$.
(ii) D_2 is a $DPIHKI - T1$ if and only if $1 \in 2 \circ 1$.

Proof. (i) We prove theorem for D_1, the proof of D_3 is the same as D_1. Let $2 \circ 2 \neq \{0\}$. Then by Theorems 2.15 (ii) and 3.3 we conclude that D_1 is a $DPIHKI - T1$. Conversely, on the contrary let $2 \circ 2 = \{0\}$. Then $1 \circ (((1 \circ 0) \circ (1 \circ 2)) \circ (1 \circ 2)) = 1 \circ ((1 \circ 2) \circ 2) = 1 \circ (2 \circ 2) = 1 \circ 0 = \{1\} = D_1$ and $1 \circ ((1 \circ 2) \circ (1 \circ 2)) = D_1$, while $1 \circ ((1 \circ 0) \circ (1 \circ 2)) = 1 \circ (1 \circ 2) = 1 \circ 2 = \{2\} \nsubseteq D_1$. Thus D_1 is not a $DPIHKI - T1$, which is a contradiction. Therefore $2 \circ 2 \neq \{0\}$.
(ii) Let D_2 be a $DPIHKI - T1$. We prove that $1 \in 2 \circ 1$. On the contrary, let $1 \notin 2 \circ 1$. Then $1 \circ (((1 \circ 0) \circ (1 \circ 2)) \circ (1 \circ 0)) = 1 \circ ((1 \circ 2) \circ 1) = 1 \circ (2 \circ 1) \subseteq 1 \circ \{0, 2\} = \{1, 2\} = D_2$ and $1 \circ ((1 \circ 2) \circ (1 \circ 0)) = 1 \circ (2 \circ 1) \subseteq \{1, 2\} = D_2$, while $1 \circ ((1 \circ 0) \circ (1 \circ 0)) = 1 \circ \{0, 1, 2\} = \{0, 1, 2\} \nsubseteq D_2$. Thus D_2 is not a $DPIHKI - T1$, which is a contradiction. So $1 \in 2 \circ 1$. Conversely, let $1 \in 2 \circ 1$. Then by $(HK2)$ we have $2 \circ 1 = (1 \circ 2) \circ 1 = (1 \circ 1) \circ 2 = \{0, 1, 2\} \circ 2 = (0 \circ 2) \bigcup (2 \circ 2) \bigcup \{2\}$. So $1 \in 0 \circ 2$ or $1 \in 2 \circ 2$ and $2 \in 2 \circ 1$. If $1 \in 0 \circ 2$, then by Theorems 2.16 (ii-d) and 3.3, we conclude that D_2 is a $DPIHKI - T1$. If $1 \in 2 \circ 2$, we

prove that D_2 is a $DPIHKI-T1$. On the contrary let D_2 does not be a $DPIHKI-T1$. Then there are $x, y, z \in H$ such that

$$1 \circ (((1 \circ x) \circ (1 \circ y)) \circ (1 \circ z)) \subseteq D_2, \quad (1)$$

and

$$1 \circ ((1 \circ y) \circ (1 \circ z)) \subseteq D_2, \quad (2)$$

while

$$1 \circ ((1 \circ x) \circ (1 \circ z)) \not\subseteq D_2. \quad (3)$$

If $y = 0$ and $z = 2$, then (1) does not hold for all $x \in H$, which is a contradiction.
For the other $y, z \in H$, by some manipulations, we see that (2) does not hold, which is a contradiction.

Now we give some examples about the above theorem.
Example 4.12. Consider the following tables :

H_1	0	1	2
0	$\{0,2\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

H_3	0	1	2
0	$\{0,2\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_4	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0, 1, 2\}$. Moreover:
(a) In H_1, D_1, D_2 and D_3 are $DPIHKI-T1$.
(b) In H_2, D_2 is a $DPIHKI-T1$, while D_1 and D_3 are not.
(c) In H_3, D_1 and D_3 are $DPIHKI-T1$, while D_2 is not.
(d) In H_4, D_1, D_2 and D_3 are not $DPIHKI-T1$.

Theorem 4.13. Let $1 \circ 1 = \{0, 2\}$ and $1 \circ 2 = \{1\}$. Then $D_1(D_2, D_3)$ is a $DPIHKI-T1$ if and only if $2 \circ 1 \neq \{0, 1\}$ or $0 \circ 1 \neq \{0\}$.

Proof. We prove theorem for D_1 the proofs of D_2 and D_3 are similar to D_1. Let D_1 be a $DPIHKI-T1$. We prove that $2 \circ 1 \neq \{0, 1\}$ or $0 \circ 1 \neq \{0\}$. On the contrary let $2 \circ 1 = \{0, 1\}$ and $0 \circ 1 = \{0\}$. Then $1 \circ (((1 \circ 1) \circ (1 \circ 0)) \circ (1 \circ 0)) = 1 \circ ((\{0, 2\} \circ 1) \circ 1) = 1 \circ (\{0, 1\} \circ 1) = 1 \circ \{0, 2\} = \{1\} = D_1$ and $1 \circ ((1 \circ 0) \circ (1 \circ 0)) = 1 \circ \{0, 2\} = D_1$,while $1 \circ ((1 \circ 1) \circ (1 \circ 0)) = 1 \circ (\{0, 2\} \circ 1) = 1 \circ \{0, 1\} = \{0, 1, 2\} \not\subseteq D_1$. Thus D_1 is not a $DPIHKI-T1$, which is a contradiction. Conversely, let $2 \circ 1 \neq \{0, 1\}$ or $0 \circ 1 \neq \{0\}$ and on the contrary let D_1 do not be a $DPIHKI-T1$. Then there are $x, y, z \in H$ such that

$$1 \circ (((1 \circ x) \circ (1 \circ y)) \circ (1 \circ z)) \subseteq D_1, \quad (1)$$

and

$$1 \circ ((1 \circ y) \circ (1 \circ z)) \subseteq D_1, \quad (2)$$

while

$$1 \circ ((1 \circ x) \circ (1 \circ z)) \not\subseteq D_1. \quad (3)$$

Now similar to the proof of Theorem 4.3 (i), we will see that one of (l), (2) or (3) does not hold, which is a contradiction. Therefore D_1 is a $DPIHKI-T1$.

Now we give some examples about the above theorem.

Example 4.14. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$

H_3	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

H_4	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0,1,2\}$. Moreover:
(a) In H_1, H_2 and H_3, D_1, D_2 and D_3 are $DPIHKI-T1$.
(b) In H_4, D_1, D_2 and D_3 are not $DPIHKI-T1$.

Theorem 4.15. Let $1\circ 1=\{0,2\}$ and $1\circ 2=\{2\}$. Then:
(i) $D_1(D_3)$ is a $DPIHKI-T1$ if and only if $2\circ 2\neq\{0\}$.
(ii) If $1\notin 2\circ 2$, then D_2 is not a $DPIHKI-T1$.
(iii) If $2\circ 2=\{0,1,2\}$, then D_2 is a $DPIHKI-T1$.
(iv) If $2\circ 2=\{0,1\}$, then D_2 is a $DPIHKI-T1$ if and only if $1\in 0\circ 1$.

Proof. (i) We prove theorem for D_1, the proof of D_3 is similar to D_1. Let $2\circ 2\neq\{0\}$. Then by Theorems 2.15 (ii) and 3.3 we conclude that D_1 is a $DPIHKI-T1$. Conversely, let D_1 be a $DPIHKI-T1$. We prove that $2\circ 2\neq\{0\}$. On the contrary let $2\circ 2=\{0\}$. Then $1\circ(((1\circ 0)\circ(1\circ 2))\circ(1\circ 2))=1\circ((1\circ 2)\circ 2)=1\circ(2\circ 2)=1\circ 0=\{1\}$ and $1\circ((1\circ 2)\circ(1\circ 2))=1\circ(2\circ 2)=\{1\}=D_1$, while $1\circ((1\circ 0)\circ(1\circ 2))=1\circ(1\circ 2)=1\circ 2=\{2\}\not\subseteq D_1$. Thus D_1 is not a $DPIHKI-T1$, which is a contradiction.
(ii) Let $1\notin 2\circ 2$. Then $1\circ(((1\circ 0)\circ(1\circ 2))\circ(1\circ 1))=1\circ((1\circ 2)\circ\{0,2\})=1\circ\{0,2\}=\{1,2\}=D_2$ and $1\circ((1\circ 2)\circ(1\circ 1))=D_2$, while $1\circ((1\circ 0)\circ(1\circ 1))=1\circ(\{1,2\})=\{0,2\}\not\subseteq D_2$. Thus D_2 is not a $DPIHKI-T1$.
(iii) Let $2\circ 2=\{0,1,2\}$. Then by Theorems 2.16 (ii-c_2) and 3.3 we have D_2 is a $DPIHKI-T1$.
(iv) The proof is similar to the proof of Theorem 4.3 (i).

Now we give some examples about the above theorem.
Example 4.16. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0\}$

H_3	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

H_4	0	1	2
0	$\{0\}$	$\{0,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

H_5	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0,1,2\}$. Also:
(a) In H_1, H_3 and H_5, D_1 and D_3 are $DPIHKI-T1$, while D_2 is not.
(b) In H_2, D_1, D_2 and D_3 are not $DPIHKI-T1$.
(c) In H_4, D_1, D_2 and D_3 are $DPIHKI-T1$.

Theorem 4.17. Let $1\circ 1=\{0,2\}$ and $1\circ 2=\{1,2\}$. Then:
(i) D_1 and D_3 are $DPIHKI-T1$.
(ii) D_2 is a $DPIHKI-T1$ if and only if $0\circ 1\neq\{0\}$ or $2\circ 1\neq\{0,1\}$.

Proof. We prove theorem for D_1 the proof of D_3 is the same as D_1. If $2\circ 1\neq\{0\}$ or $0\circ 1\neq\{0\}$, then by Theorems 2.15 (iii-c) and 3.3 we conclude that D_1 is a $DPIHKI-T1$. If $2\circ 1=\{0\}$ and $0\circ 1=\{0\}$, then

$$(0\circ 2)\bigcup(2\circ 2)=(1\circ 1)\circ 2=(1\circ 2)\circ 1=\{1,2\}\circ 1=\{0,2\}\bigcup(2\circ 1)=\{0,2\}. \quad (1)$$

Now we prove that D_1 is a $DPIHKI-T1$. On the contrary, let D_1 do not be a $DPIHKI-T1$. Then there are $x,y,z\in H$ such that

$$1\circ(((1\circ x)\circ(1\circ y))\circ(1\circ z))\subseteq D_1, \quad (2)$$

and

$$1\circ((1\circ y)\circ(1\circ z))\subseteq D_1, \quad (3)$$

while

$$1\circ((1\circ x)\circ(1\circ z))\nsubseteq D_1. \quad (4)$$

If $x=1$ and $z=0$, then (4) does not hold, which is a contradiction.
If $x\in\{0,1,2\}$ and $z\in\{1,2\}$ or $x\in\{0,2\}$ and $z=0$, then by some calculations and using (1), we can see that (2) or (3) does not hold, which is not true.
(ii) Let D_2 be a $DPIHKI-T1$. We prove that $0\circ 1\neq\{0\}$ or $2\circ 1\neq\{0,1\}$. On the contrary let $0\circ 1=\{0\}$ and $2\circ 1=\{0,1\}$. Then $1\circ(((1\circ 1)\circ(1\circ 0))\circ(1\circ 0))=1\circ(\{0,1\}\circ 1)=1\circ\{0,2\}=\{1,2\}=D_2$ and $1\circ((1\circ 0)\circ(1\circ 0))=D_2$, while $1\circ((1\circ 1)\circ(1\circ 0))=1\circ\{0,1\}=\{0,1,2\}\nsubseteq D_2$. Thus D_2 is not a $DPIHKI-T1$, which is a contradiction. The proof of the converse is similar to the proof of (i).

Now we give some examples about the above theorem.
Example 4.18. Consider the following tables :

H_1	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

H_2	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

H_3	0	1	2
0	$\{0,2\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

H_4	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$

Then each of the above tables gives a hyper K-algebra structure on $\{0,1,2\}$. Moreover:
(a) In H_1, H_3 and H_4, D_1, D_2 and D_3 are $DPIHKI-T1$.

(b) In H_2, D_1 and D_3 are $DPIHKI-T1$, while D_2 is not.

Remark 4.19. Note that Theorems 4.1, 4.3, 4.5, 4.7, 4.9, 4.11, 4.13, 4.15 and 4.17 give a classification of hyper K-algebras of order 3 in which D_1, D_2 or D_3 is a $DPIHKI-T1$.

5 Some Relations Between $DPIHKI-T1$, $T2$, $T3$ And $T4$

Theorem 5.1. Let $1\circ 1\neq\{0\}$ and $1\circ 2=\{2\}$. Then D_1 is a $DPIHKI-T1$ if and only if it is a $DPIHKI-T2$.

Proof. The proof follows from Theorems 2.7(ii), 4.9(i), 4.11(i) and 4.15(i).

Theorem 5.2. Consider the following statements :
(i) $1\circ 1=\{0\}$ and $1\in 1\circ 2$,
(ii) $1\circ 1=\{0,2\}$, $1\circ 2=\{1,2\}$, $2\circ 1=\{0\}$ and $0\circ 1=\{0\}$.
Then under each of the above statements D_1 is a $DPIHKI-T1$, while it is not a $DPIHKI-T2$.

Proof. D_1 is a $DPIHKI-T1$, by Theorems 4.1, 4.5 and 4.17(i). And it is not of type 2, by Theorems 2.7(i,iii-a,c),

Example 5.3. The following tables show some hyper K-algebra structures on $\{0,1,2\}$, such that D_1 is a $DPIHKI-T1$, but it is not a $DPIHKI-T2$.

H_1	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

H_2	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,2\}$

H_3	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$

Theorem 5.4. Consider the following statements :
(i) $1\circ 1=\{0\}$ and $1\in 1\circ 2$,
(ii) $1\circ 1=\{0,2\}$, $1\circ 2=\{2\}$, $2\circ 2=\{0,1,2\}$,
(iii) $1\circ 1=\{0,2\}$, $1\circ 2=\{2\}$, $2\circ 2=\{0,1\}$ and $1\in 0\circ 1$,
(iv) $1\circ 1=\{0,1,2\}$, $1\circ 2=\{2\}$ and $1\in 2\circ 1$.
Then under each of the above statements D_2 is a $DPIHKI-T1$, while it is not a $DPIHKI-T2$.

Proof. D_2 is a $DPIHKI-T1$, by Theorems 4.1, 4.5, 4.15(iii,iv) and 4.11 ii), respectively. While it is not type 2, by Theorem 2.8.

Example 5.5. The following tables show some hyper K-algebra structures on $\{0,1,2\}$, such that D_2 is a $DPIHKI-T1$, but it is not a $DPIHKI-T2$.

H_1	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0\}$

H_2	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1,2\}$

H_3	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

Theorem 5.6. Let $1 \circ 2 = \{2\}$ and $2 \in 1 \circ 1$. Then D_3 is a $DPIHKI-T1$ if and only if it is a $DPIHKI-T2$.

Proof. The proof follows from Theorems 2.9(ii), 4.11(i) and 4.15(i).

Theorem 5.7. Consider the following statements :
(i) $1 \circ 1 = \{0,1\}$, $1 \circ 2 = \{2\}$ and $2 \in 2 \circ 2$,
(ii) $1 \circ 1 = \{0\}$, $1 \circ 2 = \{2\}$, $2 \in (2 \circ 2) \bigcap (2 \circ 1)$,
(iii) $1 \circ 1 = \{0,2\}$, $1 \circ 2 = \{1\}$ and $2 \circ 1 \neq \{0,1\}$ or $0 \circ 1 \neq \{0\}$,
(iv) $1 \circ 2 = \{1,2\}$, $1 \circ 1 \subseteq \{0,1\}$ and $2 \in 2 \circ 1$,
(v) $1 \circ 1 = \{0,2\}$, $1 \circ 2 = \{1,2\}$ and $(2 \circ 1) \bigcup (0 \circ 1) = \{0\}$,
Then under each of the above statements D_3 is a $DPIHKI-T1$, while it is not a $DPIHKI-T2$.

Proof. Theorem 4.9(iii) (4.3(iii), 4.13) together with the statement (i) ((ii), (iii)) implies that D_3 is a $DPIHKI-T1$, while Theorem 2.9(ii)(Theorem 2.9(i)) implies that it is not a $DPIHKI-T2$ in the cases of (i) and (ii)(case of(iii)). Also by using Theorems 4.7 and 4.5(ii) together with the statement (iv) we get that D_3 is a $DPIHKI-T1$, while Theorem 2.9(iii-a) implies that it is not a $DPIHKI-T2$. Finally Theorem 4.17(i) and statement (v) imply that D_3 is a $DPIHKI-T1$, while Theorem 2.9(iii-c) implies that it is not a $DPIHKI-T2$.

Theorem 5.8. Let $1 \circ 1 = \{0,1\}$ or $1 \circ 1 = \{0,1,2\}$. Then D_1 is a $DPIHKI-T1$ if and only if it is a $DPIHKI-T4$.

Proof. The proof follows from Theorems 2.15, 4.7, 4.9(i) and 4.11(i).

Theorem 5.9. Consider the following statements :
(i) $1 \circ 1 = \{0\}$ and $1 \in 1 \circ 2$,
(ii) $1 \circ 1 = \{0\}$, $1 \circ 2 = \{2\}$, $2 \circ 2 \neq \{0\}$,
(iii) $1 \circ 1 = \{0,2\}$, $1 \circ 2 = \{1,2\}$, $(2 \circ 1) \bigcup (0 \circ 1) = \{0\}$.
Then under each of the above statements D_1 is a $DPIHKI-T1$, while it is not a $DPIHKI-T4$.

Proof. Theorems 4.1, 4.5(i) and statement (i) imply that D_1 is a $DPIHKI-T1$, while Theorem 2.15(i,iii-a) implies that it is not a $DPIHKI-T4$. By using Theorem 4.3(i) and statement (ii) we get that D_1 is a $DPIHKI-T1$, while Theorem 2.15(ii) implies that it is not a $DPIHKI-T4$. Finally Theorem 4.17(i) and statement (iii) imply that D_1 is a

$DPIHKI-T1$, while Theorem 2.15(iii-c) implies that it is not a $DPIHKI-T4$.

Theorem 5.10. Let $1\circ 1=\{0,1\}$. Then D_2 is a $DPIHKI-T1$ if and only if it is a $DPIHKI-T4$.

Proof. The proof follows from Theorems 2.16(i,ii-b,iii-a), 4.7 and 4.9(ii).

Theorem 5.11. Consider the following statements :
(i) $1\circ 1=\{0\}$ and $1\in 1\circ 2$,
(ii) $1\circ 1=\{0,2\}$, $1\circ 2=\{1\}$, $2\circ 1\neq\{0,1\}$ and $0\circ 1\neq\{0\}$.
Then under each of the above statements D_2 is a $DPIHKI-T1$, while it is not a $DPIHKI-T4$.

Proof. D_2 is a $DPIHKI-T1$, by Theorems 4.1, 4.5 and 4.13, respectively and it is not of type 4 , by Theorems 2.16(i,iii-b)

Theorem 5.12. Let $1\circ 2=\{2\}$. Then D_3 is a $DPIHKI-T1$ if and only if it is a $DPIHKI-T4$.

Proof. The proof follows from Theorems 2.17(ii), 4.15(i) , 4.11(i), 4.9(iii) and 4.3(iii).

Theorem 5.13. Consider the following statements :
(i) $1\circ 1=\{0,2\}$ and $1\circ 2=\{1\}$, $2\circ 1\neq\{0,1\}$ or $0\circ 1\neq\{0\}$,
(ii) $1\circ 1=\{0\}$, $1\circ 2=\{1,2\}$, $2\in 2\circ 1$ and $2\notin 2\circ 2$,
(iii) $1\circ 1=\{0,2\}$, $1\circ 2=\{1,2\}$, $(0\circ 1)\bigcup(2\circ 1)=\{0\}$.
Then under each of the above statements D_3 is a $DPIHKI-T1$, while it is not a $DPIHKI-T4$.

Proof. D_3 is a $DPIHKI-T1$, by Theorems 4.13, 4.5(ii) and 4.17(i), while D_3 it is not of type 4, by Theorem 2.17(i,iii-b,c).

References

[1] R.A. Borzooei, P. Corsini and M.M. Zahedi, "*Some kinds of positive Implicative hyper K-ideals,*" J. Discrete Mathematics and Cryptography, 6(2003), 97-108.

[2] R.A. Borzooei, A. Hasankhani, M.M. Zahedi and Y.B. Jun, "*On hyper K-algebras*" Math. Japon. Vol. 52, No. 1 (2000), 113-121.

[3] R.A. Borzooei and M.M. Zahedi, "*Positive Implicative hyper K-ideals,*" Scientiae Mathematicae Japonicae, Vol. 53, No. 3 (2001), 525-533.

[4] Y. Imai and K. Iseki, "*On axiom systems of propositional calculi*" XIV Proc. Japan Academy, 42 (1966), 19-22.

[5] K. Iseki and S. Tanaka, "*An introduction to the theory of BCK-algebras*", Math. Japon, 23 (1978), 1-26.

[6] F. Marty, "*Sur une generalization de la notion de groups*", 8th congress Math. Scandinaves, Stockholm, (1934), 45-49.

[7] J. Meng and Y.B. Jun, "BCK-algebras", *Kyung Moonsa, Seoul, Korea*, (1994).

[8] L. Torkzadeh and M.M. Zahedi, "*Dual Positive Implicative Hyper K-Ideals of Type 3*", J. Quasigroups and Related Systems, 9(2002), 85-106.

[9] L. Torkzadeh and M.M. Zahedi, "*Dual Positive Implicative Hyper K-Ideals of Type 4*", Scientiae Mathematicae Japonicae, 59 No.3(2004), 591-603, e9, 583-595.

[10] M.M. Zahedi, R.A. Borzooei, Y.B. Jun and A. Hasankhani, "*Some results on hyper K-algebras*", Scientiae Mathematicae, Vol. 3, No. 1 (2000), 53-59.

[11] M.M. Zahedi, L. Torkzadeh and R.A. Borzooei "*Dual Positive Implicative Hyper K-Ideals of Type 2*", Submitted.

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